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Preface

This volume is an outgrowth of the conference "Representation Theory of Algebraic Groups and Quantum Groups 06" held at Nagoya University, June 12–17, 2006, as the 6th International Conference by the Graduate School of Mathematics, Nagoya University.

This conference was planned as a continuation of the conference of the same name held at Sophia University, Tokyo in the summer of 2001. The aim of the conference was to focus on recent developments in the representation theory of algebraic groups and quantum groups established after the previous conference. During the conference, 23 lectures were delivered by invited speakers, which cover topics such as the modular representations of algebraic groups, representations of quantum groups and crystal bases, representations of affine Lie algebras, representations of Lie algebras in positive characteristic, representations of Hecke algebras, representations of double affine Hecke algebras and Cherednik algebras. There were 132 participants at the conference, including 48 from abroad.

This volume contains 13 articles (plus one appendix) contributed by invited speakers from the conference.

We would like to thank all the participants for their participation and interest in the conference, and in particular the speakers who agreed to write articles for this volume. We also thank the Grant-in-Aid for Scientific Research by the Japan Society for the Promotion of Science for financial support.

Last but not least, we wish to thank Ms. Kazuko Kozaki and other secretaries from the Graduate School of Mathematics, Nagoya University for their dedicated support of the conference, and also for the editorial work done on this volume.

July 2008 *Toshiaki Shoji*

Editorial Committee
Akihiko Gyoja
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Program

The 6th International Conference
Hosted by the Graduate School of Mathematics, Nagoya University

Representation Theory of Algebraic Groups and Quantum Groups 06

Period: June 12–17, 2006

Place: Noyori Conference Hall
Nagoya University
Nagoya, Japan

Organization: T. Shoji (*Chair*),
A. Gyoja, H. Nakajima, K. Shinoda, T. Tanisaki

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Tropical R for affine geometric crystals

13:45–14:45 ***G. I. Lehrer*** (*University of Sydney*)
Endomorphism algebras of tensor powers

15:00–16:00 ***Y. Saito*** (*University of Tokyo*)
On Hecke algebras associated to elliptic root systems

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Representations of W-algebras

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Exotic hearts for triangulated categories of coherent sheaves on cotangent bundles of flag varieties

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q-characters and crystal bases of finite dimensional representations of quantum affine algebras

13:45–14:45 ***R. Rouquier*** (*University of Leeds*)
Deligne–Lusztig varieties and modular representations

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Runner Removal Morita equivalence

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Representations of the Witt–Jacobson algebras in prime characteristic

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Non-recursive characterization of Kleschev bipartitions

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An exotic Deligne–Langlands correspondence for symplectic groups

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Elliptic Hall algebras and double affine Hecke algebras

15:00–16:00 ***T. Suzuki*** (*RIMS, Kyoto University*)
Conformal field theoy and double affine Hecke algebras

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Linear quivers, generic extensions and Kashiwara operators

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11:15–12:15 ***H. H. Andersen*** (*University of Aarhus*)
Some quotient categories of modular representations

Quotient Categories of Modular Representations

Henning Haahr Andersen

Abstract Let G be a reductive algebraic group over a field of prime characteristic p. We prove some results on the invariance under translations by p-multiples of weights for the composition factor multiplicities in Weyl modules, respectively for the Weyl factor multiplicities in indecomposable tilting modules for G. Our methods rely on using appropriate quotient categories. Our setup and arguments work as well for quantum groups at roots of unity.

Keywords Modular representations · Quotient categories · Weyl modules · Tilting modules

Mathematics Subject Classifications (2000): 20G05, 17B37

1 Introduction

Let G be a reductive algebraic group over a field k of characteristic $p > 0$. Denote by T a maximal torus in G and let λ be a dominant character of T (relative to some choice of positive roots). Then it has been observed that many data about representations of G with highest weight λ are preserved when we add a p-multiple of another dominant weight. For instance, we proved in Corollary 3.1 of [2] that if $\Delta(\lambda)$ and $L(\lambda)$ denote the Weyl module and the simple module, respectively, with highest weight λ, then the equalities

$$[\Delta(\lambda) : L(\nu)] = [\Delta(\lambda + p\mu) : L(\nu + p\mu)]$$

for composition factor multiplicities hold *generically* in the lowest dominant p^2-alcove.

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The purpose of this chapter is to explore such invariance further. We shall replace p by any p-power, and we shall not need the genericity assumption in the above result nor shall we stick only to the lowest alcove. In fact, it will be an important feature of our results that they are true for weights close to one or more walls of the dominant chamber and without any upper bounds. On the other hand, we do need to stay in the "vicinity" of our starting point λ – the results we obtain are simply not true when we move further away from λ.

As an illustration of the kind of results we prove, let us state

Theorem 1.1. *Let λ be a dominant weight and choose $r > 0$ such that λ belongs to the lowest p^r-alcove.*

i) *Let μ be a dominant weight and α a simple root with $\langle \mu, \alpha^\vee \rangle > 0$. Then we have for all Weyl group elements x and y for which $p^r\mu + x\cdot\lambda$, $p^r\mu + y\cdot\lambda$ are dominant*

$$[\Delta(p^r\mu + x\cdot\lambda) : L(p^r\mu + y\cdot\lambda)] = [\Delta(p^r(\mu+\omega_\alpha)+x\cdot\lambda) : L(p^r(\mu+\omega_\alpha)+y\cdot\lambda)].$$

ii) *Let μ be a weight for which $\mu - \rho$ is dominant. Then we have for all simple roots α and for all Weyl group elements x and y*

$$(T(p^r\mu + x\cdot\lambda) : \Delta(p^r\mu + y\cdot\lambda)) = (T(p^r(\mu+\omega_\alpha)+x\cdot\lambda) : \Delta(p^r(\mu+\omega_\alpha)+y\cdot\lambda)).$$

Here, ω_α is the fundamental weight corresponding to α and $T(\nu)$ denotes the indecomposable tilting module with highest weight ν; see Sect. 2 below where further unexplained notation may be found.

The methods we use to obtain our invariance under p^r-shifts involve a comparison between G-modules and G_rB-modules. Here, G_r denotes the rth Frobenius kernel in G and B is a Borel subgroup. On the category of G_rB-modules, it is obvious that tensoring with a p^r-multiple of a character for B gives an equivalence. After the necessary preliminaries collected in Sects. 2 and 3, we establish an equivalence between certain quotient categories of G- and G_rB-modules in Sect. 4. Our main results including the proof of Theorem 1.1 are then found in Sects. 5 and 6 below. Section 7 concerns the quantum case, see below.

The quotient categories we work with are generalizations of those considered by Soergel in [16] and more recently by Khomenko in [12]. These two papers were our main motivation for the present work.

Our approach also works for quantum groups at roots of unity (Sect. 7). When the characteristic is zero, the multiplicities we encounter in this case are already known via Kazhdan–Lusztig theory, see [11, 14, 15]. It is conjectured by Lusztig [13] that when $p > 2(h-1)$, then for restricted weights the modular irreducible characters equal their quantized counter parts. This has been proved for $p \gg 0$ in [7], and hence in this case we get specific knowledge of those composition factors of an arbitrary Weyl module $\Delta(\lambda)$ which have highest weights "close" to λ. Similarly, we conjectured in [4] that the characters of modular tilting modules in the lowest p^2-alcove equal their quantized analogue. If this conjecture is verified, then our results here give precise information about the Weyl factors of $T(\lambda)$ when the highest weights of these factors are "close" to λ. For this, we use [15].

2 Finite Dimensional G-Modules

2.1 Let G and T be as in the introduction. Denote by $X = X(T)$, the character group of T. Fix a Borel subgroup B in G containing T and let R, respectively R^+, denote the root system for (G, T), respectively the positive system in R such that $-R^+$ is the set of roots of B. We denote by $S \subset R^+$ the set of simple roots. The set of dominant weights $X^+ \subset X$ consists then of all $\lambda \in X$ for which $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in S$. Here $\alpha^\vee$ denotes the co-root of α. Denote by $\mathcal{C}_G$ the category of finite dimensional G-modules. Similarly, $\mathcal{C}_B$ is the category of finite dimensional B-modules and likewise for other subgroups of G.

2.2 We have $B = TU$ where U is the unipotent radical of B. Any $\lambda \in X$ extends to B by setting $\lambda(U) = 1$. Therefore, we may consider λ as a one-dimensional B-module. We then define

$$\nabla(\lambda) = \mathrm{Ind}_B^G \lambda \in \mathcal{C}_G.$$

Note that $\nabla(\lambda) = 0$ unless $\lambda \in X^+$. For $\lambda \in X^+$, there is a unique simple submodule in $\nabla(\lambda)$. We call this $L(\lambda)$. When λ runs through X^+, this gives us all simple modules in $\mathcal{C}_G$.

When $M \in \mathcal{C}_G$, we denote by M^* the dual module. For $\lambda \in X^+$, we then have $L(\lambda)^* \simeq L(-w_0\lambda)$ where w_0 is the longest element in the Weyl group W for (G, T). We define

$$\Delta(\lambda) = \nabla(-w_0\lambda)^*, \ \ \lambda \in X^+$$

and see that $L(\lambda)$ is the unique simple quotient of $\Delta(\lambda)$. We call $\Delta(\lambda)$ the Weyl module corresponding to λ.

2.3 A tilting module for G is a module $Q \in \mathcal{C}_G$ which allows both a ∇- and a Δ-filtration, i.e., we have submodules F_i and M_j of Q such that

$$0 = F_0 \subset F_1 \subset \cdots \subset F_r = Q = M_s \supset M_{s-1} \supset \cdots \supset M_1 \supset M_0 = 0$$

with $F_i/F_{i-1} \simeq \nabla(\lambda_i)$ for some $\lambda_i \in X^+$, $i = 1, 2, \ldots, r$ and $M_j/M_{j-1} \simeq \Delta(\mu_j)$ for some $\mu_j \in X^+$, $j = 1, 2, \ldots, s$.

Weight considerations show that in the above situation we will always have $r = s$ and $\{\lambda_1, \lambda_2, \ldots, \lambda_r\} = \{\mu_1, \mu_2, \ldots, \mu_s\}$.

In general, when a module $M \in \mathcal{C}_G$ has a ∇-filtration, respectively a Δ-filtration, then we write $(M : \nabla(\lambda))$, respectively $(M : \Delta(\lambda))$ for the number of times $\nabla(\lambda)$ or $\Delta(\lambda)$ occurs in this filtration. The above observation about the occurrences of

Weyl modules and their duals in the filtrations of a tilting module Q can then be stated simply as follows:

$$(Q : \nabla(\lambda)) = (Q : \Delta(\lambda)) \text{ for all } \lambda \in X^+.$$

For each $\lambda \in X^+$, there exists a unique indecomposable tilting module $T(\lambda)$ with highest weight λ. An arbitrary tilting module in $\mathcal{C}_G$ is a direct sum of certain of these $T(\lambda)$'s, see [8].

Remark 2.1. It is one of the major unsolved problems in modular representation theory to determine the multiplicities $(T(\mu) : \nabla(\lambda)),\ \ \mu, \lambda \in X^+$. Even when the irreducible characters for G are known, e.g., when $p \gg 0$ [7] or when the rank of G is 2, only a tiny piece of this problem has been solved. For more details on this as well as a conjecture for the first step, see [4].

2.4 Recall that any finite dimensional T-module M splits into a direct sum of its weight spaces $M_\lambda = \{m \in M \mid tm = \lambda(t)m,\ t \in T\},\ \lambda \in X$. We say that λ is a weight of M if $M_\lambda \neq 0$. The character of M is the element

$$\operatorname{ch} M = \sum_{\lambda \in X} (\dim_k M_\lambda) e^\lambda \in \mathbb{Z}[X].$$

Note that if $M \in \mathcal{C}_G$, then the Weyl group W permutes its weight spaces, i.e., $\operatorname{ch} M \in \mathbb{Z}[X]^W$. Since each of the modules $L(\lambda), \nabla(\lambda), \Delta(\lambda)$, and $T(\lambda)$ has λ as its unique highest weight, it follows that their characters $\{\operatorname{ch} L(\lambda)\}_{\lambda \in X^+}, \{\operatorname{ch} \nabla(\lambda) = \operatorname{ch} \Delta(\lambda)\}_{\lambda \in X^+}$, and $\{\operatorname{ch} T(\lambda)\}_{\lambda \in X^+}$ are bases for $\mathbb{Z}[X]^W$. The numbers $\{[\Delta(\lambda) : L(\mu)]\}_{\lambda, \mu \in X^+}$ and $\{(T(\lambda) : \Delta(\mu))\}_{\lambda, \mu \in X^+}$ are the transition matrices between these bases.

2.5 As usual ρ denotes half the sum of the positive roots. The dot action of W on X is given by $w \cdot \lambda = w(\lambda + \rho) - \rho, w \in W, \lambda \in X$.

The affine Weyl group is the group generated by the reflections $s_{\alpha,n}$ with $\alpha \in R^+$ and $n \in \mathbb{Z}$. Here, $s_{\alpha,n}$ is given by $s_{\alpha,n} \cdot \lambda = s_\alpha \cdot \lambda + np\alpha$. We equip X with the usual ordering $<$ coming from R^+.

Recall that if $\mu, \nu \in X$, then we say that μ is strongly linked to ν and write $\mu \uparrow \nu$ if there is a sequence of weights

$$\mu = \mu_0 < \mu_1 < \cdots < \mu_r = \nu$$

with $\mu_i = s_{\alpha_i, n_i} \cdot \mu_{i-1}$ for some $r \geq 0, \alpha_i \in R^+, n_i \in \mathbb{Z}$. Note that the condition $\mu_{i-1} < \mu_i$ is equivalent to $n_i p > \langle \mu_{i-1} + \rho, \alpha_i^\vee \rangle$.

The strong linkage principle [1] says that if $[\Delta(\nu) : L(\mu)] \neq 0$ for some $\nu, \mu \in X^+$, then $\mu \uparrow \nu$. It also gives that if $(T(\nu) : \Delta(\mu)) \neq 0$, then $\mu \uparrow \nu$.

3 Finite Dimensional G_rT- and G_rB-Modules

Let F be the Frobenius homomorphism on G. When $r \in \mathbb{N}$, we let G_r to denote the kernel of F_r. Then G_r is a normal infinitesimal subgroup scheme of G. Similarly, B_r will denote the Frobenius kernel in B. See [10] for further details on these constructions.

In this section, we fix $r \in \mathbb{N}$. We shall recall some basic facts about the categories $\mathcal{C}_{G_rT}$ and $\mathcal{C}_{G_rB}$ of finite dimensional G_rT- and G_rB-modules.

3.1 The set of r-restricted weights is the subset

$$X_r = \{\lambda \in X \mid 0 \leq \langle \lambda, \alpha^\vee \rangle < p^r,\ \alpha \in S\}.$$

Each $\lambda \in X$ can be written uniquely $\lambda = \lambda^0 + p^r\lambda^1$ with $\lambda^0 \in X_r$ and $\lambda^1 \in X$. In the rest of this chapter the notation λ^0 and λ^1 will always refer to this decomposition of λ.

3.2 In analogy with Sect. 2, we define for $\lambda \in X$

$$\nabla_r(\lambda) = \mathrm{Ind}_{B_rT}^{G_rT}\, \lambda \ \in \mathcal{C}_{G_rT}.$$

This time we have $\nabla_r(\lambda) \neq 0$ for all $\lambda \in X$. More precisely, $\dim_k \nabla_r(\lambda) = p^{r|R^+|}$ for all $\lambda \in X$. There is a unique simple submodule in $\nabla_r(\lambda)$. We denote it is $L_r(\lambda)$ and all simple modules in $\mathcal{C}_{G_rT}$ have this form for some $\lambda \in X$.

A key fact ([10], II.9) about simple G_rT-modules is that we have

$$L_r(\lambda) \simeq L(\lambda^0) \otimes p^r\lambda^1, \quad \lambda \in X. \tag{1}$$

Here, $L(\lambda^0)$ is (the restriction to G_rT of) the simple G-module with highest weight λ^0, see Sect. 2, and the T-character $p^r\lambda^1$ is made into a G_rT-module by extending it trivially on G_r.

3.3 Let B' be the Borel subgroup in G opposite to B. Then we set

$$\Delta_r(\lambda) = \mathrm{Ind}_{B'_rT}^{G_rT}(\lambda - 2(p^r-1)\rho) \ \in \mathcal{C}_{G_rT}.$$

We have that $L_r(\lambda)$ is the unique simple quotient of $\Delta_r(\lambda)$, see [10], II.9.

3.4 A tilting module in $\mathcal{C}_{G_rT}$ is a module that has both ∇_r- and Δ_r-filtration. It turns out that this property is equivalent to being injective [10]. We denote by $Q_r(\lambda)$ the injective hull of $L_r(\lambda)$ in $\mathcal{C}_{G_rT}$. This module has highest weight $w_0\cdot\lambda^0 + p^r(\lambda^1 + 2\rho)$. Hence, it also deserves the notation $T_r(w_0\cdot\lambda^0 + p^r(\lambda^1 + 2\rho))$.

Note that in the special case $\lambda = (p^r-1)\rho$, we have

$$L_r((p^r-1)\rho) \simeq \nabla_r((p^r-1)\rho) \simeq \Delta_r((p^r-1)\rho) \simeq Q_r((p^r-1)\rho) \simeq T_r((p^r-1)\rho).$$

This is the rth Steinberg module. In particular, it is a G-module (sometimes denoted St_r). It is conjectured that more generally the tilting modules $T_r(\lambda)$ with $\lambda \in (p^r - 1)\rho + X_r$ should all have a G-structure. More precisely, if Res denotes the restriction functor from $\mathcal{C}_G$ to $\mathcal{C}_{G_r T}$, then

Conjecture 3.1 (Donkin [8]). $T_r(\lambda) \simeq \mathrm{Res}(T(\lambda))$ for all $\lambda \in (p^r - 1)\rho + X_r$.

This conjecture is known to be true for $p \geq 2(h-1)$, where h is the Coxeter number for R.

3.5 We have $\mathrm{Ext}^j_{\mathcal{C}_{G_r T}}(\Delta_r(\lambda), \nabla_r(\mu)) = \delta_{j,0}\delta_{\lambda,\mu}k$ for all $\lambda, \mu \in X$, [10]. It follows that for any tilting module $Q \in \mathcal{C}_{G_r T}$, we have $(Q : \nabla_r(\lambda)) = \dim_k \mathrm{Hom}_{\mathcal{C}_{G_r T}}(\Delta_r(\lambda), Q)$. In particular, if $Q = Q_r(\mu)$ we deduce the reciprocity law, see [10], II.11

$$(Q_r(\mu) : \nabla_r(\lambda)) = [\Delta_r(\lambda) : L_r(\mu)] \text{ for all } \lambda, \mu \in X.$$

3.6 Most of the above have straightforward analogues for $G_r B$. For each $\lambda \in X$, we set

$$\tilde{\nabla}_r(\lambda) = \mathrm{Ind}_B^{G_r B}\, \lambda \ \in \mathcal{C}_{G_r B}.$$

Each of these modules contains a unique simple submodule and we denote these as $\tilde{L}_r(\lambda)$. To restrict to $G_r T$ amounts to just removing the $\tilde{}$. In particular, $\tilde{L}_r(\lambda)$ remains irreducible when restricted to $G_r T$. This follows from (1) in 3.2.

4 Quotient Categories

4.1 Consider a subset $Y \subset X$. For $\mu \in X$, we write $\mu \leq Y$ if there exists $\lambda \in Y$ such that $\mu \leq \lambda$.

Define a full subcategory $\mathcal{C}_G(\leq Y)$ of $\mathcal{C}_G$ by

$$\mathcal{C}_G(\leq Y) = \{M \in \mathcal{C}_G \mid [M : L(\mu)] = 0 \text{ unless } \ \mu \leq Y\}.$$

Clearly, if $\mu \leq Y$, then the modules $L(\mu), \nabla(\mu), \Delta(\mu)$, and $T(\mu)$ all belong to $\mathcal{C}_G(\leq Y)$.

Similarly, let $\mathcal{C}_G(< Y)$ be the full subcategory of $\mathcal{C}_G(\leq Y)$ consisting of those G-modules $M \in \mathcal{C}_G(\leq Y)$ for which $[M : L(\mu)] = 0$ whenever $\mu \in Y$. This is a Serre subcategory, i.e., it is closed under formations of submodules, quotients, and extensions. We can therefore define the quotient category

$$\mathcal{Q}_G(Y) = \mathcal{C}_G(\leq Y)/\mathcal{C}_G(< Y).$$

The objects in $\mathcal{Q}_G(Y)$ coincide with those in $\mathcal{C}_G(\leq Y)$. To describe the morphisms, we first define for each G-module M the following two submodules:

$$M^+(Y) = \text{ the maximal submodule of } M \text{ belonging to } \mathcal{C}_G(< Y)$$

and

$$M^-(Y) = \text{ the minimal submodule of } M \text{ such that } M/M^-(Y) \in \mathcal{C}_G(< Y).$$

Note that for any $N \in \mathcal{C}_G(< Y)$ we then have

$$\mathrm{Hom}_G(M^-(Y), N) = 0 = \mathrm{Hom}_G(N, M/M^+(Y)). \qquad (*)$$

If now $M, P \in \mathcal{C}_G(\leq Y)$ then we define

$$\mathrm{Hom}_{\mathcal{Q}_G(Y)}(M, P) = \mathrm{Hom}_G(M^-(Y), P/P^+(Y)).$$

It is easy to check that compositions of morphisms exist in $\mathcal{Q}_G(Y)$: If also $Q \in \mathcal{C}_G(\leq Y)$ and $f \in \mathrm{Hom}_{\mathcal{Q}_G(Y)}(M, P), g \in \mathrm{Hom}_{\mathcal{Q}_G(Y)}(P, Q)$, then we claim

(1) $f(M^-(Y)) \subset (P^+(Y) + P^-(Y))/P^+(Y)$
(2) $g(P^+(Y) \cap P^-(Y)) = 0.$

To see that (1) holds, we just have to note that

$$(P/P^+(Y))/((P^+(Y)+P^-(Y))/P^+(Y)) \simeq P/(P^+(Y)+P^-(Y)) \in \mathcal{C}_G(< Y),$$

so that by $(*)$ $\mathrm{Hom}_G(M^-(Y), P/(P^+(Y) + P^-(Y))) = 0$. Similarly (2) follows because $P^+(Y) \cap P^-(Y) \in \mathcal{C}_G(<Y)$ and hence by $(*)$ we get $\mathrm{Hom}_G(P^+(Y) \cap P^-(Y), Q/Q^+(Y)) = 0$.

Because of (1) and (2) we can form the composite

$$M^-(Y) \xrightarrow{f} (P^+(Y)+P^-(Y))/P^+(Y) \simeq P^-(Y)/P^+(Y) \cap P^-(Y) \xrightarrow{\bar{g}} Q/Q^+(Y),$$

where $\bar{g}$ is the homomorphism obtained from g via (2). This is then $g \circ f \in \mathrm{Hom}_{\mathcal{Q}_G(Y)}(M, Q)$.

We denote the natural functor $\mathcal{C}_G(\leq Y) \to \mathcal{Q}_G(Y)$ by q^Y. It is the identity on objects and if $f : M \to N$ is a morphism in $\mathcal{C}_G(\leq Y)$, then $q^Y(f) : M^-(Y) \to N/N^+(Y)$ is the composite $M^-(Y) \subset M \xrightarrow{f} N \to N/N^+(Y)$.

Note that q^Y is an exact functor with $q^Y(N) = 0$ for all $N \in \mathcal{C}_G(< Y)$. In particular, we obtain isomorphisms $M^-(Y) \simeq q^Y(M) \simeq M/M^+(Y)$ in $\mathcal{Q}_G(Y)$.

Remark 4.1. (i) Clearly, $\mathcal{Q}_G(\emptyset) = 0$ while $\mathcal{Q}_G(\{0\})$ is the category of finite dimensional vector spaces over k, and $\mathcal{Q}_G(X^+) = \mathcal{C}_G$.
(ii) For more information about (more general) quotient categories, see, e.g., [9].

4.2 We continue to denote by Y a subset of X. In analogy with the above, we define $\mathcal{C}_{G_rB}(\leq Y)$ to be the full subcategory of $\mathcal{C}_{G_rB}$ consisting of those M for which $[M : \tilde{L}_r(\mu)] = 0$ unless $\mu \leq Y$. Moreover, we let $\mathcal{C}_{G_rB}(< Y)$ denote the full subcategory of $\mathcal{C}_{G_rB}(\leq Y)$ consisting of those M which satisfy $[M : \tilde{L}_r(\mu)] = 0$ if $\mu \in Y$. We then have the quotient category $\mathcal{Q}_{G_rB}(Y) = \mathcal{C}_{G_rB}(\leq Y)/\mathcal{C}_{G_rB}(< Y)$.

4.3 Inside $\mathcal{C}_{G_rB}(\leq Y)$ and $\mathcal{C}_{G_rB}(< Y)$, we have full subcategories $\mathcal{C}^+_{G_rB}(\leq Y)$ and $\mathcal{C}^+_{G_rB}(< Y)$, respectively, whose objects are those M in these categories for which $[M : \tilde{L}_r(\mu)] = 0$ for $\mu \notin X^+$. The quotient category $\mathcal{C}^+_{G_rB}(\leq Y)/\mathcal{C}^+_{G_rB}(< Y)$ is denoted $\mathcal{Q}^+_{G_rB}(Y)$.

Let $p^+ : \mathcal{C}_{G_rB}(\leq Y) \to \mathcal{C}^+_{G_rB}(\leq Y)$ denote the functor which takes a module M into its maximal quotient belonging to $\mathcal{C}^+_{G_rB}(< Y)$. This is clearly a right exact functor which takes the subcategory $\mathcal{C}_{G_rB}(< Y)$ into $\mathcal{C}^+_{G_rB}(< Y)$. The induced functor $\mathcal{Q}_{G_rB}(Y) \to \mathcal{Q}^+_{G_rB}(Y)$ is also denoted p^+.

4.4

Theorem 4.2. *Suppose $Y \subset X$ has the property that if $\mu \in Y$ and $\nu > 0$, then $\mu - p^r\nu \notin Y$. Then the composite*

$$\mathcal{C}_G(\leq Y) \xrightarrow{\mathrm{Res}} \mathcal{C}_{G_rB}(\leq Y) \xrightarrow{p^+} \mathcal{C}^+_{G_rB}(\leq Y)$$

of the restriction functor Res *and the above projection functor p^+ induces an equivalence $\mathcal{Q}_G(Y) \simeq \mathcal{Q}^+_{G_rB}(Y)$. The quasi-inverse functor is induced by* $\mathrm{Ind}^G_{G_rB} : \mathcal{C}^+_{G_rB}(\leq Y) \to \mathcal{C}_G(\leq Y)$.

Proof. Let $\mu \in X^+$. Then the G_rB-composition factors of $L(\mu)$ are $L(\mu^0) \otimes p^r\nu = \tilde{L}_r(\mu^0 + p^r\nu)$, where ν runs through the multiset of weights of $L(\mu^1)$. In fact, we obtain a G_rB-composition series for $L(\mu)$ by taking a B-filtration of $L(\mu^1)$ with one-dimensional quotients, twisting this by F^r, and then tensoring by $L(\mu^0)$. Moreover, $p^+ \circ \mathrm{Res}\, L(\mu) = L(\mu^0) \otimes (p^+_B L(\mu^1))^{(r)}$, where $p^+_B : \mathcal{C}_B \to \mathcal{C}^+_B$ is the projection functor taking a finite dimensional B-module into its maximal quotient with dominant weights. It follows immediately that $p^+ \circ \mathrm{Res}$ takes $\mathcal{C}_G(< Y)$ into $\mathcal{C}^+_{G_rB}(< Y)$.

Note that for any $\mu \in X$, we have $\mathrm{Ind}^G_{G_rB}\, \tilde{L}_r(\mu) \simeq L(\mu^0) \otimes \mathrm{Ind}^G_B(\mu^1)^{(r)}$ [2]. This is 0 unless $\mu \in X^+$, and for such weights we see in particular that all G-composition factors $L(\nu)$ of $\mathrm{Ind}^G_{G_rB}\, \tilde{L}_r(\mu)$ have $\nu = \mu - p^r\eta$ for some $\eta \geq 0$. Thus, $\mathrm{Ind}^G_{G_rB}(\mathcal{C}_{G_rB}(< Y)) \subset \mathcal{C}_G(< Y)$. This gives us induced functors

$$\mathcal{Q}_G(Y) \xrightarrow{p^+ \circ \mathrm{Res}} \mathcal{Q}^+_{G_rB}(Y) \xrightarrow{\mathrm{Ind}^G_{G_rB}} \mathcal{Q}_G(Y).$$

To prove the theorem, we need to check that the two composites $(p^+ \circ \mathrm{Res}) \circ \mathrm{Ind}_{G_rB}^G$ and $\mathrm{Ind}_{G_rB}^G \circ (p^+ \circ \mathrm{Res})$ both coincide with the identity functors on the respective quotient categories.

Let first $M \in \mathcal{C}_G(\leq Y)$. Note that if $M \neq 0$, then also $p^+M \neq 0$. In fact, M has some simple G-quotient $L(\mu)$ and in turn $L(\mu)$ has $\tilde{L}_r(\mu)$ as (unique) simple G_rB-quotient. Hence, p^+M surjects onto $\tilde{L}_r(\mu)$. It follows that the natural map $M \to \mathrm{Ind}_{G_rB}^G(p^+M)$ is non-zero.

Now if $M = L(\mu)$, then by the above $p^+M = L(\mu^0) \otimes (p_B^+ L(\mu^1))^{(r)}$ and hence $\mathrm{Ind}_{G_rB}^G(p^+L(\mu)) \simeq L(\mu^0) \otimes (\mathrm{Ind}_B^G(p_B^+L(\mu^1)))^{(r)}$. Since all weights of $p_B^+L(\mu^1)$ are dominant, we see by Kempf's vanishing theorem that $\mathrm{Ind}_B^G(p_B^+L(\mu^1))$ has a G-filtration with quotients $\mathrm{Ind}_B^G(\nu)$ where ν runs through this set of dominant weights (with multiplicities). In particular, all ν occuring satisfy $\nu \leq \mu^1$ with equality occuring exactly once. It follows that in this case the natural map $L(\mu) \to \mathrm{Ind}_{G_rB}^G(p^+L(\mu))$ is an injection with cokernel in $\mathcal{C}_G(< Y)$.

We then proceed by induction on the length of M. If $0 \to M_1 \to M \to M_2 \to 0$ is an exact sequence in $\mathcal{C}_G(\leq Y)$, then applying p^+ we obtain the exact sequence $p^+M_1 \to p^+M \to p^+M_2 \to 0$ in $\mathcal{C}_{G_rB}^+(\leq Y)$. Now by Kempf's vanishing theorem $\mathrm{Ind}_{G_rB}^G$ is exact on $\mathcal{C}_{G_rB}^+(\leq Y)$, so the rows of the following commutative diagram are exact

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & 0 \\
 & \mathrm{Ind}_{G_rB}^G\, p^+M_1 \arrow[r] & \mathrm{Ind}_{G_rB}^G\, p^+M \arrow[r] & \mathrm{Ind}_{G_rB}^G\, p^+M_2 \arrow[r] & 0.
\end{tikzcd}$$

The injectivity of the left and right vertical maps implies the injectivity of the one in the middle. Similarly, once the cokernels of the same two maps are in $\mathcal{C}_G(< Y)$, we see that this is also true for the middle map. This proves that the composite $\mathrm{Ind}_{G_rB}^G \circ (p^+ \circ \mathrm{Res})$ is the identity on $\mathcal{Q}_G(Y)$.

To prove that the other composite $p^+ \circ \mathrm{Res} \circ \mathrm{Ind}_{G_rB}^G$ is the identity on $\mathcal{Q}_{G_rB}^+(Y)$, we shall for $M \in \mathcal{C}_{G_rB}(\leq Y)$ consider the evaluation map $\mathrm{Ind}_{G_rB}^G M \to M$. If $M = \tilde{L}_r(\mu)$, then this map is the surjection $L(\mu^0) \otimes \nabla(\mu^1)^{(r)} \to \tilde{L}_r(\mu)$ induced by the evaluation $\nabla(\mu^1) \to \mu^1$. The kernel N is clearly in $\mathcal{C}_{G_rB}(< Y)$ and hence also $p^+N \in \mathcal{C}_{G_rB}^+(< Y)$. It follows that $(p^+ \circ \mathrm{Res} \circ \mathrm{Ind}_{G_rB}^G)\tilde{L}_r(\mu)$ is isomorphic to $\tilde{L}_r(\mu)$ in $\mathcal{Q}_{G_rB}^+(Y)$.

Now suppose $0 \to M_1 \to M \to M_2 \to 0$ is an exact sequence in $\mathcal{C}_{G_rB}^+(\leq Y)$. Then we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] & M \arrow[r] & M_2 \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ind}_{G_rB}^G\, M_1 \arrow[r] \arrow[u] & \mathrm{Ind}_{G_rB}^G\, M \arrow[r] \arrow[u] & \mathrm{Ind}_{G_rB}^G\, M_2 \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

The bottom row is exact because $\mathrm{Ind}_{G_rB}^G$ is exact on $\mathcal{C}^+_{G_rB}(\leq Y)$ (by Kempf's vanishing theorem). The vertical homomorphisms are the evaluation maps. We see from the diagram that if the first and third of these are surjective, so is the middle one. Also if the kernels of the first and third are in $\mathcal{C}_{G_rB}(< Y)$, so is the kernel in the middle. Hence, induction on the length of M finishes the proof.

5 Multiplicities in Weyl Modules

In this section, we shall prove some identities (see Corollary 5.10) for composition factor multiplicities in Weyl modules including the ones in part (i) of Theorem 1.1. We first derive a more general inequality in Corollary 5.6. Our method consists of passing to quotient categories as introduced in Sect. 4.

Definition 5.1. A subset $Y \subset X$ is called convex if for any two elements $\lambda, \mu \in Y$ the interval $[\lambda, \mu] = \{\nu \in X \mid \lambda \leq \nu \leq \mu\}$ is also contained in Y.

Example 5.2. (i) A point in X is convex. More generally, any collection of incompatible points in X is convex.
(ii) Of course any interval $[\lambda, \mu]$ is convex.
(iii) If Y is an arbitrary subset of X, then its convex hull is $\hat{Y} = \bigcup_{\lambda,\mu \in Y}[\lambda, \mu]$. This is the smallest convex subset of X containing Y.

Remark 5.3. We may replace the order relation $\leq$ above by the strong linkage relation $\uparrow$. Then we similarly talk about intervals and convex sets with respect to this relation.

Example 5.4. Let C denote the bottom dominant alcove and fix $\lambda \in C$. Then the orbit $W \cdot \lambda$ is a convex subset with respect to $\uparrow$. In fact, if $y, w \in W$, then $w \cdot \lambda$ is strongly linked to $y \cdot \lambda$ if and only if $y \leq w$ in the Bruhat order on W. Hence, $W \cdot \lambda = [w_0 \cdot \lambda, \lambda]$, where w_0 denotes the longest element in W (and where the interval in question is with respect to the strong linkage order). If $\nu \in X$, then we have similar statements about the subset $W \cdot \lambda + p\nu$.

In the rest of this section, we fix $r > 0$ and we assume from now on that Y has the following properties

(a) Y is convex,
(b) Whenever $\lambda \in Y, \alpha \in S$ we have $\lambda - p^r\alpha \notin Y$.

Proposition 5.5. *Let $\lambda \in Y$ and assume that $\lambda = \mu_1 + p^r\mu_2$ with $\mu_1, \mu_2 \in X^+$. Then the natural homomorphism $\nabla(\mu_1) \otimes L(\mu_2)^{(r)} \to \nabla(\lambda)$ is an injection in $\mathcal{Q}_G(Y)$.*

Proof. Let $K(\mu_2)$ denote the kernel of the B-projection $L(\mu_2) \to \mu_2$. Then the short exact sequence in $\mathcal{C}_B$

$$0 \to \mu_1 \otimes K(\mu_2)^{(r)} \to \mu_1 \otimes L(\mu_2)^{(r)} \to \lambda \to 0$$

gives rise to the exact sequence

$$0 \to H^0(\mu_1 \otimes K(\mu_2)^{(r)}) \to \nabla(\mu_1) \otimes L(\mu_2)^{(r)} \to \nabla(\lambda)$$

in $\mathcal{C}_G$. Here, we have used the tensor identity for the middle term. The last map is the natural homomorphism in question and we are therefore done if we prove

$$H^0(\mu_1 \otimes K(\mu_2)^{(r)}) \in \mathcal{C}_G(< Y).$$

But any weight η of $H^0(\mu_1 \otimes K(\mu_2)^{(r)})$ satisfies $\eta \leq \mu_1 + p^r\psi$ for some weight ψ of $K(\mu_2)$. Then $\psi < \mu_2$ and therefore $\eta \leq \mu_1 + p^r(\mu_2 - \alpha) = \lambda - p^r\alpha$ for some $\alpha \in R^+$. If $\eta \in Y$, then the convexity of Y implies that also $\lambda - p^r\alpha \in Y$. This contradicts (b).

Corollary 5.6. *For all $\lambda, \mu \in Y$ and $\nu \in X^+$, we have*

$$[\nabla(\lambda) : L(\mu)] \leq [\nabla(\lambda + p^r\nu) : L(\mu + p^r\nu)].$$

Proof. Since Y satisfies conditions (a) and (b) in 5.4, so does $Y + p^r\nu$. Applying Proposition 5.5 to $\lambda + p^r\nu$, we get an inclusion $\nabla(\lambda) \otimes L(\nu)^{(r)} \hookrightarrow \nabla(\lambda + p^r\nu)$ in $\mathcal{Q}_G(Y + p^r\nu)$. Hence

$$\begin{aligned}[\nabla(\lambda + p^r\nu) : L(\mu + p^r\nu)] &\geq [\nabla(\lambda) \otimes L(\nu)^{(r)} : L(\mu + p^r\nu)]\\ &\geq [\nabla(\lambda) : L(\mu)][L(\mu) \otimes L(\nu)^{(r)} : L(\mu + p^r\nu)] = [\nabla(\lambda) : L(\mu)].\end{aligned}$$

To improve on the results in Corollary 5.6, we first need

Lemma 5.7. *Let $\lambda \in X$ and assume $H^1(\lambda) \neq 0$.*

(i) *There exists a unique $\alpha \in S$ with $\langle \lambda + \rho, \alpha^\vee \rangle < 0$.*
(ii) *Each weight μ of $H^1(\lambda)$ satisfies $\mu \leq s_\alpha \cdot \lambda$ (with α determined by* i)). *More precisely, if $L(\mu)$ is a composition factor of $H^1(\lambda)$, then μ is strongly linked to $s_\alpha \cdot \lambda$.*

Proof. (i) If $\langle \lambda + \rho, \alpha^\vee \rangle \geq 0$ for all $\alpha \in S$, then Kempf's vanishing theorem tells us that $H^i(\lambda) = 0$ for all $i > 0$. Hence, our assumption implies that $S' = \{\alpha \in S \mid \langle \lambda + \rho, \alpha^\vee \rangle < 0\}$ is non-empty. Let P denote the parabolic subgroup in G containing B corresponding to S'. When we apply Kempf's vanishing theorem combined with Serre duality to P/B, we get $H^j(P/B, \lambda) = 0$ for $j < \dim P/B$. The Leray spectral sequence

$$H^r(G/P, H^s(P/B, \lambda)) \implies H^{r+s}(G/B, \lambda)$$

shows that therefore also $H^j(\lambda) = 0$ for all $j < \dim P/B$. Hence, our assumption $H^1(\lambda) \neq 0$ implies $\dim P/B = 1$, i.e., $S' = \{\alpha\}$ for some $\alpha \in S$.

(ii) Let P be as before. According to the arguments in (i), this means that $P = P_\alpha$, the minimal parabolic subgroup corresponding to α. Then the spectral sequence above gives $H^1(\lambda) \simeq H^0(G/P, H^1(P/B, \lambda))$. Therefore, a weight μ of $H^1(\lambda)$ must satisfy $\mu \leq \eta$ for some weight η of $H^1(P/B, \lambda)$. But the weights of $H^1(P/B, \lambda)$ are $\lambda + \alpha, \lambda + 2\alpha, \dots, s_\alpha \cdot \lambda$ so that we do have $\mu \leq \eta \leq s_\alpha \cdot \lambda$ as desired.

To get the "more precise" statement, suppose that $L(\mu)$ is a composition factor of $H^1(\lambda)$. Then the above isomorphism shows that there must be a P_α composition factor $L_\alpha(\eta)$ of $H^1(P_\alpha/B, \lambda)$ such that $L(\mu)$ is a composition factor of $H^0(G/P_\alpha, L_\alpha(\eta))$. But such an η is strongly linked to $s_\alpha \cdot \lambda$, and since $L_\alpha(\eta) \subset H^0(P_\alpha/B, \eta)$ we see that $L(\mu)$ is also a composition factor of $H^0(\eta)$. Hence, μ is strongly linked to η and therefore also to $s_\alpha \cdot \lambda$.

Proposition 5.8. *Suppose Y satisfies* a *and* b *from* 5.4. *Let $\lambda \in Y$ and assume $\lambda = \nu + p^r\mu$ with $\nu, \mu \in X^+$ such that $S' = \{\alpha \in S \mid \langle \mu, \alpha^\vee \rangle \neq 0\} \subset \{\alpha \in S \mid \langle \nu, \alpha^\vee \rangle \geq p^r\}$. Then the natural homomorphism $\nabla(\nu) \otimes L(\mu)^{(r)} \to \nabla(\lambda)$ is an isomorphism in $\mathcal{Q}_G(Y)$.*

Proof. Arguing as in the proof of Proposition 5.5, we obtain an exact sequence

$$0 \to H^0(\nu \otimes K(\mu)^{(r)}) \to \nabla(\nu) \otimes L(\mu)^{(r)} \to \nabla(\lambda) \to H^1(\nu \otimes K(\mu)^{(r)})$$

in $\mathcal{C}_G$. We saw in Proposition 5.5 that the first term belongs to $\mathcal{C}_G(< Y)$. Now we prove that with our additional hypothesis, we also have that the last term belongs to that subcategory.

So suppose η is a weight of $H^1(\nu \otimes K(\mu)^{(r)})$. Then there exists a weight ζ of $K(\mu)$ such that η is a weight of $H^1(\nu \otimes p^r\zeta)$. Hence, by Lemma 5.7 we have $\eta \leq s_\alpha \cdot (\nu + p^r\zeta)$ for some $\alpha \in S$.

Assume first that $\langle \nu, \alpha^\vee \rangle < p^r$. Since $s_\alpha(\zeta)$ is a weight of $L(\mu)$, we must have $s_\alpha(\zeta) \leq \mu$. Our assumption implies that μ is fixed by s_α and hence we have strict inequality $s_\alpha(\zeta) < \mu$ (ζ being a weight of $K(\mu)$). By the above, this gives $\eta \leq s_\alpha \cdot (\nu + p^r\zeta) = s_\alpha \cdot \nu + p^r s_\alpha(\zeta) < \nu + p^r s_\alpha(\zeta) \leq \nu + p^r(\mu - \beta)$ for some $\beta \in R^+$. This makes it impossible for η to belong to Y because if it did then $[\eta, \lambda] \subset Y$ and Y would contain both λ and $\lambda - p^r\beta$.

Next consider the case $\langle \nu, \alpha^\vee \rangle \geq p^r$. Then we have $\eta \leq s_\alpha \cdot \nu + p^r s_\alpha(\zeta) < \nu - p^r\alpha + p^r s_\alpha(\zeta) \leq \nu - p^r\alpha + p^r\mu = \lambda - p^r\alpha$. Again we see that $\eta \notin Y$.

Proposition 5.9. *Let again Y satisfy* a *and* b *from* 5.4. *Then for any $M \in \mathcal{C}_G(\leq Y)$, $\mu \in X^+$ and $\nu \in Y$ we have*

$$[M \otimes L(\mu)^{(r)} : L(\nu + p^r\mu)] = [M : L(\nu)].$$

Proof. It is enough to prove this for $M = L(\lambda)$, $\lambda \in Y$. Note that $[L(\lambda) : L(\nu)] = 0 = [L(\lambda) \otimes L(\mu)^{(r)} : L(\nu + p^r \mu)]$ unless $\lambda^0 = \nu^0$. When $\lambda^0 = \nu^0$, we have

$$[L(\lambda) \otimes L(\mu)^{(r)} : L(\nu + p^r \mu)] = [L(\lambda^1) \otimes L(\mu) : L(\nu^1 + \mu)].$$

These numbers are non-zero only when $\nu^1 \leq \lambda^1$. If this inequality is strict, then we have $\nu \leq \lambda - p^r \alpha$ for some positive root α. By assumptions, this means $\nu \notin Y$. Hence, we must have equality and in this case our equality is clear (both sides are 1).

Corollary 5.10. *Suppose Y and $\lambda = \nu + p^r \mu \in Y$ are as in Proposition* 5.8. *Then we have*

$$[\nabla(\lambda) : L(\eta)] = [\nabla(\nu) : L(\eta - p^r \mu)]$$

for all $\eta \in Y$.

Proof. By Proposition 5.8, we have $\nabla(\lambda) \simeq \nabla(\nu) \otimes L(\mu)^{(r)}$ in $\mathcal{Q}_G(Y)$. When we combine this with Proposition 5.9, we get the corollary.

Remark 5.11. The result stated in Theorem 1.1 i is the special case of this corollary where $Y = \widehat{W \cdot \nu} + p^r \omega_\alpha$ (with convex closure taken with respect to the strong linkage relation) for ν a weight in the bottom dominant p^r-alcove and $\alpha \in S$ with $\langle \nu, \alpha^\vee \rangle > 0$. Of course this reflects just one choice for Y and μ. There are many other configurations where the conditions in this corollary are satisfied.

6 Tilting Modules

Recall from Sect. 2.3 the basic facts and notations concerning tilting modules. In this chapter, we keep $r \in \mathbb{N}$ fixed and we compare some of the Weyl factors in the indecomposable tilting module $T(\lambda)$ with the corresponding factors in $T(\lambda + p^r \mu)$. Here, μ is an arbitrary dominant weight, whereas for λ we need that it lies above the rth Steinberg weight.

6.1 Let $\lambda \in (p^r - 1)\rho + X^+$. Then we write $\lambda = \tilde{\lambda}^0 + p^r \tilde{\lambda}^1$ with $\tilde{\lambda}^0 \in (p^r - 1)\rho + X_r$ and $\tilde{\lambda}^1 \in X^+$. Note that if $\lambda = \lambda^0 + p^r \lambda^1$ is our decomposition for λ from Sect. 3.1, then we have $\tilde{\lambda}^0 = \lambda^0 + p^r \rho'$ and $\tilde{\lambda}^1 = \lambda^1 - \rho'$ where ρ' is the sum of those fundamental weights ω_α for which $\langle \lambda^0, \alpha^\vee \rangle < p^r - 1$.

With this notation, we have

Theorem 6.1 (Donkin [8]). *Suppose $p \geq 2h - 2$. Then there is for each $\lambda \in (p^r - 1)\rho + X^+$ an isomorphism of G-modules*

$$T(\lambda) \simeq T(\tilde{\lambda}^0) \otimes T(\tilde{\lambda}^1)^{(r)}.$$

Remark 6.2. This theorem is in fact a consequence of Conjecture 3.1. In the following, we assume this conjecture (alternatively the sceptical reader may add the assumption $p \geq 2h - 2$ from now on).

6.2

Proposition 6.3. *Let $\lambda \in (p^r - 1)\rho + X^+$. If $\lambda \in Y$ for some $Y \in X$ which satisfies conditions* a *and* b *in* 5.4*, then we have an isomorphism*

$$T(\lambda) \simeq T(\tilde{\lambda}^0) \otimes L(\tilde{\lambda}^1)^{(r)}$$

in the quotient category $\mathcal{Q}_G(Y)$.

Proof. Let $0 = F_0 \subset F_1 \subset \cdots \subset F_s = T(\tilde{\lambda}^1)$ be a composition series. Then there is a unique i with $F_i/F_{i-1} \simeq L(\tilde{\lambda}^1)$. For $j \neq i$, we have $F_j/F_{j-1} \simeq L(\mu_j)$ for some $\mu_j < \tilde{\lambda}^1$. Therefore, $T(\tilde{\lambda}^0) \otimes (F_j/F_{j-1})^{(r)} \in \mathcal{C}_G(< Y)$. The proposition follows easily.

Corollary 6.4. *Let $\lambda \in (p^r - 1)\rho + X^+$. Suppose $\lambda \in Y$ for some $Y \subset X$ which satisfies conditions* a) *and* b) *in* 5.4*. If $\nu \in Y$ and $\mu \in X^+$ satisfy the assumptions in Proposition* 5.8*, then we have*

$$(T(\lambda) : \Delta(\nu)) = (T(\lambda + p^r\mu) : \Delta(\nu + p^r\mu)).$$

Proof. By Theorem 6.1, we have $T(\lambda + p^r\mu) \simeq T(\tilde{\lambda}^0) \otimes T(\tilde{\lambda}^1 + \mu)^{(r)}$. Via Proposition 6.3 we then get an isomorphism $T(\lambda + p^r\mu) \simeq T(\tilde{\lambda}^0) \otimes L(\tilde{\lambda}^1 + \mu)^{(r)}$ in $\mathcal{Q}_G(Y')$ with $Y' = Y + p^r\mu$. If $L(\zeta)$ is a composition factor of $L(\tilde{\lambda}^1) \otimes L(\mu)$ different from $L(\tilde{\lambda}^1 + \mu)$, we have $\zeta < \tilde{\lambda}^1 + \mu$. Hence for such ζ, we have $T(\tilde{\lambda}^0) \otimes L(\zeta)^{(r)} \in \mathcal{C}_G(< Y')$. So we have $T(\lambda + p^r\mu) \simeq T(\tilde{\lambda}^0) \otimes L(\tilde{\lambda}^1)^{(r)} \otimes L(\mu)^{(r)}$ in $\mathcal{Q}_G(Y')$. We shall now combine this with Propositions 5.5 and 5.8 to prove the corollary.

First recall that (for any $\lambda, \nu \in X^+$) we have

$$(T(\lambda) : \Delta(\nu)) = \dim \mathrm{Hom}_G(\Delta(\nu), T(\lambda)).$$

Now in our case we see by the above considerations that

$$\mathrm{Hom}_G(\Delta(\nu + p^r\mu), T(\lambda + p^r\mu)) = \mathrm{Hom}_G(\Delta(\nu + p^r\mu), T(\lambda) \otimes L(\mu)^{(r)}),$$

because of the fact that $\mathrm{Hom}_G(\Delta(\nu + p^r\mu), T(\lambda) \otimes L(\mu')^{(r)}) = 0 = \mathrm{Ext}^1_G(\Delta(\nu + p^r\mu), T(\lambda) \otimes L(\mu')^{(r)})$ for all $\mu' < \mu$ (otherwise $\nu + p^r\mu \leq \lambda + p^r\mu'$, which is impossible by our assumptions on Y).
Take now a ∇-filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_r = T(\lambda)$$

of $T(\lambda)$ ordered such that if η_j is the highest weight of F_j/F_{j-1}, then $\eta_j < \eta_i$ implies $i < j$. Choose then $F = F_{i_1}$ such that $\nu = \eta_{i_1+1}$ and $\nu \neq \eta_j$ for $j \leq i_1$. Then $\mathrm{Hom}_G(\Delta(\nu + p^r\mu), \nabla(\eta_j) \otimes L(\mu)^{(r)}) = 0 = \mathrm{Ext}^1_G(\Delta(\nu + p^r\mu), \nabla(\eta_j) \otimes L(\mu)^{(r)})$ for all $j \leq i_1$ (because $\nu + p^r\mu \not\leq \eta_j + p^r\mu$).

Hence, $\mathrm{Hom}_G(\Delta(\nu + p^r\mu), F \otimes L(\mu)^{(r)})) = 0$ and $\mathrm{Hom}_G(\Delta(\nu + p^r\mu), T(\lambda) \otimes L(\mu)^{(r)})) = \mathrm{Hom}_G(\Delta(\nu + p^r\mu), (T(\lambda)/F) \otimes L(\mu)^{(r)}))$. Now we may assume that our filtration has $\eta_j = \nu$ for $j = i_1 + 1, \ldots, i_2$ and $\eta_j \neq \nu$ for $j > i_2$. We set $F' = F_{i_2}$. Noting that $\mathrm{Hom}_G(\Delta(\nu + p^r\mu), M) = 0$ for all $M \in \mathcal{C}_G(< Y')$ (because $\nu + p^r\mu$ is not a weight of such M), we conclude from Proposition 5.5 that $\mathrm{Hom}_G(\Delta(\nu + p^r\mu), \nabla(\eta) \otimes L(\mu)^{(r)}) \subset \mathrm{Hom}_G(\Delta(\nu + p^r\mu), \nabla(\eta + p^r\mu)) = 0$ for all $\eta \neq \nu$. This means that $\mathrm{Hom}_G(\Delta(\nu + p^r\mu), (T(\lambda)/F_2) \otimes L(\mu)^{(r)}) = 0$ so that we get $\mathrm{Hom}_G(\Delta(\nu + p^r\mu), (T(\lambda)/F_1) \otimes L(\mu)^{(r)}) = \mathrm{Hom}_G(\Delta(\nu + p^r\mu), (F_2/F_1) \otimes L(\mu)^{(r)})$. By Proposition 5.8, the last term has dimension $i_2 - i_1 = (T(\lambda) : \Delta(\nu))$ and we are done.

7 The Quantum Case

7.1 Let U_q denote the quantum group associated with our root system R. We shall assume that q is a primitive lth root of 1 in an arbitrary field k, and that U_q is constructed from the Lusztig divided power method by specializing the quantum parameter to q. We refer to [6] for details.

For each $\lambda \in X^+$, we then have a simple module $L_q(\lambda)$, a Weyl and a dual Weyl module $\Delta_q(\lambda)$ and $\nabla_q(\lambda)$, and an indecomposable tilting module $T_q(\lambda)$ for U_q, all having highest weight λ. These are obtained by "quantizing" the corresponding constructions of modules for G, see [4–6].

Replacing p^r by l, we can now imitate all the previous arguments and obtain analogous results for U_q. We leave the details to the reader and state only the following analogues of Corollaries 5.10 and 6.4.

7.2

Corollary 7.1. *Let $Y \subset X$ be convex such that if $\mu \in Y$ and $\alpha \in R$, then $\mu + l\alpha \notin Y$. Let $\lambda \in Y$ and assume $\lambda = \nu + l\mu$ with $\nu, \mu \in X^+$ such that $\{\alpha \in S \mid \langle \mu, \alpha^\vee \rangle \neq 0\} \subset \{\alpha \in S \mid \langle \nu, \alpha^\vee \rangle \geq l\}$. Then we have*

$$[\nabla_q(\lambda) : L_q(\eta)] = [\nabla_q(\nu) : L_q(\eta - l\mu)]$$

for all $\eta \in Y$.

Corollary 7.2. *Let Y be as in Corollary 7.1 and suppose $\lambda \in (l-1)\rho + X^+$ belongs to Y. Then for each $\eta \in Y$ and each $\mu \in X^+$, we have*

$$(T_q(\lambda) : \Delta_q(\eta)) = (T_q(\lambda + l\mu) : \Delta_q(\eta + l\mu)).$$

Remark 7.3. (a) When the characteristic of k is 0 and l is not too small, then the multiplicities $[\Delta_q(\lambda) : L_q(\mu)]$ are known to be given by the Kazhdan–Lusztig algorithm for all $\lambda, \mu \in X^+$, see [11, 14]. However this algorithm does not immediately reveal the identities in Corollary 7.1.

(b) Still assuming that the characteristic of k is 0, we have that $T_q(\lambda)$ is projective for each $\lambda \in (l-1)\rho + X^+$, see [3]. In this case, we have a reciprocity law relating the multiplicities $(T_q(\lambda) : \Delta_q(\mu))$ to the composition factor multiplicities in $\Delta_q(\mu)$.

(c) When the characteristic of k is positive, the problem of determining both $[\Delta_q(\lambda) : L_q(\mu)]$ and $(T_q(\lambda) : \Delta_q(\eta))$ is just as open as the corresponding problems for G.

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Dipper–James–Murphy's Conjecture for Hecke Algebras of Type B_n

Susumu Ariki and Nicolas Jacon

Dedicated to Toshiaki Shoji and Ken-ichi Shinoda on their 60th birthdays

Abstract We prove a conjecture by Dipper, James, and Murphy that a bipartition is restricted if and only if it is Kleshchev. Hence, the restricted bipartitions naturally label the crystal graphs of level 2 irreducible integrable $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$-modules and the simple modules of Hecke algebras of type B_n in the non semisimple case.

Keywords Hecke algebra · Kleshchev bipartition · Dipper–James–Murphy conjecture

Mathematics Subject Classifications (2000): Primary 17B37; Secondary 20C08, 05E99

1 Introduction

Let F be a field, q and Q invertible elements of F. The Hecke algebra of type B_n is the F-algebra defined by generators $T_0, \dots, T_{n-1}$ and relations

$$(T_0 - Q)(T_0 + 1) = 0, \quad (T_i - q)(T_i + 1) = 0 \ (1 \leqslant i < n)$$
$$(T_0T_1)^2 = (T_1T_0)^2, \quad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \ (1 \leqslant i < n-1)$$
$$T_iT_j = T_jT_i \ (j \geqslant i+2).$$

We denote it by $\mathcal{H}_n(Q, q)$, or $\mathcal{H}_n$ for short. The representation theory of $\mathcal{H}_n$ in the semisimple case was studied by Hoefsmit, which had applications in determining generic degrees and Lusztig's a-values. Motivated by the modular representation theory of $U_n(q)$ and $Sp_{2n}(q)$ in the nondefining characteristic case, Dipper,

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James, and Murphy began the study of the modular case more than a decade ago. The first task was to obtain classification of simple modules. For this, they constructed Specht modules which are indexed by the set of bipartitions [7]. The work shows in particular that Hecke algebras of type B_n are cellular algebras in the sense of Graham and Lehrer.[1] Then they conjectured that the simple modules were labeled by (Q, e)-restricted bipartitions. Their philosophy to classify the simple $\mathcal{H}_n$-modules resembles the highest weight theory in Lie theory: let $\mathfrak{g}$ be a semisimple Lie algebra. It has a commutative Lie subalgebra $\mathfrak{h}$, the Cartan subalgebra. One-dimensional $\mathfrak{h}$-modules are called weights (by abuse of notion). When a $\mathfrak{g}$-module admits a simultaneous generalized eigenspace decomposition with respect to $\mathfrak{h}$, the decomposition is called the (generalized) weight space decomposition. Let Λ be a weight. Suppose that a $\mathfrak{g}$-module M has the property that

(i) Λ appears in the weight space decomposition of M,
(ii) If N is a proper $\mathfrak{g}$-submodule of M, then Λ does not appear in the weight space decomposition of N.

Then the standard argument shows that M has a unique nonzero irreducible quotient. In fact, Verma modules enjoy the property and their irreducible quotients give a complete set of simple objects in the BGG category. Now, we turn to the Hecke algebra $\mathcal{H}_n$. Define the Jucys–Murphy elements $t_1, \dots, t_n$ by $t_1 = T_0$ and $t_{i+1} = q^{-1} T_i t_i T_i$, for $1 \leqslant i \leqslant n-1$. They generate a commutative subalgebra $\mathcal{A}_n$ of the Hecke algebra $\mathcal{H}_n$, and $\mathcal{A}_n$ plays the role of the Cartan subalgebra: one-dimensional $\mathcal{A}_n$-modules are called *weights*, and the generalized simultaneous eigenspace decomposition of an $\mathcal{H}_n$-module is called the *weight space decomposition*. Any weight is uniquely determined by the values at $t_1, \dots, t_n$ of the weight, and the sequence of these values in this order is called the *residue sequence*. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ be a bipartition (see Sect. 2.1) and let $\mathbf{t}$ be a standard bitableau of shape $\boldsymbol{\lambda}$ (see Definition 2.9). Then, $\mathbf{t}$ defines a weight whose values at t_i are given by $c_i q^{b_i - a_i}$ where a_i and b_i are the row number and the column number of the node of $\mathbf{t}$ labelled by i respectively, $c_i = -Q$ if the node is in $\lambda^{(1)}$ and $c_i = 1$ if the node is in $\lambda^{(2)}$. By the theory of seminormal representations in the semisimple case and the modular reduction, a weight appears in some $\mathcal{H}_n$-module if and only if it is obtained from a bitableau this way.

Suppose that there is a weight obtained from a bitableau $\mathbf{t}$ of shape $\boldsymbol{\lambda}$ such that it does not appear in $S^{\boldsymbol{\mu}}$ when $\boldsymbol{\mu} \lhd \boldsymbol{\lambda}$. If such a bitableau exists, we say that $\boldsymbol{\lambda}$ is (Q, e)*-restricted*. This is a clever generalization of the notion of e-restrictedness. Recall that a partition $\lambda = (\lambda_0, \lambda_1, \dots)$ is called *e-restricted* if $\lambda_{i+1} - \lambda_i < e$, for all $i \geqslant 0$. Recall also that we have the similar Specht module theory for Hecke algebras of type A. Using Jucys–Murphy elements of the Hecke algebra of type A, we can define weights as well. Then, a partition is e-restricted if and only if there is a weight obtained from a tableau of shape λ such that it does not appear in S^{μ} when $\mu \lhd \lambda$.

[1] This result has been recently generalized by Geck in [8].

Recall from [7] that

$$[S^{\boldsymbol{\lambda}}] = [D^{\boldsymbol{\lambda}}] + \sum_{\boldsymbol{\mu} \lhd \boldsymbol{\lambda}} d_{\boldsymbol{\lambda}\boldsymbol{\mu}} [D^{\boldsymbol{\mu}}],$$

where the summation is over $\boldsymbol{\mu}$ such that $D^{\boldsymbol{\mu}} \neq 0$, $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ are decomposition numbers, and $\sum_{\boldsymbol{\mu} \lhd \boldsymbol{\lambda}} d_{\boldsymbol{\lambda}\boldsymbol{\mu}}[D^{\boldsymbol{\mu}}]$ is represented by the radical of the bilinear form on $S^{\boldsymbol{\lambda}}$. As $D^{\boldsymbol{\mu}}$ is a surjective image of $S^{\boldsymbol{\mu}}$, it implies that the weight does not appear in the radical, while it appears in $S^{\boldsymbol{\lambda}}$. Therefore, $D^{\boldsymbol{\lambda}} \neq 0$ if $\boldsymbol{\lambda}$ is (Q, e)-restricted. Unlike the case of the BGG category, we may have $D^{\boldsymbol{\lambda}} = 0$, and it is important to know when it occurs. When $-Q$ is not a power of q, a bipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ is (Q, e)-restricted if and only if both $\lambda^{(1)}$ and $\lambda^{(2)}$ are e-restricted. Thus, we know when a bipartition is (Q, e)-restricted. Further, [6, Theorem 4.18] implies that $D^{\boldsymbol{\lambda}} \neq 0$ if and only if $\boldsymbol{\lambda}$ is (Q, e)-restricted, that is, simple $\mathcal{H}_n$-modules are labelled by (Q, e)-restricted bipartitions. Now we suppose that $-Q$ is a power of q. More precisely, we suppose that

1. q is a primitive eth root of unity with $e \geqslant 2$,
2. $-Q = q^m$, for some $0 \leqslant m < e$.

in the rest of the chapter. We call (Q, e)-restricted bipartitions *restricted* bipartitions. They conjectured in this case that $D^{\boldsymbol{\lambda}} \neq 0$ if and only if $\boldsymbol{\lambda}$ is restricted, and it has been known as the Dipper–James–Murphy conjecture for Hecke algebras of type B_n.

Later, connection with the theory of canonical bases in deformed Fock spaces in the sense of Hayashi and Misra-Miwa was discovered by Lascoux–Leclerc–Thibon [11], and its proof in the framework of cyclotomic Hecke algebras [1] allowed the first author and Mathas [2, 4] to label simple $\mathcal{H}_n$-modules by the nth layer of the crystal graph of the level 2 irreducible integrable $\mathfrak{g}(A^{(1)}_{e-1})$-module $L_v(\Lambda_0 + \Lambda_m)$. In the theory, the crystal graph is realized as a subcrystal of the crystal of bipartitions, and the nodes of the crystal graph are called *Kleshchev* bipartitions. More precise definition is given in the next section and $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ is Kleshchev if and only if $\lambda^{(2)} \otimes \lambda^{(1)}$ belongs to the subcrystal $B(\Lambda_0 + \Lambda_m)$ of $B(\Lambda_0) \otimes B(\Lambda_m)$, where the crystals $B(\Lambda_0)$ and $B(\Lambda_m)$ are realized on the set of e-restricted partitions. Now, $D^{\boldsymbol{\lambda}} \neq 0$ if and only if $\boldsymbol{\lambda}$ is Kleshchev by [2]. Hence, we obtained the classification of simple $\mathcal{H}_n$-modules, or more precisely description of the set $\{\boldsymbol{\lambda} \mid D^{\boldsymbol{\lambda}} \neq 0\}$, through a different approach, and the Dipper–James–Murphy conjecture in the modern language is the statement that the Kleshchev bipartitions are precisely the restricted bipartitions.

The aim of this chapter is to prove the Dipper–James–Murphy conjecture. Recall that Lascoux, Leclerc, and Thibon considered Hecke algebras of type A, and they showed that if λ is a e-restricted partition then we can find $a_1, \ldots, a_p$ and $i_1, \ldots, i_p$ such that we may write

$$f_{i_1}^{(a_1)} \ldots f_{i_p}^{(a_p)} \emptyset = \lambda + \sum_{\nu \rhd \lambda} c_{\nu,\lambda}(v) \nu$$

in the deformed Fock space, where $c_{\nu,\lambda}(v)$ are Laurent polynomials. This follows from the ladder decomposition of a partition. Then LLT algorithm proves that Kleshchev partitions are precisely e-restricted partitions. The second author [10] proved the similar formula for FLOTW multipartitions in the Jimbo–Misra–Miwa–Okado higher level Fock space using certain a-values instead of the dominance order. Recall that Geck and Rouquier gave another method to label simple $\mathcal{H}_n$-modules by bipartitions. The result shows that the parametrizing set of simple $\mathcal{H}_n$-modules in the Geck–Rouquier theory, which is called the *canonical basic set*, is precisely the set of the FLOTW bipartitions. Our strategy to prove the conjecture is to give the analogous formula for Kleshchev bipartitions. To establish the formula, a nonrecursive characterization of Kleshchev bipartitions given by the first author, Kreiman and Tsuchioka [5], plays a key role.

The chapter is organized as follows. In the first section, we briefly recall the definition of Kleshchev bipartitions. We also recall the main result of [5]. In the second section, we use this result to give an analogue for bipartitions of the ladder decomposition. Finally, the last section gives a proof for the conjecture.

2 Preliminaries

In this section, we recall the definition of Kleshchev bipartitions together with the main result of [5] which gives a nonrecursive characterization of these bipartitions. We fix m as in the introduction. Namely, the parameter Q of the Hecke algebra is $Q = -q^m$ with $0 \leqslant m < e$.

2.1 First Definitions

Recall that a partition λ is a sequence of weakly decreasing nonnegative integers $(\lambda_0, \lambda_1, \ldots)$ such that $|\lambda| = \sum_{i \geqslant 0} \lambda_i$ is finite. If $\lambda_i = 0$ for $i \geqslant r$, then we write $\lambda = (\lambda_0, \ldots, \lambda_{r-1})$. A *bipartition* $\boldsymbol{\lambda}$ is an ordered pair of partitions $(\lambda^{(1)}, \lambda^{(2)})$. $|\boldsymbol{\lambda}| = |\lambda^{(1)}| + |\lambda^{(2)}|$ is called the *rank* of $\boldsymbol{\lambda}$. The empty bipartition $(\emptyset, \emptyset)$ is the only bipartition of rank zero. The *diagram* of $\boldsymbol{\lambda}$ is the set

$$\{(a, b, c) \mid 1 \leqslant c \leqslant 2,\ 0 \leqslant b \leqslant \lambda_a^{(c)} - 1\} \subseteq \mathbb{Z}^3_{\geqslant 0}.$$

We often identify a bipartition with its diagram. The *nodes* of $\boldsymbol{\lambda}$ are the elements of the diagram. Let $\gamma = (a, b, c)$ be a node of $\boldsymbol{\lambda}$. Then the *residue* of γ is defined by

$$\operatorname{res}(\gamma) = \begin{cases} b - a + m \pmod e & \text{if } c = 1, \\ b - a \pmod e & \text{if } c = 2. \end{cases}$$

By assigning residues to the nodes of a bipartition, we view a bipartition as a colored diagram with colors in $\mathbb{Z}/e\mathbb{Z}$.

Example 2.1. Put $e = 4$, $m = 2$ and $\boldsymbol{\lambda} = ((3,2),(4,2,1))$. Then the colored diagram associated with $\boldsymbol{\lambda}$ is as follows.

$$\left(\begin{array}{|c|c|c|} \hline 2 & 3 & 0 \\ \hline 1 & 2 \\ \cline{1-2} \end{array} \, , \; \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline 3 & 0 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array} \right)$$

If γ is a node with residue i, we say that γ is an *i-node*. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be two bipartitions such that $\boldsymbol{\mu} = \boldsymbol{\lambda} \sqcup \{\gamma\}$. Then, we denote $\boldsymbol{\mu}/\boldsymbol{\lambda} = \gamma$ and if $\operatorname{res}(\gamma) = i$, we say that γ is a *removable i-node* of $\boldsymbol{\mu}$. We also say that γ is an *addable i-node* of $\boldsymbol{\lambda}$ by abuse of notion.[2]

Let $i \in \mathbb{Z}/e\mathbb{Z}$. We choose a total order on the set of removable and addable i-nodes of a bipartition. Let $\gamma = (a,b,c)$ and $\gamma' = (a',b',c')$ be removable or addable i-nodes of a bipartition. We say that γ is *above* γ' if either $c = 1$ and $c' = 2$, or $c = c'$ and $a < a'$.[3]

Let $\mathcal{F}$ be the vector space over $\mathbb{Q}$ such that the basis is given by the set of all bipartitions. We color the nodes of bipartitions as above. We call it the (level two) *Fock space*. We may equip it with $\widehat{sl}_e$-module structure in which the action of the Chevalley generators is given by

$$e_i\boldsymbol{\lambda} = \sum_{\boldsymbol{\nu}:\operatorname{res}(\boldsymbol{\lambda}/\boldsymbol{\nu})=i} \boldsymbol{\nu}, \quad f_i\boldsymbol{\lambda} = \sum_{\boldsymbol{\nu}:\operatorname{res}(\boldsymbol{\nu}/\boldsymbol{\lambda})=i} \boldsymbol{\nu}.$$

Using the total order on the set of removable and addable i-nodes given above, we deform the $\widehat{sl}_e$-module structure to $U_v(\widehat{sl}_e)$-module structure on the deformed Fock space $\mathcal{F} \otimes_{\mathbb{Q}} \mathbb{Q}(v)$, which is the tensor product of two level 1 deformed Fock spaces. We refer to [3] for the details.

2.2 Kleshchev Bipartitions

Recall that the crystal basis of the deformed Fock space is given by the basis vectors of the deformed Fock space. Hence, it defines a crystal structure on the set of bipartitions. We call it *the crystal of bipartitions*. As is explained in [3], the map $(\lambda^{(1)}, \lambda^{(2)}) \mapsto \lambda^{(2)} \otimes \lambda^{(1)}$ identifies the crystal of bipartitions with the tensor product of the crystal of partitions of highest weight Λ_0 and that of highest weight Λ_m. As is already mentioned in the introduction, Kleshchev bipartitions are those bipartitions which belongs to the same connected component as the empty

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2 An addable i-node of $\boldsymbol{\lambda}$ is not a node of $\boldsymbol{\lambda}$.

3 We now know that there are more than one Specht module theory, and different Specht module theories prefer different total orders on the set of i-nodes of a bipartition. Our choice of the total order is the one prefered by Dipper–James–Murphy's Specht module theory.

bipartition in the crystal of bipartitions. Equivalently, Kleshchev bipartitions are those bipartitions which may be obtained from the empty bipartition by applying the Kashiwara operators successively. Rephrasing it in combinatorial terms, we have a recursive definition of Kleshchev bipartitions as follows.

Let $\boldsymbol{\lambda}$ be a bipartition and let γ be an i-node of $\boldsymbol{\lambda}$, we say that γ is a *normal* i-node of $\boldsymbol{\lambda}$ if, whenever η is an addable i-node of $\boldsymbol{\lambda}$ below γ, there are more removable i-nodes between η and γ than addable i-nodes between η and γ. If γ is the highest normal i-node of $\boldsymbol{\lambda}$, we say that γ is a *good* i-node of $\boldsymbol{\lambda}$. When γ is a good i-node, we denote $\boldsymbol{\lambda} \setminus \{\gamma\}$ by $\tilde{e}_i\boldsymbol{\lambda}$.

Definition 2.2. A bipartition $\boldsymbol{\lambda}$ is *Kleshchev* if either $\boldsymbol{\lambda} = (\emptyset, \emptyset)$ or there exists $i \in \mathbb{Z}/e\mathbb{Z}$ and a good i-node γ of $\boldsymbol{\lambda}$ such that $\boldsymbol{\lambda} \setminus \{\gamma\}$ is Kleshchev.

Note that the definition depends on m. The reader can prove easily using induction on $n = |\lambda^{(1)}| + |\lambda^{(2)}|$ that if $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ is Kleshchev, then both $\lambda^{(1)}$ and $\lambda^{(2)}$ are e-restricted. By general property of crystal bases, the following is clear.

Lemma 2.3. *Suppose that $\boldsymbol{\lambda}$ is a Kleshchev bipartition, γ a good i-node of $\boldsymbol{\lambda}$, for some i. Then $\tilde{e}_i\boldsymbol{\lambda} = \boldsymbol{\lambda} \setminus \{\gamma\}$ is Kleshchev.*

In [5], the first author, Kreiman and Tsuchioka have given a different characterization of Kleshchev bipartitions.

Let λ be a partition. Then the set of *beta numbers of charge* h, where we only use $h = 0$ or $h = m$ in the chapter, is by definition the set J_h of decreasing integers

$$j_0 > j_1 > \cdots > j_k > \cdots$$

defined by $j_k = \lambda_k + h - k$, for $k \geqslant 0$. The charge h also defines a coloring of nodes: $\operatorname{res}(\gamma) = b - a + h \pmod{e}$ where a and b are the row number and the column number of a node γ, respectively.

An addable i-node of λ corresponds to $x \in J_h$ such that $x + e\mathbb{Z} = i$ and $x + 1 \notin J_h$. We call x an addable i-node of J_h. Similarly, a removable i-node of λ corresponds to $x \in J_h$ such that $x + e\mathbb{Z} = i + 1$ and $x - 1 \notin J_h$. We call x a removable i-node of J_h.

We define the abacus display of J_h in the usual way. The i^{th} runner of the abacus is $\{x \in \mathbb{Z} \mid x + e\mathbb{Z} = i\}$, for $i \in \mathbb{Z}/e\mathbb{Z}$.

Definition 2.4. Let λ be an e-restricted partition, and $J_m(\lambda)$ the corresponding set of beta numbers of charge m. We write J_m for $J_m(\lambda)$ and define

$$U(J_m) = \{x \in J_m \mid x - e \notin J_m\}.$$

If λ is an e-core, then we define $\operatorname{up}_m(\lambda) = \lambda$. Otherwise let $p = \max U(J_m)$ and define

$$V(J_m) = \{x > p \mid x \neq p \pmod{e},\ x - e \in J_m,\ x \notin J_m\}.$$

Note that $V(J_m)$ is nonempty since λ is e-restricted. Let $q = \min V(J_m)$. Then we define

$$\mathrm{up}(J_m) = (J_m \setminus \{p\}) \sqcup \{q\}$$

and we denote the corresponding partition by $\mathrm{up}_m(\lambda)$.

In [5], it is shown that $\mathrm{up}_m(\lambda)$ is again e-restricted and we reach an e-core after applying up_m finitely many times.

Definition 2.5. Let λ be an e-restricted partition. Apply up_m repeatedly until we reach an e-core. We denote the resulting e-core by $\mathrm{roof}_m\,(\lambda)$.

Definition 2.6. Let λ be an e-restricted partition, $J_0(\lambda)$ the corresponding set of beta numbers of charge 0. We write J_0 for $J_0(\lambda)$ and define

$$U(J_0) = \{x \in J_0 \mid x - e \notin J_0\}.$$

If λ is an e-core, then we define $\mathrm{down}_0(\lambda) = \lambda$. Otherwise let $p' = \min U(J_0)$ and define

$$W(J_0) = \{x > p' - e \mid x \in J_0,\ x + e \notin J_0\} \cup \{p'\}.$$

It is clear that $W(J_0)$ is nonempty. Let $q' = \min W(J_0)$. Then we define

$$\mathrm{down}(J_0) = (J_0 \setminus \{q'\}) \sqcup \{p' - e\}$$

and we denote the corresponding partition by $\mathrm{down}_0(\lambda)$.

In [5], it is shown that $\mathrm{down}_0(\lambda)$ is again e-restricted and we reach an e-core after applying down_0 finitely many times.

Definition 2.7. Let λ be an e-restricted partition. Apply down_0 repeatedly until we reach an e-core. We denote the resulting e-core by $\mathrm{base}_0\,(\lambda)$.

Finally, let λ be an e-restricted partition, $J_0^{\max}$ the set of beta numbers of charge 0 for $\mathrm{base}_0(\lambda)$. Define $M_i(\lambda)$, for $i \in \mathbb{Z}/e\mathbb{Z}$, by

$$M_i(\lambda) = \max\{x \in J_0^{\max} \mid x + e\mathbb{Z} = i\}.$$

We write $M_i(\lambda)$ in decreasing order

$$M_{i_1}(\lambda) > M_{i_2}(\lambda) > \cdots > M_{i_e}(\lambda).$$

Then $J_0^{\max} \sqcup \{M_{i_k}(\lambda) + e\}_{1 \leqslant k \leqslant m}$ is the set of beta numbers of charge m, for some partition. We denote the partition by $\tau_m(\mathrm{base}_0(\lambda))$.

Now, the characterization of Kleshchev bipartitions is as follows.

Theorem 2.8 ([5]). *Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ be a bipartition such that both $\lambda^{(1)}$ and $\lambda^{(2)}$ are e-restricted. Then $\boldsymbol{\lambda}$ is Kleshchev if and only if*

$$\mathrm{roof}_m(\lambda^{(1)}) \subseteq \tau_m(\mathrm{base}_0(\lambda^{(2)})).$$

2.3 The Dipper–James–Murphy Conjecture

We recall the dominance order for bipartitions. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ and $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)})$ be bipartitions. In this chapter, we write $\boldsymbol{\mu} \trianglelefteq \boldsymbol{\lambda}$ if

$$\sum_{k=1}^{j} \lambda_k^{(1)} \geqslant \sum_{k=1}^{j} \mu_k^{(1)} \text{ and } |\lambda^{(1)}| + \sum_{k=1}^{j} \lambda_k^{(2)} \geqslant |\mu^{(1)}| + \sum_{k=1}^{j} \mu_k^{(2)},$$

for all $j \geqslant 0$.

Definition 2.9. Let $\boldsymbol{\lambda}$ be a bipartition of rank n. A *standard bitableau of shape* $\boldsymbol{\lambda}$ is a sequence of bipartitions

$$\emptyset = \boldsymbol{\lambda}[0] \subseteq \boldsymbol{\lambda}[1] \subseteq \cdots \subseteq \boldsymbol{\lambda}[n] = \boldsymbol{\lambda}$$

such that the rank of $\boldsymbol{\lambda}[k]$ is k, for $0 \leqslant k \leqslant n$. Let $\mathbf{t}$ be a standard bitableau of shape $\boldsymbol{\lambda}$. Then the *residue sequence* of $\mathbf{t}$ is the sequence

$$(\mathrm{res}(\gamma[1]), \ldots, \mathrm{res}(\gamma[n])) \in (\mathbb{Z}/e\mathbb{Z})^n,$$

where $\gamma[k] = \boldsymbol{\lambda}[k]/\boldsymbol{\lambda}[k-1]$, for $1 \leqslant k \leqslant n$.

A standard bitableau may be viewed as filling of the nodes of $\boldsymbol{\lambda}$ with numbers $1, \ldots, n$: we write k in the node $\gamma[k]$, for $1 \leqslant k \leqslant n$.

Definition 2.10. A bipartition $\boldsymbol{\lambda}$ is $(-q^m, e)$-*restricted*, or *restricted* for short, if there exists a standard bitableau $\mathbf{t}$ of shape $\boldsymbol{\lambda}$ such that the residue sequence of any standard bitableau of shape $\boldsymbol{\nu} \lhd \boldsymbol{\lambda}$ does not coincide with the residue sequence of $\mathbf{t}$.

Conjecture 2.11 ([7, Conj. 8.13]). A bipartition $\boldsymbol{\lambda}$ is Kleshchev if and only if it is restricted.

3 Properties of Kleshchev Bipartitions

The aim of this section is to prove some combinatorial results concerning Kleshchev bipartitions.

3.1 Admissible Sequence

Definition 3.1. Let $i \in \mathbb{Z}/e\mathbb{Z}$. We say that a sequence of removable i-nodes $R_1, \ldots, R_s$ (where $s \geqslant 1$) of a bipartition $\boldsymbol{\lambda}$ is an *admissible sequence of i-nodes for* $\boldsymbol{\lambda}$ if

- $R_1, \ldots, R_s$ are the lowest s removable i-nodes of $\boldsymbol{\lambda}$ and every addable i-node is above all of these nodes, and
- If there is a removable i-node R above $R_1, \ldots, R_s$, there must exist an addable i-node below R.

The following lemma shows the existence of an admissible sequence of i-nodes, for some i, for a Kleshchev bipartition: choose i as in the lemma and read addable and removable i-nodes in the total order of nodes. Suppose that $\boldsymbol{\lambda}$ has at least one addable i-node and let η be the lowest addable i-node. Then removable i-nodes below η form an admissible sequence of i-nodes. If $\boldsymbol{\lambda}$ does not have an addable i-node, all removable i-nodes of $\boldsymbol{\lambda}$ form an admissible sequence of i-nodes.

Lemma 3.2. *Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ be a nonempty Kleshchev bipartition. Then there exists $i \in \mathbb{Z}/e\mathbb{Z}$ and a removable i-node γ such that if η is an addable i-node of $\boldsymbol{\lambda}$, then η is above γ.*

Proof. Recall that both $\lambda^{(1)}$ and $\lambda^{(2)}$ are e-restricted. There are two cases to consider.

- Assume that $\lambda^{(2)}$ is not the empty partition. Let $\lambda^{(2)}_j$ be the last row. Define $\gamma = (j, \lambda^{(2)}_j - 1, 2)$ and $i = \mathrm{res}(\gamma)$. Since $\lambda^{(2)}$ is e-restricted, the residue of the addable node $(j+1, 0, 2)$ is not i. Hence, all the addable i-node of $\boldsymbol{\lambda}$ are above γ.
- Assume that $\lambda^{(2)}$ is the empty partition. Let $\lambda^{(1)}_j$ be the last row. Define $\gamma = (j, \lambda^{(1)}_j - 1, 1)$ and $i = \mathrm{res}(\gamma)$. Since $\lambda^{(1)}$ is e-restricted, the residue of the addable node $(j+1, 0, 1)$ is not i. We show that the residue of the addable node $(0, 0, 2)$ is not i. Suppose to the contrary that the residue is i. As $\boldsymbol{\lambda}$ is Kleshchev, we may delete good nodes successively to obtain the empty bipartition. Hence, γ must be deleted at some point in the process. However, it can never be a good node since we always have an addable i-node $(0, 0, 2)$ just below it and there is no removable i-node between them, and so we have a contradiction.

Hence the claim follows. □

Lemma 3.3. *Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ be a nonempty Kleshchev bipartition and $R_1, \ldots, R_s$ an admissible sequence of i-nodes for $\boldsymbol{\lambda}$. Define $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)})$ by $\boldsymbol{\lambda} = \boldsymbol{\mu} \sqcup \{R_1, \ldots, R_s\}$. Then $\boldsymbol{\mu}$ is Kleshchev.*

Proof. Recall that $\lambda^{(1)}$ and $\lambda^{(2)}$ are both e-restricted. We claim that $\mu^{(1)}$ and $\mu^{(2)}$ are both e-restricted. We only prove that $\mu^{(1)}$ is e-restricted as the proof for $\mu^{(2)}$ is the same. Suppose that $\mu^{(1)}$ is not e-restricted. Since $\lambda^{(1)}$ is e-restricted, it occurs only when there exists j such that $\lambda^{(1)}_j = \lambda^{(1)}_{j+1} + e - 1$, $\mu^{(1)}_j = \lambda^{(1)}_j$, $\mu^{(1)}_{j+1} = \lambda^{(1)}_{j+1} - 1$ and $\mathrm{res}(j+1, \lambda^{(1)}_{j+1} - 1, 1) = i$. But then $\mathrm{res}(j, \lambda^{(1)}_j - 1, 1) = i$, which implies $\mu^{(1)}_j = \lambda^{(1)}_j - 1$ by definition of $\boldsymbol{\mu}$.

First case: First, we consider the case when either $\lambda^{(2)} = \emptyset$ or $\lambda^{(2)} \neq \emptyset$ and $\lambda^{(2)}$ has no addable i-node. If $\lambda^{(1)}$ has no addable i-node, then the admissible sequence

$R_1, \dots, R_s$ is given by all the removable i-nodes of $\boldsymbol{\lambda}$, and thus all the normal i-nodes of $\boldsymbol{\lambda}$. Hence $\boldsymbol{\mu} = \tilde{e}_i^s \boldsymbol{\lambda}$, which implies that $\boldsymbol{\mu}$ is Kleshchev. Therefore, we may and do assume that $\lambda^{(1)}$ has at least one addable i-node.

Define $t \geqslant 0$ by

$$t = \min\{k \mid \mathrm{roof}_m(\lambda^{(1)}) = \mathrm{up}_m^k(\lambda^{(1)})\}.$$

We prove that $\boldsymbol{\mu}$ is Kleshchev by induction on t. Note that if $\boldsymbol{\lambda}$ is Kleshchev, then so is $(\mathrm{up}_m(\lambda^{(1)}), \lambda^{(2)})$ since

$$\mathrm{roof}_m(\mathrm{up}_m(\lambda^{(1)})) = \mathrm{roof}_m(\lambda^{(1)}) \subseteq \tau_m(\mathrm{base}_0(\lambda^{(2)})).$$

Suppose that $t = 0$. Then $\lambda^{(1)}$ is an e-core. As $\lambda^{(1)}$ has an addable i-node, $\lambda^{(1)}$ has no removable i-node. Thus, all the removable i-nodes of $\boldsymbol{\lambda}$ are nodes of $\lambda^{(2)}$. As $\lambda^{(2)}$ has no addable i-node, the admissible sequence $R_1, \dots, R_s$ is given by all the normal i-nodes of $\boldsymbol{\lambda}$. Hence, $\boldsymbol{\mu} = \tilde{e}_i^s \boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ is Kleshchev.

Suppose that $t > 0$ and that the lemma holds for $(\mathrm{up}_m(\lambda^{(1)}), \lambda^{(2)})$. Recall that we have assumed that $\lambda^{(1)}$ has an addable i-node. Let r be the minimal addable i-node of $J_m := J_m(\lambda^{(1)})$. The corresponding addable i-node of $\lambda^{(1)}$, say γ, is the lowest addable i-node of $\boldsymbol{\lambda}$, and the admissible sequence $R_1, \dots, R_s$ is given by all the removable i-nodes of $\boldsymbol{\lambda}$ that is below γ. If there is no removable i-node greater than r, then all the removable i-nodes of $\boldsymbol{\lambda}$ are normal, and by deleting them, we obtain that $\boldsymbol{\mu}$ is Kleshchev. Hence, we assume that there is a removable i-node greater than r. As $r + 1 \notin J_m$, this implies that there is $x \in U(J_m)$ on the $(i+1)^{th}$ runner such that $x > r + 1$. Let $p = \max U(J_m)$. Then $x \in U(J_m)$ implies that $p \geqslant x > r + 1$. As p moves to $q > p$, it implies that $R_1, \dots, R_s$ is an admissible sequence of i-nodes for $(\mathrm{up}_m(\lambda^{(1)}), \lambda^{(2)})$ and that

$$(\mathrm{up}_m(\lambda^{(1)}), \lambda^{(2)}) = (\mathrm{up}_m(\mu^{(1)}), \mu^{(2)}) \sqcup \{R_1, \dots, R_s\}.$$

Now, $(\mathrm{up}_m(\mu^{(1)}), \mu^{(2)})$ is Kleshchev by the induction hypothesis. Hence,

$$\mathrm{roof}_m(\mu^{(1)}) = \mathrm{roof}_m(\mathrm{up}_m(\mu^{(1)})) \subseteq \tau_m(\mathrm{base}_0(\mu^{(2)}))$$

and $\boldsymbol{\mu}$ is Kleshchev as desired.

Second case: Now, we consider the case when $\lambda^{(2)} \neq \emptyset$ and $\lambda^{(2)}$ has at least one addable i-node. Note that it forces $\lambda^{(1)} = \mu^{(1)}$ and $R_1, \dots, R_s$ are nodes of $\lambda^{(2)}$. Define $t' \geqslant 0$ by

$$t' = \min\{k \mid \mathrm{base}_0(\lambda^{(2)}) = \mathrm{down}_0^k(\lambda^{(2)})\}.$$

We prove that $\boldsymbol{\mu}$ is Kleshchev by induction on t'. Note that if $\boldsymbol{\lambda}$ is Kleshchev, then so is $(\lambda^{(1)}, \mathrm{down}_0(\lambda^{(2)}))$ since

$$\mathrm{roof}_m(\lambda^{(1)}) \subseteq \tau_m(\mathrm{base}_0(\lambda^{(2)})) = \tau_m(\mathrm{base}_0(\mathrm{down}_0(\lambda^{(2)}))).$$

If $t' = 0$, then $\lambda^{(2)}$ is an e-core and it has removable i-nodes $R_1, \dots, R_s$. Hence, $\lambda^{(2)}$ has no addable i-node and we are reduced to the previous case. Thus, we suppose that $t' > 0$ and that the lemma holds for $(\lambda^{(1)}, \mathrm{down}(\lambda^{(2)}))$. Let $J = J_0(\lambda^{(2)})$ and

$$r = \min\{x \in J \mid x + e\mathbb{Z} = i + 1,\ x - 1 \notin J\} - 1.$$

Note that r is on the ith runner. Then there exists $N \geqslant 1$ such that

$$r, r + e, \dots, r + (N - 1)e \notin J \quad \text{and} \quad r + Ne \in J.$$

Let $p' = \min U(J)$. Then $p' \leqslant r + Ne$. Suppose that p' is not on the ith runner or the $(i + 1)$th runner. If a node which is not on one of these two runners moves to $p' - e$ by the down operation, the admissible sequence $R_1, \dots, R_s$ is an admissible sequence of i-nodes for $(\lambda^{(1)}, \mathrm{down}_0(\lambda^{(2)}))$ and

$$(\lambda^{(1)}, \mathrm{down}_0(\lambda^{(2)})) = (\mu^{(1)}, \mathrm{down}_0(\mu^{(2)})) \sqcup \{R_1, \dots, R_s\}.$$

Thus, by the induction hypothesis, $(\mu^{(1)}, \mathrm{down}_0(\mu^{(2)}))$ is Kleshchev. Hence,

$$\mathrm{roof}_m(\mu^{(1)}) \subseteq \tau_m(\mathrm{base}_0(\mathrm{down}_0(\mu^{(2)}))) = \tau_m(\mathrm{base}_0(\mu^{(2)}))$$

implies that $\boldsymbol{\mu}$ is Kleshchev.

If a node in one of the two runners moves to $p' - e$, then there exists $0 \leqslant k \leqslant N - 1$ such that $r + ke + 1 \in J$, $r + (k + 1)e + 1 \notin J$ and $r + ke + 1$ moves to $p' - e$. Suppose that $k < N - 1$. Then, $r + ke \in J_0(\mu^{(2)})$ and $r + (k + 1)e \notin J_0(\mu^{(2)})$. Hence, $r + ke$ moves to $p' - e$ to obtain $\mathrm{down}_0(\mu^{(2)})$. As $r + ke + 1$ corresponds to one of $R_1, \dots, R_s$, say R_k, $R_1, \dots, \hat{R}_k, \dots, R_s$ is an admissible sequence of i-nodes for $(\lambda^{(1)}, \mathrm{down}_0(\lambda^{(2)}))$ and

$$(\lambda^{(1)}, \mathrm{down}_0(\lambda^{(2)})) = (\mu^{(1)}, \mathrm{down}_0(\mu^{(2)})) \sqcup \{R_1, \dots, \hat{R}_k, \dots, R_s\}.$$

Hence, $(\mu^{(1)}, \mathrm{down}_0(\mu^{(2)}))$ is Kleshchev by the induction hypothesis, and $\boldsymbol{\mu}$ is Kleshchev as before. Next suppose that $k = N - 1$. As $r + (N - 1)e + 1$ moves to $p' - e$, we have

$$r + (N - 1)e + 1 < p' < r + Ne,$$

$r + Ne + 1 \notin J_0$ and $r + Ne$ is an addable i-node. Let K be the set of beta numbers of charge 0 of $\mu^{(2)}$. For $x \in \mathbb{Z}$, we denote $J_{\leqslant x} := J \cap \mathbb{Z}_{\leqslant x}$ and $K_{\leqslant x} := K \cap \mathbb{Z}_{\leqslant x}$. We claim that

$$\mathrm{base}(J_{\leqslant r+Ne}) = \mathrm{base}(K_{\leqslant r+Ne}).$$

Let $p' = y_0 < y_1 < \cdots < y_l < r + Ne$ be the nodes in J which are greater than or equal to p' and smaller than $r + Ne$. We show that

$$\mathrm{base}(J_{\leqslant y_j}) = s_i\, \mathrm{base}(K_{\leqslant y_j}) \supseteq \mathrm{base}(K_{\leqslant y_j}),$$

for $0 \leqslant j \leqslant l$, where s_i means swap of the ith and $(i+1)$th runners. $J_{\leqslant p'}$ and $K_{\leqslant p'}$ are s_i-cores in the sense of [5], and direct computation shows the formula for $j = 0$. Now we use $\mathrm{base}(J_{\leqslant y_{j+1}}) = \mathrm{base}(\{y_{j+1}\} \sqcup \mathrm{base}(J_{\leqslant y_j}))$ and $\mathrm{base}(K_{\leqslant y_{j+1}}) = \mathrm{base}(\{y_{j+1}\} \sqcup \mathrm{base}(K_{\leqslant y_j}))$[4] and continue the similar computation and comparison of $\mathrm{base}(J_{\leqslant y_j})$ and $\mathrm{base}(K_{\leqslant y_j})$. At the end of the inductive step, we get

$$\mathrm{base}(J_{<r+Ne}) = s_i\, \mathrm{base}(K_{<r+Ne}) \supseteq \mathrm{base}(K_{<r+Ne}).$$

Now, one more direct computation shows

$$\mathrm{base}(\{r+Ne\} \sqcup \mathrm{base}(J_{<r+Ne})) = \mathrm{base}(\{r+Ne\} \sqcup \mathrm{base}(K_{<r+Ne})),$$

and we have the claim. Therefore, $\mathrm{base}_0(\lambda^{(2)}) = \mathrm{base}_0(\mu^{(2)})$, and we have

$$\mathrm{roof}_m(\lambda^{(1)}) \subseteq \tau_m(\mathrm{base}_0(\lambda^{(2)})) = \tau_m(\mathrm{base}_0(\mu^{(2)})).$$

Recalling that $\lambda^{(1)} = \mu^{(1)}$, we have that $\boldsymbol{\mu}$ is Kleshchev.

Suppose that p' is on one of the two runners. As $p' \leqslant r + Ne$, we have either $p' = r + Ne$ or $p' = r + ke + 1$, for some $0 \leqslant k \leqslant N - 1$. If the latter occurs, $\mathrm{down}_0(\lambda^{(2)})$ is obtained by moving a node outside the two runners to $p'-e$ or moving p' to $p' - e$, and $\mathrm{down}_0(\mu^{(2)})$ is obtained from $\mu^{(2)}$ by moving the same node outside the two runners to $p' - 1 - e$ or moving $p' - 1$ to $p' - 1 - e$, respectively. Thus, $\mathrm{down}_0(\mu^{(2)})$ is obtained from $\mathrm{down}_0(\lambda^{(2)})$ by removing the nodes of an admissible sequence of i-nodes for $(\lambda^{(1)}, \mathrm{down}_0(\lambda^{(2)}))$. Hence, $(\mu^{(1)}, \mathrm{down}_0(\mu^{(2)}))$ is Kleshchev by the induction hypothesis, and it follows that $\boldsymbol{\mu}$ is Kleshchev. If $p' = r + Ne$ and $r + Ne + 1 \in J_0$, then the same is true, and if $p' = r + Ne$ and $r + Ne + 1 \notin J_0$, then $\mu^{(2)} = \mathrm{down}_0^N(\lambda^{(2)})$, and we have

$$\mathrm{roof}_m(\lambda^{(1)}) \subseteq \tau_m(\mathrm{base}_0(\lambda^{(2)})) = \tau_m(\mathrm{base}_0(\mu^{(2)})).$$

Hence, $\boldsymbol{\mu} = (\lambda^{(1)}, \mu^{(2)})$ is Kleshchev. □

3.2 Optimal Sequence of a Kleshchev Bipartition

Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ be a Kleshchev bipartition. By the previous lemma, we may define by induction a sequence of Kleshchev bipartitions

$$\boldsymbol{\lambda} =: \boldsymbol{\lambda}[1],\ \boldsymbol{\lambda}[2], \ldots,\ \boldsymbol{\lambda}[p],\ \boldsymbol{\lambda}[p+1] = \boldsymbol{\emptyset}$$

[4] See [5, Proposition 7.8].

and a sequence of residues

$$\underbrace{i_1,\dots,i_1}_{a_1\text{ times}},\dots,\underbrace{i_p,\dots,i_p}_{a_p\text{ times}}$$

such that, for $1 \leqslant j \leqslant p$, $\boldsymbol{\lambda}[j] = \boldsymbol{\lambda}[j+1] \sqcup \{R_1^j,\dots,R_{a_j}^j\}$, and $R_1^j,\dots,R_{a_j}^j$ is an admissible sequence of i_j-nodes for $\boldsymbol{\lambda}[j]$.

We call $\underbrace{i_1,\dots,i_1}_{a_1\text{ times}},\dots,\underbrace{i_p,\dots,i_p}_{a_p\text{ times}}$ an *optimal sequence* of $\boldsymbol{\lambda}$.

Example 3.4. Keeping Example 2.1, it is easy to see that $((3,2),(4,2,1))$ is a Kleshchev bipartition, and an optimal sequence is given by

$$2,2,0,3,3,2,1,1,0,0,3,2.$$

4 Proof of the Conjecture

4.1 The Result

We are now ready to prove the conjecture. As is explained in the introduction, it is enough to prove that Kleshchev bipartitions are restricted bipartitions. To do this, we introduce certain reverse lexicographic order on the set of bipartitions.

Definition 4.1. We write $\boldsymbol{\lambda} \prec \boldsymbol{\nu}$ if either

- There exists $j \geqslant 0$ such that $\lambda_k^{(2)} = \nu_k^{(2)}$, for $k > j$, and $\lambda_j^{(2)} < \nu_j^{(2)}$, or
- There exists $j \geqslant 0$ such that $\lambda^{(2)} = \nu^{(2)}$, $\lambda_k^{(1)} = \nu_k^{(1)}$, for $k > j$, and $\lambda_j^{(1)} < \nu_j^{(1)}$.

It is clear that if $\boldsymbol{\nu} \lhd \boldsymbol{\lambda}$, then $\boldsymbol{\lambda} \prec \boldsymbol{\nu}$. Recall that the deformed Fock space is given a specific $U_v(\hat{sl}_e)$-module structure which is suitable for the Dipper–James–Murphy's Specht module theory.

Proposition 4.2. *Let $\boldsymbol{\lambda} = (\lambda^{(1)},\lambda^{(2)})$ be Kleshchev and let*

$$\underbrace{i_1,\dots,i_1}_{a_1\text{ times}},\dots,\underbrace{i_p,\dots,i_p}_{a_p\text{ times}}$$

be an optimal sequence of $\boldsymbol{\lambda}$. Then we have

$$f_{i_1}^{(a_1)}\dots f_{i_p}^{(a_p)}\emptyset = \boldsymbol{\lambda} + \sum_{\boldsymbol{\nu}\prec\boldsymbol{\lambda}} c_{\boldsymbol{\nu},\boldsymbol{\lambda}}(v)\boldsymbol{\nu},$$

for some Laurent polynomials $c_{\boldsymbol{\nu},\boldsymbol{\lambda}}(v) \in \mathbb{Z}_{\geqslant 0}[v,v^{-1}]$, in the deformed Fock space.

Proof. First note that the coefficient of $\boldsymbol{\lambda}$ is one because each admissible sequence of i_j-nodes is a sequence of normal i_j-nodes.

Now the proposition is proved by induction on p. Let $R_1^1, \dots, R_{a_1}^1$ be the admissible sequence of i_1-nodes for $\boldsymbol{\lambda}$, and let $\boldsymbol{\lambda}'$ be the Kleshchev bipartition such that

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}' \sqcup \{R_1^1, \dots, R_{a_1}^1\}.$$

By the induction hypothesis, we have

$$f_{i_2}^{(a_2)} \dots f_{i_p}^{(a_p)} \boldsymbol{\emptyset} = \boldsymbol{\lambda}' + \sum_{\boldsymbol{\nu}' \prec \boldsymbol{\lambda}'} c_{\boldsymbol{\nu},\boldsymbol{\lambda}'}(v) \boldsymbol{\nu}'.$$

Let $\boldsymbol{\nu} \neq \boldsymbol{\lambda}$ be a bipartition such that it appears in $f_{i_1}^{(a_1)} \dots f_{i_p}^{(a_p)} \boldsymbol{\emptyset}$ with nonzero coefficient. Then there exist removable i_1-nodes $R'^1_1, \dots, R'^1_{a_1}$ of $\boldsymbol{\nu}$ and a bipartition $\boldsymbol{\nu}' \preceq \boldsymbol{\lambda}'$ such that

$$\boldsymbol{\nu} = \boldsymbol{\nu}' \sqcup \{R'^1_1, \dots, R'^1_{a_1}\}.$$

As $R_1^1, \dots, R_{a_1}^1$ are the lowest a_1 i_1-nodes of $\boldsymbol{\lambda}$, $\boldsymbol{\nu}' = \boldsymbol{\lambda}'$ implies $\boldsymbol{\nu} \preceq \boldsymbol{\lambda}$. Hence we may assume $\boldsymbol{\nu}' \prec \boldsymbol{\lambda}'$. Suppose that we have $\boldsymbol{\lambda} \prec \boldsymbol{\nu}$. If $\nu'^{(2)} \neq \lambda'^{(2)}$, then choose t such that $\nu'^{(2)}_t < \lambda'^{(2)}_t$ and $\nu'^{(2)}_j = \lambda'^{(2)}_j$, for $j > t$. Then we can show

(i) $\nu_j^{(2)} = \lambda_j^{(2)}$, for $j > t$,
(ii) $\nu_{t+1}^{(2)} < \nu_t^{(2)} = \nu'^{(2)}_t + 1 = \lambda'^{(2)}_t = \lambda_t^{(2)}$,
(iii) At least one of the nodes $R_1^1, \dots, R_{a_1}^1$ is above $(t, \lambda_t^{(2)} - 1, 2)$.

The condition (ii) implies that $\operatorname{res}(t, \lambda_t^{(2)} - 1, 2) = \operatorname{res}(t, \nu'^{(2)}_t, 2) = i_1$. Thus, $(t, \lambda_t^{(2)} - 1, 2)$ is an i_1-node of $\lambda^{(2)}$. Then (iii) implies that it is not a removable node of $\lambda^{(2)}$ (otherwise it has to be removed to obtain $\boldsymbol{\lambda}'$). This implies that $\lambda_t^{(2)} = \lambda_{t+1}^{(2)}$. Thus, $\nu_{t+1}^{(2)} < \lambda_t^{(2)} = \lambda_{t+1}^{(2)}$ and (i) is contradicted.

If $\nu'^{(2)} = \lambda'^{(2)}$, then choose t such that $\nu'^{(1)}_t < \lambda'^{(1)}_t$ and $\nu'^{(1)}_j = \lambda'^{(1)}_j$, for $j > t$. Then we argue as above. □

Corollary 4.3. *The Dipper–James–Murphy conjecture is true.*

Proof. Observe that $\boldsymbol{\nu}$ appears in $f_{i_n} \dots f_{i_1} \boldsymbol{\emptyset}$ if and only if there exists a standard bitableau of shape $\boldsymbol{\nu}$ such that its residue sequence is $(i_1, \dots, i_n)$. Let $\boldsymbol{\lambda}$ be Kleshchev. Then Proposition 4.2 shows that there is a standard bitableau $\mathbf{t}$ such that if the residue sequence of $\mathbf{t}$ appears as the residue sequence of a standard bitableau of shape $\boldsymbol{\nu}$, then $\boldsymbol{\nu} \preceq \boldsymbol{\lambda}$. Suppose that the residue sequence of $\mathbf{t}$ is the residue sequence of a standard bitableau of shape $\boldsymbol{\nu} \lhd \boldsymbol{\lambda}$. As $\boldsymbol{\nu} \lhd \boldsymbol{\lambda}$ implies $\boldsymbol{\lambda} \prec \boldsymbol{\nu}$, we cannot have $\boldsymbol{\nu} \preceq \boldsymbol{\lambda}$, a contradiction. Hence $\boldsymbol{\lambda}$ is restricted. □

4.2 Remarks

We conclude the chapter with two remarks.

Remark 4.4. In the language of the Fock space theory, the proof of the fact that restricted implies Kleshchev goes as follows. The proof in the introduction is the same proof, but it was explained in a different manner. Note that we may assume that the characteristic of F is zero in the proof.

Assume that $\boldsymbol{\lambda}$ is restricted. Then there exist $i_1, \dots, i_n$ such that

(i) $\boldsymbol{\lambda}$ appears in $f_{i_1} \dots f_{i_n} \boldsymbol{\emptyset}$,
(ii) $\boldsymbol{\nu} \lhd \boldsymbol{\lambda}$ implies that $\boldsymbol{\nu}$ does not appear in $f_{i_1} \dots f_{i_n} \boldsymbol{\emptyset}$.

Let $\{G(\boldsymbol{\nu}) \mid \boldsymbol{\nu} \text{ is Kleshchev.}\}$ be the canonical basis at $v = 1$ in the Fock space. As $f_{i_1} \dots f_{i_n} \boldsymbol{\emptyset}$ is represented by a projective $\mathcal{H}_n$-module, and $G(\boldsymbol{\nu}) = [P^{\boldsymbol{\nu}}]$, the indecomposable projective $\mathcal{H}_n$-module indexed by $\boldsymbol{\nu}$, we may write

$$f_{i_1} \dots f_{i_n} \boldsymbol{\emptyset} = \sum c_{\boldsymbol{\nu}} G(\boldsymbol{\nu}),$$

where $c_{\boldsymbol{\nu}} \geqslant 0$. In particular, $c_{\boldsymbol{\nu}} > 0$ implies that $\boldsymbol{\nu}$ appears in $f_{i_1} \dots f_{i_n} \boldsymbol{\emptyset}$.

Suppose that $\boldsymbol{\lambda}$ is not Kleshchev. As $\boldsymbol{\lambda}$ appears in $f_{i_1} \dots f_{i_n} \boldsymbol{\emptyset}$, $\boldsymbol{\lambda}$ appears in $G(\boldsymbol{\nu})$ for some Kleshchev bipartition $\boldsymbol{\nu}$ with $c_{\boldsymbol{\nu}} > 0$. Since

$$G(\boldsymbol{\nu}) = \boldsymbol{\nu} + \sum_{\boldsymbol{\lambda} \rhd \boldsymbol{\nu}} d_{\boldsymbol{\lambda}\boldsymbol{\nu}} \boldsymbol{\lambda},$$

we must have $\boldsymbol{\nu} \lhd \boldsymbol{\lambda}$. However, this implies that $\boldsymbol{\nu}$ does not appear in $f_{i_1} \dots f_{i_n} \boldsymbol{\emptyset}$, which contradicts $c_{\boldsymbol{\nu}} > 0$.

Remark 4.5. There is a systematic way to produce realizations of the crystal $B(\Lambda_0 + \Lambda_m)$ on a set of bipartitions, for each choice of $\log_q(-Q)$. The bipartitions are called Uglov bipartitions. Recent results of Geck [8] and Geck and the second author [9] show that Uglov bipartitions naturally label simple $\mathcal{H}_n$-modules, and Rouquier's theory of the BGG category of rational Cherednik algebras as quasihereditary covers of Hecke algebras naturally explains the existence of various Specht module theories which depends on $\log_q(-Q)$.

We conjecture that Uglov bipartitions satisfy an analogue of Proposition 4.2 except that the dominance order is replaced by an appropriate a-value in the sense of [9, Proposition 2.1]. As is mentioned in the introduction, it is known that this conjecture is true in the case where Uglov bipartitions are FLOTW bipartitions [10, Proposition 4.6].

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On Domino Insertion and Kazhdan–Lusztig Cells in Type B_n

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Abstract Based on empirical evidence obtained using the CHEVIE computer algebra system, we present a series of conjectures concerning the combinatorial description of the Kazhdan–Lusztig cells for type B_n with unequal parameters. These conjectures form a far-reaching extension of the results of Bonnafé and Iancu obtained earlier in the so-called asymptotic case. We give some partial results in support of our conjectures.

Keywords Coxeter groups · Kazhdan-Lusztig cells · Domino insertion

Mathematics Subject Classifications (2000): Primary 20C08, Secondary 05E10

1 Introduction and the Main Conjectures

Let W be a Coxeter group, Γ be a totally ordered abelian group and $L\colon W \to \Gamma_{\geqslant 0}$ be a weight function, in the sense of Lusztig [30, Sect. 3.1]. This gives rise to various pre-order relations on W, usually denoted by $\leqslant_{\mathcal{L}}$, $\leqslant_{\mathcal{R}}$ and $\leqslant_{\mathcal{LR}}$. Let $\sim_{\mathcal{L}}$, $\sim_{\mathcal{R}}$ and $\sim_{\mathcal{LR}}$ be the corresponding equivalence relations. The equivalence classes are called the left, right and two-sided cells of W, respectively. They were first defined by Kazhdan and Lusztig [26] in the case where L is the length function on W (the "equal parameter case"), and by Lusztig [29] in general. They play a fundamental role, for example, in the representation theory of finite or p-adic groups of Lie type; see the survey in [30, Chap. 0].

Our aim is to understand the dependence of the Kazhdan–Lusztig cells on the weight function L. We shall be interested in the case where W is a finite Coxeter group. Then unequal parameters can only arise in type $I_2(m)$ (dihedral), F_4 or B_n. Now types $I_2(m)$ and F_4 can be dealt with by computational methods; see [14].

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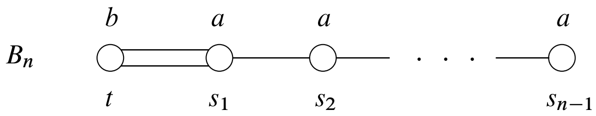
Thus, as far as finite Coxeter groups are concerned, the real issue is to study type B_n with unequal parameters. And in any case, this is the most important case with respect to applications to finite classical groups (unitary, symplectic and orthogonal). Quite recently, new connections between Kazhdan–Lusztig cells in type B_n and the theory of rational Cherednik algebras appeared in the work of Gordon and Martino [23].

The purpose of this paper is to present a series of conjectures that would completely and explicitly determine the Kazhdan–Lusztig cells in type B_n for any positive weight function L. We will also establish some relative results in support of these conjectures. So let now $W = W_n$ be a Coxeter group of type B_n, with generating set $S_n = \{t, s_1, \dots, s_{n-1}\}$ and Dynkin diagram as given below; the "weights" $a, b \in \Gamma_{>0}$ attached to the generators of W_n uniquely determine a weight function $L = L_{a,b}$ on W_n.



If b is "large" with respect to a, more precisely, if $b > (n-1)a$, then we are in the "asymptotic case" studied in [5] (see also [3, Proposition 5.1 and Corollary 5.2] for the determination of the exact bound). In general, we expect that the combinatorics governing the cells in type B_n are provided by the

"*domino insertion of a signed permutation into a* 2*-core*";

see [27,28,33] (see also Sect. 3). Having fixed $r \geqslant 0$, let δ_r be the partition with parts $(r, r-1, \dots, 0)$ (a 2-core). Let $\mathcal{P}_r(n)$ be the set of partitions $\lambda \vdash (\frac{1}{2}r(r+1) + 2n)$ such that λ has 2-core δ_r. Then the domino insertion with respect to δ_r gives a bijection from W_n onto the set of all pairs of standard domino tableaux of the same shape $\lambda \in \mathcal{P}_r(n)$. We write this bijection as $w \to (P^r(w), Q^r(w))$; see [27, Sect. 2] for a detailed description.

The following conjectures have been verified for $n \leqslant 6$ by explicit computation using CHEVIE [21] and the program Coxeter developed by du Cloux [7]. For the basic definitions concerning Kazhdan–Lusztig cells, see Lusztig [30].

Conjecture A. *Assume that* $\Gamma = \mathbb{Z}$, $a = 2$ *and* $b = 2r + 1$ *where* $r \geqslant 0$. *Then the following hold.*

(a) $w, w' \in W_n$ *lie in the same Kazhdan–Lusztig left cell if and only if* $Q^r(w) = Q^r(w')$.
(b) $w, w' \in W_n$ *lie in the same Kazhdan–Lusztig right cell if and only if* $P^r(w) = P^r(w')$.
(c) $w, w' \in W_n$ *lie in the same Kazhdan–Lusztig two-sided cell if and only if all of* $P^r(w)$, $Q^r(w)$, $P^r(w')$, $Q^r(w')$ *have the same shape.*

Remark 1.1. The 2-core δ_r, the set of partitions $\mathcal{P}_r(n)$ and the parameters $a = 2$, $b = 2r + 1$ (where $\Gamma = \mathbb{Z}$) naturally arise in the representation theory of the finite unitary groups $\mathrm{GU}_N(q)$, where $N = \frac{1}{2}r(r+1) + 2n$. The Hecke algebra of type B_n with parameters $q^{2r+1}, q^2, \ldots, q^2$ appears as the endomorphism algebra of a certain induced cuspidal representation. The irreducible representations of this endomorphism algebra parametrize the unipotent representations of $\mathrm{GU}_N(q)$ indexed by partitions in $\mathcal{P}_r(n)$; see [6, Sect. 13.9]. In this case, Conjecture A(c) is somewhat more precise than [30, Conjecture 25.3 (b)] (see Sect. 3.3 for more details).

Conjecture A$^+$. *Let $r \geqslant 0$ and assume that a, b are any elements of $\Gamma_{>0}$ such that $ra < b < (r+1)a$. Then the statements in Conjecture A still hold. That is, the Kazhdan–Lusztig (left, right, two-sided) cells for this choice of parameters coincide with those obtained for the special values $a = 2$ and $b = 2r + 1$ (where $\Gamma = \mathbb{Z}$).*

Remark 1.2. Assume we are in the setting of Conjecture A or A$^+$. If $w \in W_n$, let $\lambda(w) \in \mathcal{P}_r(n)$ denote the shape of $P^r(w)$ (or $Q^r(w)$). Let $\trianglelefteq$ denote the dominance order on partitions. The following property of the pre-order $\leqslant_{\mathcal{LR}}$ has been checked for $n \leqslant 4$ using CHEVIE [21]:

$$w \leqslant_{\mathcal{LR}} w' \quad \text{if and only if} \quad \lambda(w) \trianglelefteq \lambda(w') \tag{c$^+$}$$

Remark 1.3. Assume that the statement concerning the left cells in Conjecture A (or A$^+$) is true. Since $P^r(w^{-1}) = Q^r(w)$ (see for instance [27, Lemma 7]), this would imply that the statement concerning the right cells is also true. However, it is not clear that the partition into two-sided cells easily follows from the knowledge of the partitions into left and right cells. Indeed, it is conjectured (but not proved in general) that the relation $\sim_{\mathcal{LR}}$ is generated by $\sim_{\mathcal{L}}$ and $\sim_{\mathcal{R}}$. This would follow from Lusztig's Conjectures (P4), (P9), (P10) and (P11).

Remark 1.4. If $b > (n-1)a$ ("asymptotic case"), then domino insertion is equivalent to the generalized Robinson–Schensted correspondence in [5, Sect. 3] (see Theorem 3.13). Thus, Conjectures A and A$^+$ hold in this case [5, Theorem 7.7], [3, Corollary 3.6 and Remark 3.7]. Also, the refinement (c$^+$) proposed in Remark 1.2 holds in this case if w and w' have the same t-length [3, Theorem 3.5 and Remark 3.7] (the *t-length* of an element $w \in W_n$ is the number of occurrences of t in a reduced decomposition of w).

Remark 1.5. Assume that Conjectures A and A$^+$ hold. Then we also conjecture that the Kazhdan–Lusztig basis of the Iwahori–Hecke algebra $\mathcal{H}_n$ associated with W_n and the weight function $L_{a,b}$ is a *cellular basis* in the sense of Graham–Lehrer [24]. See Sect. 2.2 for a more precise statement and applications to the representation theory of non-semisimple specialisations of $\mathcal{H}_n$.

We define the equivalence relation $\simeq_r$ on elements of W_n as follows: we write $w \simeq_r w'$ if and only if $Q^r(w) = Q^r(w')$. An equivalence class for the relation $\simeq_r$ is called a *left r-cell*. In other words, left r-cells are the fibers of the map Q^r. Similarly, we define *right r-cells* as the fibers of the map P^r and *two-sided r-cell* as the fibers of the map $\breve{\ } : W_n \to \mathcal{P}_r(n)$.

Conjectures A and A$^+$ deal with the Kazhdan–Lusztig cells for parameters such that $ra < b < (r+1)a$. The next conjecture is concerned with the Kazhdan–Lusztig cells whenever $b \in \mathbb{N}^* a$.

Conjecture B. *Assume that $b = ra$ for some $r \geqslant 1$. Then the Kazhdan–Lusztig left (resp. right, resp. two-sided) cells of W_n are the smallest subsets of W_n which are at the same time unions of left (resp. right, resp. two-sided) $(r-1)$-cells and left (resp. right, resp. two-sided) r-cells.*

We will give a combinatorially more precise version of Conjecture B in Sect. 4.

Remark 1.6. (a) If $r \geqslant n$, since the left r-cells and the left $(r-1)$-cells coincide, then the Conjecture B holds ("asymptotic case", see Remark 1.4).

(b) There is one case which is not covered by Conjectures A, A$^+$ or B: it is when $b > ra$ for every $r \in \mathbb{N}$. But this case is exactly the case which is dealt with in [5, Theorem 7.7] (and [3, Corollary 3.6] for the determination of two-sided cells) and it leads to the same partition into left and two-sided cells as the case where $(a,b) = (2, 2n-1)$ for instance (see Remark 1.4).

(c) The fundamental difference between the cases where $b \in \{a, 2a, \ldots, (n-1)a\}$ and $b \notin \{a, 2a, \ldots, (n-1)a\}$ is already apparent in [30, Chap. 22], where the "constructible representations" are considered. Conjecturally, these are precisely the representations given by the various left cells of W. By [30, Chap. 22], the constructible representations are all irreducible if and only if $b \notin \{a, 2a, \ldots, (n-1)a\}$.

(d) Again, in Conjecture B, the statement concerning left cells is equivalent to the statement concerning right cells. However, the statement concerning two-sided cells would then follow if one could prove that the relation $\sim_{\mathcal{LR}}$ is generated by the relations $\sim_{\mathcal{L}}$ and $\sim_{\mathcal{R}}$.

(e) Conjectures A$^+$ and B are consistent with analogous results for type F_4 (see [14] as far as Conjecture A$^+$ is concerned; Geck also checked that an analogue of Conjecture B holds in type F_4).

In Sect. 2, we will discuss representation-theoretic issues related to Conjecture A. In Sects. 3 and 4, we will present a number of partial results in support of our conjectures.

2 Leading Matrix Coefficients and Cellular Bases

Let W be a finite Coxeter group with generating set S. Let Γ be a totally ordered abelian group. Let $L\colon W \to \Gamma$ be a weight function in the sense of Lusztig [30, Sect. 3.1]. Thus, we have $L(ww') = L(w) + L(w')$ for all $w, w' \in W$ such that $l(ww') = l(w) + l(w')$ where $l\colon W \to \mathbb{N}$ is the usual length function with respect to S (where $\mathbb{N} = \{0, 1, 2, \ldots\}$). Let $A = \mathbb{Z}[\Gamma]$ be the group ring of Γ. It will be denoted exponentially: in other words, $A = \bigoplus_{\gamma \in \Gamma} \mathbb{Z} v^\gamma$ and $v^\gamma v^{\gamma'} = v^{\gamma+\gamma'}$. If $\gamma_0 \in \Gamma$, let $A_{>\gamma_0} = \bigoplus_{\gamma > \gamma_0} \mathbb{Z} v^\gamma$. We define similarly $A_{\geqslant\gamma_0}$, $A_{<\gamma_0}$ and $A_{\leqslant\gamma_0}$.

Let $\mathcal{H} = \mathcal{H}_A(W, S, L)$ be the corresponding Iwahori–Hecke algebra. Then $\mathcal{H}$ is free over A with basis $(T_w)_{w \in W}$; the multiplication is given by the rule

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) = l(w)+1, \\ T_{sw} + (v^{L(s)} - v^{-L(s)}) T_w & \text{if } l(sw) = l(w)-1, \end{cases}$$

where $w \in W$ and $s \in S$. For basic properties of W and $\mathcal{H}$, we refer to [20].

2.1 Leading Matrix Coefficients

We now recall the basic facts concerning the leading matrix coefficients introduced in [12]. First, since Γ is an ordered group, the ring A is integral. Similarly, the group algebra $\mathbb{R}[\Gamma]$ is integral; we denote by $K = \mathbb{R}(\Gamma)$ its field of fractions.

Extending scalars from A to the field K, we obtain a finite dimensional K-algebra $\mathcal{H}_K = K \otimes_A \mathcal{H}$, with basis $(T_w)_{w\in W}$. It is well known that $\mathcal{H}_K$ is split semisimple and abstractly isomorphic to the group algebra of W over K; see, for example, [18, Remark 3.1]. Let $\mathrm{Irr}(\mathcal{H}_K)$ be the set of irreducible characters of $\mathcal{H}_K$. We write this set in the form

$$\mathrm{Irr}(\mathcal{H}_K) = \{\chi_\lambda \mid \lambda \in \Lambda\},$$

where Λ is some finite indexing set. If $\lambda \in \Lambda$, we denote by d_λ the degree of χ_λ. We have a symmetrizing trace $\tau\colon \mathcal{H}_K \to K$ defined by $\tau(T_1) = 1$ and $\tau(T_w) = 0$ for $1 \neq w \in W$; see [20, Sect. 8.1]. The fact that $\mathcal{H}_K$ is split semisimple yields that

$$\tau = \sum_{\lambda\in\Lambda} \frac{1}{c_\lambda}\chi_\lambda \qquad \text{where } 0 \neq c_\lambda \in \mathbb{R}[\Gamma].$$

The elements c_λ are called the *Schur elements*. There is a unique $a(\lambda) \in \Gamma_{\geqslant 0}$ and a positive real number r_λ such that

$$c_\lambda \in r_\lambda\, v^{-2a(\lambda)} + A_{>-2a(\lambda)};$$

see [12, Definition 3.3]. The number $a(\lambda)$ is called the a-invariant of χ_λ. Using the *orthogonal representations* defined in [12, Sect. 4], we obtain the *leading matrix coefficients* $c_{w,\lambda}^{ij} \in \mathbb{R}$ for $\lambda \in \Lambda$ and $1 \leqslant i, j \leqslant d_\lambda$. See [12, Sect. 4] for further general results concerning these coefficients.

Following [18, Definition 3.3], we say that

- $\mathcal{H}$ is *integral* if $c_{w,\lambda}^{ij} \in \mathbb{Z}$ for all $\lambda \in \Lambda$ and $1 \leqslant i, j \leqslant d_\lambda$;
- $\mathcal{H}$ is *normalized* if $r_\lambda = 1$ for all $\lambda \in \Lambda$.

The relevance of these notions is given by the following result.

Theorem 2.1 (See [12, Sect. 4] and [18, Lemma 3.8]). *Assume that $\mathcal{H}$ is integral and normalized.*

(a) *We have $c_{w,\lambda}^{ij} \in \{0, \pm 1\}$ for all $w \in W$, $\lambda \in \Lambda$ and $1 \leqslant i, j \leqslant d_\lambda$.*

(b) *For any $\lambda \in \Lambda$ and $1 \leqslant i, j \leqslant d_\lambda$, there exists a unique $w \in W$ such that $c_{w,\lambda}^{ij} \neq 0$; we denote that element by $w = w_\lambda(i, j)$. The correspondence $(\lambda, i, j) \mapsto w_\lambda(i, j)$ defines a bijective map*

$$\{(\lambda, i, j)\} \mid \lambda \in \Lambda, 1 \leqslant i, j \leqslant d_\lambda\} \longrightarrow W.$$

(c) *For a fixed* $\lambda \in \Lambda$ *and* $1 \leqslant k \leqslant d_\lambda$,

(i) $\mathfrak{L}_{\lambda,k} := \{w_\lambda(i,k) \mid 1 \leqslant i \leqslant d_\lambda\}$ *is contained in a left cell;*
(ii) $\mathfrak{R}_{\lambda,k} := \{w_\lambda(k,j) \mid 1 \leqslant j \leqslant d_\lambda\}$ *is contained in a right cell.*

Remark 2.2. Assume that Lusztig's conjectures **(P1)–(P15)** in [30, Sect. 14.2] hold for $\mathcal{H}$. Assume also that $\mathcal{H}$ is normalized and integral. Combining [15, Corollary 4.8] and [18, Lemma 3.10], we conclude that the sets $\mathfrak{L}_{\lambda,k}$ and $\mathfrak{R}_{\lambda,k}$ are precisely the left cells and the right cells of W, respectively.

Now let $W = W_n$ be the Coxeter group of type B_n as in Sect. 1; let $\mathcal{H}_n$ be the associated Iwahori–Hecke algebra with respect to the weight function $L = L_{a,b}$ where $a, b \geqslant 0$.

Proposition 2.3. *Assume that* $a > 0$ *and* $b \notin \{a, 2a, \ldots, (n-1)a\}$. *Then* $\mathcal{H}_n$ *is integral and normalized.*

Proof. The fact that $\mathcal{H}_n$ is normalized follows from the explicit description of $a(\lambda)$ in [30, Proposition 22.14]. To show that $\mathcal{H}_n$ is integral, we follow again the discussion in [18, Example 3.6] where we showed that $\mathcal{H}_n$ is integral if $b > (n-1)a$. So we may, and we will, assume from now on that $b < (n-1)a$. Since $b \notin \{a, 2a, \ldots, (n-1)a\}$, there exists a unique $r \geqslant 0$ such that $ra < b < (r+1)a$. Given $\lambda \in \Lambda$, let $\tilde{S}^\lambda$ be the Specht module constructed by Dipper–James–Murphy [8]. There is a non-degenerate $\mathcal{H}_n$-invariant bilinear form $\langle\ ,\ \rangle_\lambda$ on $\tilde{S}^\lambda$. Let $\{f_t \mid t \in \mathbb{T}_\lambda\}$ be the orthogonal basis constructed in [8, Theorem 8.11], where $\mathbb{T}_\lambda$ is the set of all standard bitableaux of shape λ. Using the recursion formula in [9, Proposition 3.8], it is straightforward to show that, for each basis element f_t, there exist integers $s_t, a_{ti}, b_{tj}, c_{tk}, d_{tl} \in \mathbb{Z}$ such that $a_{ti} \geqslant 0$, $b_{tj} \geqslant 0$, and

$$\langle f_t, f_t \rangle_\lambda = v^{2s_t a} \cdot \frac{\prod_i (1 + v^{2a} + \cdots + v^{2a_{ti}a})}{\prod_j (1 + v^{2a} + \cdots + v^{2b_{tj}a})} \cdot \frac{\prod_k \big(1 + v^{2(b+c_{tk}a)}\big)}{\prod_l \big(1 + v^{2(b+d_{tl}a)}\big)}.$$

In [18, Example 3.6], we noticed that we also have $b + c_{tk}a > 0$ and $b + d_{tl}a > 0$ if $b > (n-1)a$, and this allowed us to deduce that $\mathcal{H}_n$ is integral in that case. Now, if we only assume that $ra < b < (r+1)a$, then $b + c_{tk}a$ and $b + d_{tl}a$ will no longer be strictly positive, but at least we know that they cannot be zero. Thus, there exist $h_t, h_t', m_{tk}, m_{tl}' \in \mathbb{Z}$ such that

$$\prod_k \big(1 + v^{2(b+c_{tk}a)}\big) = v^{2h_t} \prod_k \big(1 + v^{2m_{tk}}\big) \qquad \text{where } m_{tk} > 0,$$
$$\prod_l \big(1 + v^{2(b+d_{tl}a)}\big) = v^{2h_t'} \prod_l \big(1 + v^{2m_{tl}'}\big) \qquad \text{where } m_{tl}' > 0.$$

Hence, setting

$$\tilde{f}_t := v^{-s_t a - h_t + h_t'} \cdot \Big(\prod_j (1 + v^{2a} + \cdots + v^{2b_{tj}a})\Big) \cdot \Big(\prod_l (1 + v^{2m_{tl}'})\Big) \cdot f_t,$$

we obtain $\langle \tilde{f}_t, \tilde{f}_t \rangle_\lambda \in 1 + v\mathbb{Z}[v]$ for all t. We can then proceed exactly as in [18, Example 3.6] to conclude that $\mathcal{H}_n$ is integral. □

The above result, in combination with Theorem 2.1, provides a first approximation to the left and right cells of W_n. By Remark 2.2, the sets $\mathcal{L}_{\lambda,k}$ and $\mathcal{R}_{\lambda,k}$ should be precisely the left and right cells, respectively. In this context, Conjecture A would give an explicit combinatorial description of the correspondence $(\lambda, i, j) \mapsto w_\lambda(i, j)$.

2.2 Cellular Bases

Let us assume that we are in the setting of Conjecture A. As announced in Remark 1.5, we believe that then the Kazhdan–Lusztig basis of $\mathcal{H}_n$ will be cellular in the sense of Graham–Lehrer [24]. To state this more precisely, we have to introduce some further notation. Let $(C_w)_{w\in W_n}$ be the Kazhdan–Lusztig basis of $\mathcal{H}_n$; the element C_w is uniquely determined by the conditions that

$$\overline{C}_w = C_w \qquad \text{and} \qquad C_w \equiv T_w \mod \mathcal{H}_{n,>0},$$

where $\mathcal{H}_{n,>0} = \sum_{w\in W_n} A_{>0}T_w$ and the bar denotes the ring involution defined in [30, Lemma 4.2]. Furthermore, let $* : \mathcal{H}_n \to \mathcal{H}_n$ be the unique anti-automorphism such that $T_w^* = T_{w^{-1}}$ for all $w \in W_n$. We also have $C_w^* = C_{w^{-1}}$ for any $w \in W_n$.

Now assume that $a > 0$ and $b \notin \{a, 2a, \dots, (n-1)a\}$. If $b < (n-1)a$, let $r \geqslant 0$ be such that $ra < b < (r+1)a$. If $b > (n-1)a$, let r be any natural number greater than or equal to $n - 1$.

We set $\Lambda_r := \mathcal{P}_r(n)$ and consider the partial order on Λ_r given by the dominance order $\trianglelefteq$ on partitions. For $\lambda \in \Lambda_r$, let $M_r(\lambda)$ denote the set of standard domino tableaux of shape λ. If $(S, T) \in M_r(\lambda) \times M_r(\lambda)$, let $C_r(S, T) := C_w$ where $(S, T) = (P^r(w), Q^r(w))$.

Conjecture C. *With the above notation, $(\Lambda_r, M_r, C_r, *)$ is a cell datum in the sense of Graham–Lehrer* [24, Definition 1.1].

The existence of a cellular structure has strong representation-theoretic applications. For the remainder of this section, assume that Conjecture C is true. Let $\theta\colon A \to k$ be a ring homomorphism into a field k. Extending scalars from A to k, we obtain a k-algebra $\mathcal{H}_{n,k} := k \otimes_A \mathcal{H}_n$ which will no longer be semisimple in general. The theory of cellular algebras [24] provides, for every $\lambda \in \Lambda_r$, a *cell module* S^λ of $\mathcal{H}_{n,k}$, endowed with an $\mathcal{H}_{n,k}$-equivariant bilinear form ϕ^λ. We set

$$D^\lambda := S^\lambda/\operatorname{rad}\phi^\lambda \qquad \text{for every } \lambda \in \Lambda_r.$$

Let $\Lambda_r^\circ := \{D^\lambda \mid \lambda \in \Lambda_r \text{ such that } \phi^\lambda \neq 0\}$. Then we have

$$\operatorname{Irr}(\mathcal{H}_{n,k}) = \{D^\lambda \mid \lambda \in \Lambda_r^\circ\}; \qquad \text{see Graham–Lehrer [24, Theorem 3.4].}$$

Thus, we obtain a natural parametrization of the irreducible representations of $\mathcal{H}_{n,k}$ by the set $\Lambda_r^\circ \subseteq \Lambda_r$.

Remark 2.4. Assume that $b > (n-1)a > 0$. Then Conjecture C holds by [16, Corollary 6.4]. In this case, the set Λ_r° is determined explicitly by Dipper–James–Murphy [8] and Ariki [1]. Finally, it is shown in [22] that the cell modules S^λ are canonically isomorphic to the Specht modules defined by Dipper–James–Murphy [8].

Remark 2.5. Now consider arbitrary values of a, b such that $a > 0$ and $b \notin \{a, 2a, \dots, (n-1)a\}$. Then, assuming that the conjectured relation (c^+) in Remark 1.2 holds, a description of the set Λ_r follows from the results of Geck–Jacon [19] on *canonical basic sets*. Indeed, one readily shows that the set Λ_r° coincides with the *canonical basic set* determined by [19]. Thus, by the results of [19], we have explicit combinatorial descriptions of Λ_r° in all cases. Note that these descriptions heavily depend on a, b and $\theta\colon A \to k$.

It is shown in [17] that, if $a = 2$ and $b = 1$ or 3, then the sets Λ_r° parametrize the modular principal series representations of the finite unitary groups.

3 Domino Insertion

The aim of this section is to describe the domino insertion algorithm and to provide some theoretical evidences for Conjecture A. For this purpose we will see W_n as the group of permutations w of $\{-1, -2, \dots, -n\} \cup \{1, 2, \dots, n\}$ such that $w(-i) = -w(i)$ for any i. The identification is as follows: t corresponds to the transposition $(1, -1)$ and s_i to $(i, i+1)(-i, -i-1)$. If $r \leqslant n$, we identify W_r with the subgroup of W_n generated by $S_r = \{t, s_1, s_2, \dots, s_{r-1}\}$. The symmetric group of degree n will be denoted by $\mathfrak{S}_n$: when necessary, we shall identify it in the natural way with the subgroup of W_n generated by $\{s_1, s_2, \dots, s_{n-1}\}$. Let $t_1 = t$ and, if $1 \leqslant i \leqslant n-1$, let $t_{i+1} = s_i t_i s_i$. As a signed permutation, t_i is just the transposition $(i, -i)$.

Remark 3.1. Since we shall be interested in various descent sets of elements of W_n, we state here for our future needs the following two easy facts. Let $w \in W_n$. Then the following hold.

(a) If $1 \leqslant i \leqslant n-1$, then $\ell(ws_i) > \ell(w)$ if and only if $w(i) < w(i+1)$.
(b) If $1 \leqslant i \leqslant n$, then $\ell(wt_i) > \ell(w)$ if and only if $w(i) > 0$.

3.1 Partitions and Tableaux

We refer to [27,33] for further details of the material in this section. We shall assume some familiarity with (standard) Young tableaux.

Let $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{l(\lambda)} > 0)$ be a partition of $n = |\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_{l(\lambda)}$. We will not distinguish between a partition λ and its *Young diagram* (often denoted $D(\lambda)$). Our Young diagrams will be drawn in the English notation so that the boxes are upper-left justified. When λ and μ are partitions satisfying $\mu \subset \lambda$, we will use λ/μ to denote the shape corresponding to the set-difference of the diagrams of λ and μ. We call λ/μ a *domino* if it consists of exactly two squares sharing an edge.

The 2-*core* (or just core) $\widetilde{\lambda}$ of a shape λ is obtained by removing dominoes from λ, keeping the shape a partition, until this is no longer possible. The partition $\widetilde{\lambda}$ does not depend on how these dominoes are removed. Every 2-core has the shape of a staircase $\delta_r = (r, r-1, \ldots, 0)$ for some integer $r \geqslant 0$.

We denote the set of partitions by $\mathcal{P}$ and the set of partitions with 2-core δ_r by $\mathcal{P}_r$. The set of all partitions λ satisfying the conditions:

$$\widetilde{\lambda} = \delta_r \text{ and } |\lambda| = |\delta_r| + 2n$$

will be denoted by $\mathcal{P}_r(n)$. Note that $\mathcal{P} = \bigcup_{r,n} \mathcal{P}_r(n)$ is a disjoint union.

A (*standard*) *domino tableau* D of *shape* $\lambda \in \mathcal{P}_r(n)$ consists of a tiling of the shape $\lambda/\widetilde{\lambda}$ by dominoes and a filling of the dominoes with the integers $\{1, 2, \ldots, n\}$, each used exactly once, so that the numbers are increasing when read along either the rows or columns. The *value* of a domino is the number written inside it. We will denote by dom_i the domino with the value i inside. We will also write $\mathrm{sh}(D) = \lambda$ for the shape of D. An equivalent description of the domino tableau D is as the sequence of partitions $\{\widetilde{\lambda} = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^n = \lambda\}$, where $\mathrm{sh}(\mathrm{dom}_i) = \lambda^i/\lambda^{i-1}$. If the values of the dominoes in a tableau D are not restricted to the set $\{1, 2, \ldots, n\}$ (but each value occurs at most once), we will call D an *injective domino tableau*.

We now describe a number of operations on standard Young and domino tableaux needed in the sequel. One may obtain a standard Young tableau $T = T(D)$ from a domino tableau D by replacing a domino with the value i in D by two boxes containing $\bar{i}$ and i in T. The boxes are placed so that T is standard with respect to the order $\bar{1} < 1 < \bar{2} < 2 < \cdots$. If D has shape λ, then $T(D)$ will have shape $\lambda/\widetilde{\lambda}$. Suppose now that Y is a standard Young tableau of shape $\widetilde{\lambda}$ filled with letters smaller than any of the letters occurring in D. Define $T_Y(D)$ by "filling" in the empty squares in $T(D)$ with the tableau Y.

Let T be a standard Young tableau and i a letter occurring in T. The *conversion* process proceeds as follows (see [25, 33]). Replace the letter i in T with another letter j. The resulting tableau may not be standard; so we repeatedly swap j with its neighbours until the tableau is standard. We say that the value i has been converted to j.

Now let T be any standard Young tableau filled with barred $\bar{i}$ and non-barred letters i. Define T^{neg} by successively converting barred letters $\bar{i}$ to negative letters $-i$, starting with the smallest letters. The main fact that we shall need is that the operation "neg" is invertible. We refer the reader to [33] for a full discussion of these operations.

3.2 The Barbasch–Vogan Domino Insertion Algorithm

The Robinson–Schensted correspondence establishes a bijection

$$\pi \leftrightarrow (P(\pi), Q(\pi))$$

between permutations $\pi \in \mathfrak{S}_n$ and pairs of standard tableaux with the same shape and size n (see [34]). Domino insertion generalizes this by replacing the symmetric group with the hyperoctahedral group. It depends on the choice of a core δ_r, and establishes a bijection between W_n and pairs (P^r, Q^r) of standard domino tableaux of the same shape $\lambda \in \mathcal{P}_r(n)$. There are in fact many such bijections, but we will be concerned only with the algorithm introduced by Barbasch and Vogan [2] and later given a different description by Garfinkle [10]. We now describe this algorithm following the more modern expositions [27, 33].

Let D be an injective domino tableau with shape λ such that $i > 0$ is a value which does not occur in D. We describe the insertion $E = D \leftarrow i$ (or $E = D \leftarrow -i$) of a horizontal (vertical) domino with value i into D. Let $D_{<i} \subset D$ denote the sub-domino tableau of D containing all dominoes with values less than i. If λ has a 2-core $\tilde{\lambda}$, then we will always assume that $\tilde{\lambda} \subset \mathrm{sh}(D_{<i})$. Let $E_{\leqslant i}$ be the domino tableau obtained from $D_{<i}$ by adding an additional vertical domino in the first column or an additional horizontal domino in the first row labeled i.

For $j > i$, we define $E_{\leqslant j}$, supposing that $E_{\leqslant j-1}$ is known. If D contains no domino labeled j, then $E_{\leqslant j} = E_{\leqslant j-1}$; otherwise let dom_j denote the domino in D labeled j. Let $\mu = \mathrm{sh}(E_{\leqslant j-1})$. We now distinguish four cases:

1. If $\mu \cap \mathrm{dom}_j = \emptyset$ do not touch, then we set $E_{\leqslant j} = E_{\leqslant j-1} \cup \mathrm{dom}_j$.
2. If $\mu \cap \mathrm{dom}_j = (k, l)$ is exactly one square in the kth row and lth column, then we add a domino containing j to $E_{\leqslant j-1}$ to obtain the tableau $E_{\leqslant j}$ which has shape $\mu \cup \mathrm{dom}_j \cup (k+1, l+1)$.
3. If $\mu \cap \mathrm{dom}_j = \mathrm{dom}_j$ and dom_j is horizontal, then we bump the domino dom_j to the next row, by setting $E_{\leqslant j}$ to be the union of $E_{\leqslant j-1}$ with an additional (horizontal) domino with value j one row below that of dom_j.
4. If $\mu \cap \mathrm{dom}_j = \mathrm{dom}_j$ and dom_j is vertical, then we bump the domino dom_j to the next column, by setting $E_{\leqslant j}$ to be the union of $E_{\leqslant j-1}$ with an additional (vertical) domino with value j one column to the right of dom_j.

Finally, we let $E = \lim_{j\to\infty} E_{\leqslant j}$.

Let $w = w(1)w(2)\cdots w(n) \in W_n$ be a hyperoctahedral permutation written in one-line notation. Thus, for each i, we have $w(i) \in \{\pm 1, \pm 2, \ldots, \pm n\}$; furthermore, $|w(1)||w(2)|\cdots|w(n)| \in \mathfrak{S}_n$ is a usual permutation. Let δ_r be a 2-core assumed to be fixed. Then the insertion tableau $P^r(w)$ is defined as $((\ldots((\delta_r \leftarrow w(1)) \leftarrow w(2))\cdots) \leftarrow w(n))$. The sequence of shapes obtained in the process defines another standard domino tableau called the recording tableau $Q^r(w)$ of $w \in W_n$. The insertion tableau $P^r(w)$ can of course be defined for any sequence $w = w(1)w(2)\cdots w(n)$ such that $|w(i)| \neq |w(j)|$ for $i \neq j$.

The following theorem is due to Barbasch–Vogan [2] and Garfinkle [10] when $r = 0, 1$ and extended by van Leeuwen [28] to larger cores.

Theorem 3.2. *Fix $r \geqslant 0$. The domino insertion algorithm defines a bijection between $w \in W_n$ and pairs (P, Q) of standard domino tableaux of the same shape lying in $\mathcal{P}_r(n)$. This bijection satisfies the equality $P^r(w) = Q^r(w^{-1})$.*

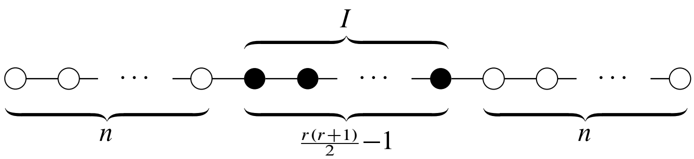
It is easy to see that the bijectivity in Theorem 3.2 together with Conjecture A^+ would imply that the relevant left cell representations are irreducible. This is consistent with the conjecture that left cell representations for W_n are irreducible for "generic parameters" and in particular if $b \notin \{a, 2a, \ldots, (n-1)a\}$ (see Proposition 2.3).

We have computational evidences for Conjectures A, A^+ and B: they were checked for $n \leqslant 6$ using CHEVIE [21] and Coxeter [7]. In the rest of this section, we shall give theoretical evidences for Conjecture A and A^+ (induction of cells, multiplication by the longest element, link to [30, Conjecture 25.3], asymptotic case, quasi-split case, right descent sets, coplactic relations).

3.3 Conjecture A and Lusztig's Conjecture 25.3

There is an alternative description (in the case where $r = 0$, 1, it is in fact the original description of Barbasch and Vogan) of domino insertion. As we will now explain, it is related to [30, Conjecture 25.3]. Let us fix in this subsection a Coxeter group (W, S) of type $A_{2n+r(r+1)/2-1}$. Let σ be the unique non-trivial automorphism of W such that $\sigma(S) = S$. If J is a subset of S, we denote by W_J the parabolic subgroup of W generated by J and let w_J denote the longest element of W_J.

Let I be the unique connected (when we view it as a subdiagram of the Dynkin diagram of (W, S)) subset of S of cardinality $r(r+1)/2 - 1$ (or 0 if $r = 0$) such that $\sigma(I) = I$:



Let $\mathcal{W}$ denote the subgroup of W consisting of all elements w such that $wW_Iw^{-1} = W_I$, and w has minimal length in wW_I (see [30, Sect. 25.1]). If Ω is a σ-orbit in $S \setminus I$, we set $s_\Omega = w_{I\cup\Omega}w_I$. If $0 \leqslant i \leqslant n-1$, let Ω_i denote the orbit of σ in $S \setminus I$ consisting of elements which are separated from I by i nodes in the Dynkin diagram. Then $\{\Omega_0, \Omega_1, \ldots, \Omega_{n-1}\}$ is the set of orbits of σ in $S \setminus I$. Moreover, there is a unique morphism of groups $\iota_r : W_n \to \mathcal{W}^\sigma$ that sends t to s_{Ω_0} and s_i to s_{Ω_i} (for $1 \leqslant i \leqslant n-1$). It is an isomorphism of groups (see [30, Sect. 25.1]).

The morphism ι_r can be described explicitly in the language of signed permutations. First, identify W with the permutation group of the following $2n + r(r+1)/2$ elements (ordered according to the ordering of S):

$$\{-n < -(n-1) < \cdots < -1 < 0_1 < 0_2 < \cdots < 0_{r(r+1)/2} < 1 < 2 < \cdots < n\},$$

so that the subgroup W_I (which is isomorphic to $\mathfrak{S}_{r(r+1)/2}$) acts on the elements $\{0_1, 0_2, \ldots, 0_{r(r+1)/2}\}$. Let $w = w(1)w(2)\cdots w(n) \in W_n$. Then the two-line notation of $\iota_r(w)$ is given by

$$\begin{pmatrix} -n & \cdots & -1 & 0_1 & \cdots & 0_{r(r+1)/2} & 1 & 2 & \cdots & n \\ -w(n) & \cdots & -w(1) & 0_1 & \cdots & 0_{r(r+1)/2} & w(1) & w(2) & \cdots & w(n) \end{pmatrix}. \tag{1}$$

Now, let $\mathbf{c}_0$ denote the two-sided cell of W_I which has "shape" δ_r. If $w, w' \in \mathfrak{S}_n$, we write $w \simeq_{\mathfrak{S}} w'$ if $Q(w) = Q(w')$ (the equivalence relation $\simeq_{\mathfrak{S}}$ defines the Robinson–Schensted left cells of $\mathfrak{S}_n$, which coincide with the Kazhdan–Lusztig left cells [26, Sect. 5]).

Theorem 3.3. *Fix $x \in \mathbf{c}_0$. Let $w \in W_n$, $r \geqslant 0$ and $\pi = \iota_r(w)x \in \mathfrak{S}_{2n+r(r+1)/2}$. Then we have*

$$(T_{P(x)}(P^r(w)))^{\mathrm{neg}} = P(\pi) \quad \textit{and} \quad (T_{Q(x)}(Q^r(w)))^{\mathrm{neg}} = Q(\pi).$$

Since neg *is invertible, in particular $w \simeq_r w'$ if and only if $\iota_r(w)x \simeq_{\mathfrak{S}} \iota_r(w')x$.*

For the construction of $T_{P(x)}(P^r(w))$ in Theorem 3.3, we are using the ordering $0_1 < 0_2 < \cdots < 0_{r(r+1)/2} < \bar{1} < 1 < \bar{2} < 2 < \cdots < \bar{n} < n$. In the case where $r = 0$ or 1, Theorem 3.3 is essentially [28, Theorem 4.2.3] with different notation. To generalize the result to all $r \geqslant 0$, we follow the approach of [33].

Proof. For the case $r = 0$, the theorem is exactly [33, Theorem 32]. We now explain, assuming familiarity with [33], how to extend the result to larger cores. It is shown in [33, Lemma 31] that a domino insertion $D \leftarrow i$ can be imitated by *doubly mixed insertion*, denoted by P_{m^*}. The proof of [33, Lemma 31] is local and remains valid when we replace $D_{<i}$ by any Young tableau of the same shape, filled with "small" letters. More precisely, their proof shows that $T_{P(x)}(P^r(w))$ can be obtained by doubly mixed insertion of a "biword" w^{dup} (explicitly defined in [33]) into $P(x)$. Thus one has

$$T_{P(x)}(P^r(w)) = P_{m^*}(x \sqcup w^{\mathrm{dup}}), \tag{2}$$

where $a \sqcup b$ denotes the word obtained from concatenating a and b. In the notation of [33], x here is a biword with no bars so that $P_{m^*}(x) = P(x)$.

Now [33, Theorem 21 and Proposition 14] connect doubly mixed insertion with usual Schensted insertion via the equation

$$P_{m^*}(u)^{\mathrm{neg}} = P(u^{\mathrm{inv\,neg\,inv\,neg}}). \tag{3}$$

The operation denoted "inv neg inv neg" in [33] applied to $x \sqcup w^{\mathrm{dup}}$ coincides with our inclusion $\iota_r(w)x$. Combining (2) and (3), one obtains

$$(T_{P(x)}(P^r(w)))^{\mathrm{neg}} = P(\pi).$$

The statement about recording tableau is obtained analogously, or using the equation $Q(\pi) = P(\pi^{-1}) = P(x^{-1}\iota_r(w)^{-1}) = P(\iota_r(w^{-1})x^{-1})$. □

Remark 3.4. Note that the last statement of Theorem 3.3 does not depend on the choice of $x \in \mathbf{c}_0$.

Corollary 3.5. *If $r \geqslant 0$ and if $(a, b) = (2, 2r+1)$, then Conjecture A*(c) *agrees with* [30, Conjecture 25.3] *for the case (W, S, I, σ) described above.*

3.4 Longest Element

Let w_0 denote the longest element of W_n: it is equal to $t_1 t_2 \ldots t_n$ (or to -1 -2 $\ldots -n$ in the one line notation). It is a classical result that two elements x and y in W_n satisfy $x \sim_{\mathcal{L}} y$ if and only if $w_0 x \sim_{\mathcal{L}} w_0 y$. The next result shows that the relations $\simeq_r$ share the same property.

Proposition 3.6. *Let $r \geqslant 0$ and let $x, y \in W_n$. Then $x \simeq_r y$ if and only if $w_0 x \simeq_r w_0 y$.*

Proof. This follows from the easy fact that $P^r(w_0 x)$ (resp. $Q^r(w_0 x)$) is the conjugate (i.e., the transpose with respect to the diagonal) of $P^r(x)$ (resp. $Q^r(x)$), and similarly for y. □

3.5 Induction of Cells

Let $m \leqslant n$. Let X_m^n denote the set of elements $w \in W_n$ which have minimal length in wW_m. It is a cross-section of W_n/W_m. By Remark 3.1, an element $x \in W_n$ belongs to X_m^n if and only if $0 < x(1) < x(2) < \cdots < x(m)$. A theorem of Geck [13] asserts that if C is a Kazhdan–Lusztig left cell of W_m (associated with the restriction of $L_{a,b}$ to W_m), then $X_m^n C$ is a union of Kazhdan–Lusztig left cells of W_n. The next result shows that the same hold if we replace *Kazhdan–Lusztig left cell* by *left r-cell.*

Proposition 3.7. *Let $r \geqslant 0$. If C is a left r-cell of W_m, then $X_m^n C$ is a union of left r-cells of W_n.*

Proof. Let $w, w' \in W_m$ and $x, x' \in X_m^n$ be such that $xw \simeq_r x'w'$ (in W_n). We must show that $w \simeq_r w'$ (in W_m). For the purpose of this proof, we shall denote by $(P_n^r(w), Q_n^r(w))$ (resp. $(P_m^r(w), Q_m^r(w))$) the pair of standard domino tableaux obtained by viewing w as an element of W_n (resp. of W_m). Then, since x is increasing on $\{1, 2, \ldots, m\}$ and takes only positive values, the dominoes filled with $\{1, 2, \ldots, m\}$ in the recording tableau $Q_n^r(xw)$ are the same as the one in the recording tableau $Q_n^r(w)$. In particular, $Q_m^r(w)$ is obtained from $Q_n^r(xw)$ by removing the dominoes filled by $\{m+1, m+2, \ldots, n\}$. Similarly, $Q_m^r(w')$ is obtained from $Q_n^r(x'w')$ by removing the dominoes filled by $\{m+1, m+2, \ldots, n\}$. Since $Q_n^r(xw) = Q_n^r(x'w')$ by hypothesis, we have that $Q_m^r(w) = Q_m^r(w')$. In other words, $w \simeq_r w'$ in W_m. □

Corollary 3.8. *Let $r \geqslant 0$ and let x and y be two elements of W_m. Then $x \simeq_r y$ in W_m if and only if $x \simeq_r y$ in W_n.*

The previous corollary shows that it is not necessary to make the ambient group precise when one studies the equivalence relation $\simeq_r$.

Geck's result [13] is valid for any Coxeter group and any parabolic subgroup. We shall investigate now the analogue of Proposition 3.7 for the parabolic subgroup $\mathfrak{S}_n$ of W_n. We denote by $X(n)$ the set of elements $w \in W_n$ which have minimal length in $w\mathfrak{S}_n$. It is a cross-section of $W_n/\mathfrak{S}_n$. By Remark 3.1, an element $w \in W_n$ belongs to $X(n)$ if and only if $w(1) < w(2) < \cdots < w(n)$.

Proposition 3.9. *Let $r \geqslant 0$ and let C be a Robinson–Schensted left cell of $\mathfrak{S}_n$. Then $X(n)C$ is a union of domino left cells for $\simeq_r$.*

Proof. Let $w, w' \in \mathfrak{S}_n$ and $x, x' \in X(n)$ be such that $xw \simeq_r x'w'$ in W_n. We must show that $w \simeq_{\mathfrak{S}} w'$. It is well known that for two words $a_1a_2\cdots a_k$ and $b_1b_2\cdots b_k$, one has $Q(a_1a_2\cdots a_k) = Q(b_1b_2\cdots b_k) \implies Q(a_ja_{j+1}\cdots a_l) = Q(b_jb_{j+1}\cdots b_l)$ for any $1 \leqslant j \leqslant l \leqslant k$. Indeed, this is Geck's result [13] for $\mathfrak{S}_n$.

By Theorem 3.3, we have $Q(\iota_r(xw)c) = Q(\iota_r(x'w')c)$ for any $c \in \mathbf{c}_0$. Treating $i_r(xw)c$ as a word using (1), we thus have

$$Q(xw(1)\ xw(2)\ \cdots\ xw(n)) = Q(x'w'(1)\ x'w'(2)\ \cdots\ x'w'(n)).$$

But $x(1) < x(2) < \cdots < x(n)$, so that this is equivalent to $Q(w) = Q(w')$. □

If $x \in X_m^n$, and if $w, w' \in W_m$ are such that $w \sim_{\mathcal{L}} w'$ in W_m, then a result of Lusztig [30, Proposition 9.13] asserts that $wx^{-1} \sim_{\mathcal{L}} w'x^{-1}$. The next result shows that the same statement holds if we replace $\sim_{\mathcal{L}}$ by $\simeq_r$.

Proposition 3.10. *Let $r \geqslant 0$, $x \in X_m^n$ and $w, w' \in W_m$ be such that $w \simeq_r w'$. Then $wx^{-1} \simeq_r w'x^{-1}$.*

Proof. Let us use here the notation of the proof of Proposition 3.7. So we assume that $P_m^r(w^{-1}) = P_m^r(w'^{-1})$ and we must show that $P_n^r(xw^{-1}) = P_n^r(xw'^{-1})$. Let $D = ((\ldots((\delta_r \leftarrow xw^{-1}(1)) \leftarrow xw^{-1}(2))\cdots) \leftarrow xw^{-1}(m))$ and $D' = ((\ldots((\delta_r \leftarrow xw'^{-1}(1)) \leftarrow xw'^{-1}(2))\cdots) \leftarrow xw'^{-1}(m))$. Since $w^{-1}(i) = i$ if $i \geqslant$

$m+1$, we have $P_n^r(xw^{-1}) = ((\ldots((D \leftarrow x(m+1)) \leftarrow x(m+2))\cdots) \leftarrow x(n))$ and similarly for $P_n^r(xw'^{-1})$. Therefore, we only need to show that $D = D'$. But, since w^{-1} stabilizes $\{\pm 1, \pm 2, \ldots, \pm m\}$ and since x is increasing on $\{1, 2, \ldots, m\}$ and takes only positive values, it follows from the domino insertion algorithm that D is obtained from $P_m^r(w^{-1})$ by applying x, or in other words replacing the domino dom_i by $\mathrm{dom}_{x(i)}$. Similarly, D' is obtained from $P_m^r(w'^{-1})$ by applying x. Since $P_m^r(w^{-1}) = P_m^r(w'^{-1})$ by hypothesis, we get that $D = D'$, as desired. □

As for Geck's result, Lusztig's result [30, Proposition 9.13] is valid for any parabolic subgroup of any Coxeter group. The next result is the analogue of Proposition 3.10 for the parabolic subgroup $\mathfrak{S}_n$ of W_n.

Proposition 3.11. *Let $r \geqslant 0$, $x \in X(n)$ and $w, w' \in \mathfrak{S}_n$ be such that $w \simeq_{\mathfrak{S}} w'$. Then $wx^{-1} \simeq_r w'x^{-1}$.*

Proof. We must show that $P^r(xw^{-1}) = P^r(xw'^{-1})$ knowing that $P(w^{-1}) = P(w'^{-1})$. For $u \in W_n$, denote by $u_{\mathfrak{S}}$ the word $u(1)\ u(2)\ \cdots\ u(n)$ and $u_{-\mathfrak{S}}$ the word $-u(n)\ -u(n-1)\ \cdots\ -u(1)$. The equation $P(w^{-1}) = P(w'^{-1})$ gives

$$P((xw^{-1})_{\mathfrak{S}}) = P((xw'^{-1})_{\mathfrak{S}}) \text{ and } P((xw^{-1})_{-\mathfrak{S}}) = P((xw'^{-1})_{-\mathfrak{S}}), \quad (4)$$

where we are comparing pairs of standard Young tableaux filled with the set of letters $\{x(1), x(2), \ldots, x(n)\}$ (resp. $(\{-x(1), -x(2), \ldots, -x(n)\})$.

By Theorem 3.3, it is enough to show that $P(\iota_r(xw^{-1})c) = P(\iota_r(xw'^{-1})c)$ for some fixed $c \in \mathbf{c}_0$. Using (1), we may write $\iota_r(xw^{-1})c$ in one-line notation as the concatenation $(xw^{-1})_{-\mathfrak{S}} \sqcup c \sqcup (xw^{-1})_{\mathfrak{S}}$. It is well known that if a, a', b are words such that $P(a) = P(a')$, then one has $P(a \sqcup b) = P(a' \sqcup b)$ (this is Lusztig's result [30, Proposition 9.11] for the symmetric group). Combining this with (4), we obtain $P((xw^{-1})_{-\mathfrak{S}} \sqcup c \sqcup (xw^{-1})_{\mathfrak{S}}) = P((xw'^{-1})_{-\mathfrak{S}} \sqcup c \sqcup (xw'^{-1})_{\mathfrak{S}})$, as desired. □

Remark 3.12. The Propositions 3.7, 3.9–3.11 generalize [4, Proposition 4.8] (which corresponds to the asymptotic case).

3.6 Asymptotic Case, Quasi-Split Case

We now prove Conjectures A for $r = 0, 1$ and $r \geqslant n-1$.

Theorem 3.13. *Conjectures A and A^+ are true for $r \geqslant n-1$.*

Proof. Let $r \geqslant n-1$, and let D be a domino tableau with shape $\lambda \in \mathcal{P}_r(n)$. The dominoes $\{\mathrm{dom}_i \mid i \in \{1, 2, \ldots, n\}\}$ can be decomposed into the two disjoint collections $\mathcal{D}_+ = \{\mathrm{dom}_i \mid \mathrm{dom}_i \text{ is horizontal}\}$ and $\mathcal{D}_- = \{\mathrm{dom}_i \mid \mathrm{dom}_i \text{ is vertical}\}$ such that all the dominoes in $\mathcal{D}_+$ lie strictly above and to the right of all the dominoes in $\mathcal{D}_-$. We call a tableau satisfying this property *segregrated*. If the collection of dominoes $\mathcal{D}_+$ is left justified, and each domino replaced by a single box, one obtains a usual Young tableau. Similarly, if the dominoes $\mathcal{D}_-$ are justified upwards and changed into boxes, one obtains a usual Young tableau.

In other words, D can be thought of as a union of two usual tableau D_+ and D_- so that the union of the values in D_+ and in D_- is the set $\{1, 2, \ldots, n\}$. To be consistent with the remaining discussion, we in fact define D_- to be the *conjugate* (reflection in the main diagonal) of the Young tableau obtained from the dominoes $\mathcal{D}_-$, as described above.

Domino insertion is compatible with this decomposition so that the following diagram commutes:

$$\begin{tikzcd} w \arrow[r, "\text{domino insertion}"] \arrow[d] & (P^r(w), Q^r(w)) \arrow[d] \\ {[w_+, w_-]} \arrow[r, "\text{RS insertion}"] & {[(P^r_+(w), Q^r_+(w)), (P^r_-(w), Q^r_-(w))]} \end{tikzcd}$$

Here w_+ denotes the subword of w consisting of positive letters and w_- denotes the subword consisting of negative letters, with the minus signs removed. In [5] and [3, Corollary 5.2], it is shown for $r \geqslant n - 1$ and $w, w' \in W_n$ that $w \sim_{\mathcal{L}} w'$ if and only if $Q(w_+) = Q(w'_+)$ and $Q(w_-) = Q(w'_-)$; similar results hold for $\sim_{\mathcal{R}}$ and $\sim_{\mathcal{LR}}$. Since $Q(w_+) = Q^r_+(w)$ and $Q(w_-) = Q^r_-(w)$, we have $w \sim_{\mathcal{L}} w'$ if and only if $Q^r(w) = Q^r(w')$, establishing Conjectures A and A$^+$ in this case for left cells. A similar argument works for right cells and two-sided cells, using also the classification of two-sided cells in [3]. □

Theorem 3.14. *Conjecture A is true if $a = 2b$ or if $3a = 2b$.*

Proof. In [29], Lusztig determined the left cells of W_n with parameters $b = (2r+1)a/2$ for $r \in \{0, 1\}$ as follows. When $r \in \{0, 1\}$, we have $I = \emptyset$ in the notation of Sect. 3.3. The equal parameter weight function L on $\mathfrak{S}_{2n+r}$ restricts to the weight function $L_{b,a}$ on $\iota_r(W_n)$, where $b = (2r + 1)a/2$. Lusztig [29, Theorem 11] shows that each left cell of W_n is the intersection of a left cell of $\mathfrak{S}_{2n+r}$ with $\iota_r(W_n)$. Thus, $w \simeq_{\mathcal{L}} w'$ in W_n if and only if $\iota_r(w) \simeq_{\mathfrak{S}} \iota_r(w')$ in $\mathfrak{S}_{2n+r}$. When $r \in \{0, 1\}$, there is no need for the element $x \in \mathbf{c}_0$ in Theorem 3.3; so one obtains Conjecture A for $r \in \{0, 1\}$. □

3.7 Right Descent Sets

If $r \geqslant 0$, let $S_n^{(r)} = \{s_1, s_2, \ldots, s_{n-1}\} \cup \{t_1, \ldots, t_r\}$ (if $r \geqslant n$, then $S_n^{(r)} = S_n^{(n)}$). If $w \in W_n$, let

$$\mathcal{R}_n^{(r)}(w) = \{s \in S_n^{(r)} \mid \ell(ws) < \ell(w)\}$$

be the *extended right descent set* of w. The following proposition is easy:

Proposition 3.15. *Let x and y be two elements of W_n. Then:*

(a) *If $x \simeq_r y$, then $\mathcal{R}_n^{(r+1)}(x) = \mathcal{R}_n^{(r+1)}(y)$.*
(b) *If $b > ra$ and if $x \sim_{\mathcal{L}} y$, then $\mathcal{R}_n^{(r+1)}(x) = \mathcal{R}_n^{(r+1)}(y)$.*

Proof. If $r \geqslant n-1$, then statements (a) and (b) are equivalent by Theorem 3.13; see also Remark 1.6(b). But, in this case, (b) has been proved in [4, Proposition 4.5]. So, let us assume from now on that $r < n-1$. We shall prove (a) and (b) together. Let us set

$$\mathcal{R}_s(x) = \{s \in \{s_1, \ldots, s_{n-1}\} \mid \ell(xs) < \ell(x)\},$$

$$\mathcal{R}_t^{(r)}(x) = \{s \in \{t_1, \ldots, t_r\} \mid \ell(xs) < \ell(x)\}.$$

Then $\mathcal{R}_n^{(r)}(x) = \mathcal{R}_s(s) \cup \mathcal{R}_t^{(r)}(x)$.

Write $x = ux'$ and $y = vy'$, with $u, v \in X_{r+1}^n$ and $x', y' \in W_{r+1}$. Since $\ell(ux') = \ell(u) + \ell(x')$ (and similarly for $ux's$ for any $s \in W_{r+1}$), we have that $\mathcal{R}_t^{(r+1)}(x) = \mathcal{R}_t^{(r+1)}(x')$. Similarly, $\mathcal{R}_t^{(r+1)}(y) = \mathcal{R}_t^{(r+1)}(y')$. But, if x and y satisfy (a) or (b), then $\mathcal{R}_t^{(r+1)}(x') = \mathcal{R}_t^{(r+1)}(y')$: indeed, this follows from the fact that (a) and (b) have been proved in the asymptotic case and, in case (a), from Proposition 3.7 and, in case (b), from [13].

Now it remains to show that $\mathcal{R}_s(x) = \mathcal{R}_s(y)$ if x and y satisfy (a) or (b). In case (b), this follows from [30, Lemma 8.6]. So assume now that $x \simeq_r y$. Write $x = u'\sigma$ and $y = v'\tau$, with $u, v \in X(n)$ and $\sigma, \tau \in \mathfrak{S}_n$. As in the previous case, we have $\mathcal{R}_s(x) = \mathcal{R}_s(\sigma)$ and $\mathcal{R}_s(y) = \mathcal{R}_s(\tau)$. Moreover, by Proposition 3.9, we have $\sigma \simeq_{\mathfrak{S}} \tau$. It is well known that it implies that $\mathcal{R}_s(\sigma) = \mathcal{R}_s(\tau)$. □

Remark 3.16. Proposition 3.15 (a) can also be deduced from [33, Lemma 33] or [27, Lemma 9].

3.8 Coplactic Relations

If x and y are two elements of W_n such that $\ell(x) \leqslant \ell(y)$, then we write $x \smile_r y$ if there exists $s \in S_n^{(0)}$ and $s' \in S_n^{(r)}$ such that $y = sx$ and $\ell(s'x) < \ell(x) < \ell(y) < \ell(s'y)$. If $\ell(x) \geqslant \ell(y)$, then we write $x \smile_r y$ if $y \smile_r x$. Let $\equiv_r$ denote the reflexive and transitive closure of $\smile_r$.

Remark 3.17. (a) If $x \equiv_r y$, then $\ell_t(x) = \ell_t(y)$. Here, $\ell_t(w)$ denotes the number of occurrences of the generator t in a reduced expression for $w \in W_n$

(b) If $r' \geqslant r$ and if $x \equiv_r y$, then $x \equiv_{r'} y$ (indeed, if $x \smile_r y$, then $x \smile_{r'} y$: this just follows from the fact that $S_n^{(r)} \subset S_n^{(r')}$). Moreover, the relations $\equiv_n$ and $\equiv_r$ are equal if $r \geqslant n$.

(c) If $r \geqslant n-1$, then $x \equiv_r y$ if and only if $x \equiv_{n-1} y$. Let us prove this statement. By (b) above, we only need to show that if $x \smile_n y$, then $x \smile_{n-1} y$. For this, we may assume that $\ell(y) > \ell(x)$. So there exists $i \in \{1, 2, \ldots, n-1\}$ and $s' \in S_n^{(n)}$ such that $y = s_i x$ and $\ell(s'x) < \ell(x) < \ell(s_i x) < \ell(s's_i x)$. If $s' \in S_n^{(n-1)}$ then we are done. So we may assume that $s' = t_n$. Therefore, the first inequality says that $x^{-1}(n) < 0$ and the last inequality says that $x^{-1}s_i(n) > 0$. This implies that $i = n-1$. Consequently, we have $x^{-1}(n) < 0$ and $x^{-1}(n-1) > 0$. But the middle inequality says that $x^{-1}(n-1) < x^{-1}(n)$; so we obtain a contradiction with the fact that $s' = t_n$.

(d) If $b > ra$ and if $r \geqslant n-1$, then it follows from Theorem 3.13, from [5, Proposition 3.8] and from (a), (b) and (c) above that the relations $\sim_{\mathcal{L}}$, $\simeq_r$ and $\equiv_r$ coincide.

Proposition 3.18. *Let x and y be two elements of W_n such that $x \equiv_r y$. Then the following hold:*

(i) $x \simeq_r y$.

(ii) *If $b > ra$, then $x \sim_{\mathcal{L}} y$.*

Proof. We may, and we will, assume that $x \smile_r y$. By symmetry, we may also assume that $\ell(y) > \ell(x)$. We shall prove (i) and (ii) together. There exists $s \in S_n^{(0)}$ and $s' \in S_n^{(r)}$ such that $y = sx$ and $\ell(s'x) < \ell(x) < \ell(y) < \ell(s'y)$. Two cases may occur:

- If $s' \in S_n^{(0)}$, then write $x = x'u^{-1}$ and $y = y'v^{-1}$ with x', $y' \in \mathfrak{S}_n$ and u, $v \in X(n)$. Then $y' = sx'$, $u = v$ and $\ell(s'x') < \ell(x') < \ell(y') < \ell(s'y')$. It is well known that it implies that $Q(x') = Q(y')$ (Knuth relations), so $x' \simeq_{\mathfrak{S}} y'$ and $x' \sim_{\mathcal{L}} y'$. Therefore, since moreover $u = v$, it follows from Proposition 3.11 (resp. [30, Proposition 9.13]) that $x \simeq_r y$ (resp. $x \sim_{\mathcal{L}} y$).
- If $s' \notin S_n^{(0)}$, then we write $s = s_i$ and $s' = t_j$. Then the relations $y = sx$ and $\ell(s'x) < \ell(x) < \ell(y) < \ell(s'y)$ imply that $x^{-1}(j) < 0$ and $x^{-1}s_i(j) > 0$. In particular, s and s' belong to W_{r+1}. Now, write $x = x'u^{-1}$ and $y = y'v^{-1}$ with x', $y' \in W_{r+1}$ and u, $v \in X_{r+1}^n$. Then $y' = sx'$, $u = v$ and $\ell(s'x') < \ell(x') < \ell(y') < \ell(s'y')$. By Remark 3.17 (d), this implies that $x' \simeq_r y'$ and, if $b > ra$, that $x' \sim_{\mathcal{L}} y'$. Therefore, since moreover $u = v$, it follows from Proposition 3.11 (resp. [30, Proposition 9.13]) that $x \simeq_r y$ (resp. $x \sim_{\mathcal{L}} y$). □

Even if we have both $\ell_t(x) = \ell_t(y)$ and $x \simeq_r y$, we do not necessarily have $x \equiv_r y$. For example, let $r = 0$, $n = 6$ and take $x = 5\ 6\ 1\ 4\ 2\ {-3}$ and $y = 5\ 6\ {-1}\ 4\ 3\ 2$.

4 Cycles and Conjecture B

4.1 Open and Closed Cycles

We now describe a more refined combinatorial structure of domino tableaux introduced by Garfinkle [10]. We will mostly follow the setup of [32].

Let D be a domino tableau with shape $\lambda \in \mathcal{P}_r(n)$. We call a square $(i, j) \in D$ *variable* if $i + j$ and r have the same parity, otherwise we call it *fixed*. If the domino dom_i contains the square (k, l), we write $D(k, l) = i$.

Now let (k, l) be the fixed square of dom_i. Suppose that dom_i occupies the squares $\{(k, l), (k + 1, l)\}$ or $\{(k, l - 1), (k, l)\}$. We define a new domino dom'_i by letting it occupy the squares

1. $\{(k, l), (k - 1, l)\}$ if $i < D(k - 1, l + 1)$,
2. $\{(k, l), (k, l + 1)\}$ if $i > D(k - 1, l + 1)$.

Otherwise dom_i occupies the squares $\{(k, l), (k, l + 1)\}$ or $\{(k - 1, l), (k, l)\}$. We define a new domino dom'_i by letting it occupy the squares

1. $\{(k, l), (k, l - 1)\}$ if $i < D(k + 1, l - 1)$,
2. $\{(k, l), (k + 1, l)\}$ if $i > D(k + 1, l - 1)$.

Now define the *cycle* $c = c(D, i)$ of D through i to be the smallest union c of dominoes satisfying that (i) $\mathrm{dom}_i \in c$ and (ii) $\mathrm{dom}_j \in c$ if $\mathrm{dom}_j \cap \mathrm{dom}'_k \neq \emptyset$ or $\mathrm{dom}'_j \cap \mathrm{dom}_k \neq \emptyset$ for some $\mathrm{dom}_k \in c$. If c is a cycle of D, let $M(D, c)$ be the domino tableau obtained from D by replacing each domino $\mathrm{dom}_i \in c$ by dom'_i. We call this procedure *moving through* c.

Theorem 4.1 ([10]). *Let D be a domino tableau and c a cycle of D. Then $M(D, c)$ is a standard domino tableau. Furthermore, if C is a set of cycles of D, then the tableau $M(D, C)$ obtained by moving through each $c \in C$ is defined unambiguously.*

We call a cycle c *closed* if $M(D, c)$ has the same shape as D; otherwise we call c *open*. Note that each (non-trivial) cycle c is in one of two positions, so that moving through is an invertible operation.

4.2 Evidence for Conjecture B

The notion of open and closed cycles allows us to state a combinatorially more precise version of Conjecture B.

Conjecture D. *Assume that $b = ra$ for some $r \geqslant 1$. Then the following hold for any $w, w' \in W_n$:*

(a) *$w \sim_{\mathcal{L}} w'$ if and only if $Q^{r-1}(w) = M(Q^{r-1}(w'), C)$ for a set C of* ***open*** *cycles.*

(b) *$w \sim_{\mathcal{R}} w'$ if and only if $P^{r-1}(w) = M(P^{r-1}(w'), C)$ for a set C of **open** cycles.*
(c) *$w \sim_{\mathcal{LR}} w'$ if and only if there exists a sequence of elements $w = w_1, w_2, \ldots, w_k = w'$ such that, for each i, some tableau with shape equal to $\mathrm{sh}(P^{r-1}(w_i)) = \mathrm{sh}(Q^{r-1}(w_i))$ can be obtained from a tableau with shape $\mathrm{sh}(P^{r-1}(w_{i-1})) = \mathrm{sh}(Q^{r-1}(w_{i-1}))$ by moving through a set of **open** cycles.*

Remark 4.2. (a) In an earlier version of this paper, part (c) of the above conjecture stated that $w \sim_{\mathcal{LR}} w'$ if and only if some tableau with shape equal to $\mathrm{sh}(P^{r-1}(w)) = \mathrm{sh}(Q^{r-1}(w))$ can be obtained from a tableau with shape $\mathrm{sh}(P^{r-1}(w')) = \mathrm{sh}(Q^{r-1}(w')$ by moving through a set of open cycles. We wish to thank Pietraho (private communication, 23 July 2007) for pointing out to us that it is actually necessary to consider sequences of elements as above.
(b) Each cycle c of a domino tableau D is in one of two positions, and by Theorem 4.1 they can be moved independently. Thus, Conjecture D would imply that every left cell for the parameters $b = ra$ would be a union of 2^d left cells for the parameters $b = \frac{(2r-1)a}{2}$. Here d is equal to the number of open cycles, which do not change the shape of the core, in one of the Q-tableaux in Conjecture D. This is consistent with the fact that if $b = ra$ with $r \geqslant 1$, then the number of irreducible components of a constructible representation is a power of 2 (see [30, Chap. 22]; see also with [29, (12.1)] for the equal parameter case).

We have the following theorem of Pietraho, obtained via a careful study of the combinatorics of cycles.

Theorem 4.3 (Pietraho [31]). *Conjecture B and Conjecture D are equivalent.*

Some special cases of Conjectures B and D are known. The case $b = a$ or $r = 1$ is known as the equal parameter case and is closely connected with the classification of primitive ideals of classical Lie algebras.

Theorem 4.4 (Garfinkle [11]). *Conjecture D is true for $r = 1$.*

The asymptotic case follows from [3, 5].

Theorem 4.5. *Conjecture D holds for $r \geqslant n$.*

Proof. Let D be a domino tableau with shape $\lambda \in \mathcal{P}_q(n)$ such that $q \geqslant n-1$. Then moving through any cycle of D changes the shape of the core of D. Thus (for left cells) the condition $Q^q(w) = M(Q^q(w'), C)$ in Conjecture D is the same as the condition $Q^q(w) = Q^q(w')$. This agrees with the classification given in [5]. A similar argument works for right cells and two-sided cells, also using the classification in [3]. □

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Runner Removal Morita Equivalences

Joseph Chuang and Hyohe Miyachi

Dedicated to Ken-ichi Shinoda and Toshiaki Shoji on the occasions of their 60th birthdays.

Appendix by Akihiko Hida and Hyohe Miyachi.

Abstract Let $U_q = U_q(\mathfrak{gl}_n)$ be Lusztig's divided power quantum general linear group over the complex field with parameter q, a root of unity. We investigate the category of finite-dimensional modules over U_q.

Lusztig's famous character formula for simple modules over U_q is written purely in terms of the affine Weyl group and its Hecke algebra, which are independent of q. Our result may be viewed as the categorical version of this independence. Moreover, our methods are valid over fields of positive characteristic.

The proof uses the modular representation theory of symmetric groups and finite general linear groups, and the notions of sl2-categorification and perverse equivalences.

Keywords Morita equivalence · Schur algebras

Mathematics Subject Classifications (2000): 16D90, 18E30, 20G42, 20G43, 20G05, 20G40, 20C08, 33D80

1 Introduction

Gordon James and Andrew Mathas [JM02] showed that certain decomposition numbers of Iwahori–Hecke algebras of symmetric groups and of the associated q-Schur

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algebras at different complex roots of unity are equal. The purpose of this chapter is to interpret these equalities as consequences of Morita equivalences.

Our results extend to q-Schur algebras over fields of positive characteristic, as long as the parameter q is in the prime subfield and we restrict to blocks 'of abelian defect'. The idea of the proof is to deal first with certain distinguished blocks, the *Rouquier blocks*, and then use special derived equivalences, called *perverse equivalences*, that arise from $\mathfrak{sl}_2$-categorifications, to make a link to other blocks. This second part is an inductive step, which is based on a result proven in a forthcoming paper [CRa].

For the Rouquier blocks, the case of ground fields of characteristic 0 is more difficult, contrary to the usual expectation. The only known method to prove structure theorems for these blocks is via local representation theory of finite groups. Over fields of positive characteristic, we require finite general linear groups – hence the insistence that the parameter q lies in the ground field. To obtain the result in characteristic 0, we use a lifting argument; a more conceptual method would be desirable.

We formulate our main result in such a way as to also allow comparison between Schur algebras at different roots of unity of the same order. Therefore as a bonus we deduce that the q-Schur algebra $\mathcal{S}_{\Bbbk,q}(r)$ over a field $\Bbbk$ is isomorphic to the q'-Schur algebra $\mathcal{S}_{\Bbbk,q'}(r)$ *as* $\Bbbk$-*algebras* if $\Bbbk$ has characteristic 0 and q and q' have the same multiplicative order. Versions of this isomorphism result (see Corollary 4) have been proven independently using other approaches: in characteristic zero by Rouquier [Rou08a] via Cherednik algebras, and for the Hecke algebras of symmetric groups in all characteristics by Brundan-Kleshchev [BK09] and by Rouquier [Rou08b] using quiver Hecke algebras.

The relevant combinatorial operations used by James and Mathas, the addition of runners on abaci and the reverse procedure of 'runner removal', have very natural interpretations in terms of alcove geometry. This is very consistent with Goodman's remark on James–Mathas's results. We are not able to exploit this point of view in our proof, because for the Rouquier blocks the combinatorics of partitions and abaci seem better suited. Nevertheless, in the final section we take a stab at a possible analogue of the main theorem for quantized enveloping algebras at complex roots of unity.

We would like to thank Matthew Fayers and the referees for pointing out mistakes in the first and second version of this chapter. We would also like to thank Akihiko Hida for his permission to append Hida-Miyachi result on GL_n to this chapter.

2 Notations and Background

2.1 Partitions and Fock Space

We associate with any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ its Young diagram $\{(i, j) \mid 1 \leq i \leq \lambda_j\} \in \mathbb{N} \times \mathbb{N}$. We denote by λ^{tr} the *conjugate* partition; its Young diagram is obtained from that of λ by interchanging the coordinates i and j.

Fix an integer $e \geq 2$. Given $a \in \{0, \dots, e-1\}$, and partitions λ and μ, we write $\lambda \to_a \mu$ if the Young diagram of μ is obtained from that of λ by adding an extra (i, j) such that $j - i \equiv a$ modulo e. Let $\mathcal{F} = \bigoplus_\lambda \mathbb{C}\lambda$ be a complex vector space with basis indexed by the set of all partitions of all nonnegative integers. We define linear operators $\mathfrak{e}_0, \dots, \mathfrak{e}_{e-1}, \mathfrak{f}_0, \dots, \mathfrak{f}_{e-1}$ on $\mathcal{F}$ by

$$\mathfrak{e}_a\lambda = \sum_{\mu \to_a \lambda} \mu \qquad \text{and} \qquad \mathfrak{f}_a\lambda = \sum_{\lambda \to_a \mu} \mu.$$

These locally nilpotent endomorphisms extend to an action of the Kac–Moody algebra $\widehat{\mathfrak{sl}}_e$. Put $s_a := \exp(-\mathfrak{f}_a)\exp(\mathfrak{e}_a)\exp(-\mathfrak{f}_a)$, an automorphism of $\mathcal{F}$. Then for any partition λ, we have $s_a\lambda = \pm\sigma_a(\lambda)$ for some partition $\sigma_a(\lambda)$. The permutations $\sigma_0, \dots, \sigma_{e-1}$ define an action of the affine Weyl group on the set of partitions.

Following James, consider an abacus with e half-infinite vertical runners, labelled $\rho_0, \dots, \rho_{e-1}$ from left to right. On runner i, we may put beads in positions labelled $i, i+e, i+2e, \dots$ from top to bottom. Let $d \geq 0$. Any partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_d)$ with at most d nonzero parts may be represented by placing d beads at positions $\lambda_1 + d - 1, \lambda_2 + d - 2, \dots, \lambda_d$. Note that λ may be recovered easily from the resulting configuration.

Sliding a bead one place up a runner into an unoccupied position corresponds to removing a rim e-hook from the Young diagram of λ. Repeating this process until no longer possible, say w times, we obtain a partition called the e-core of λ, and we say that λ has weight w. In going from the partition to its core, we remember how many times each bead on runner i is moved as a partition $\lambda^{(i)}$. The resulting e-tuple $\overline{\lambda} = \{\lambda^{(0)}, \dots, \lambda^{(e-1)}\}$ is called the e-quotient of λ.

The actions of $\mathfrak{e}_a$, $\mathfrak{f}_a$ and σ_a described above have an easy interpretation on an abacus with d beads. Let $i \in \{0, \dots, e-1\}$ such that $i \equiv a + d$ modulo e. Then $\lambda \to_a \mu$ if and only if (the abacus representation) of μ can be obtained from that of λ by moving a bead in ρ_{i-1} one position to the right into an unoccupied position in ρ_i. And σ_a acts by interchanging the configuration of beads on runners ρ_{i-1} and ρ_i.

2.2 Representations of Schur Algebras

Let $\Bbbk$ be a field and let $q \in \Bbbk^\times$. Let $e = e(q)$ be the least integer $i \geq 2$ such that $1 + q + \dots + q^{i-1} = 0$ in $\Bbbk$. Hence, e is the characteristic of $\Bbbk$ if $q = 1$, and is the multiplicative order of q otherwise. According to Kleshchev, e is sometimes called a 'quantum characteristic' for Hecke algebras.

The Schur algebra $\mathcal{S}_{\Bbbk,q}(d, r)$ is a quasihereditary algebra with simple modules $L(\lambda)$ indexed by the partitions of r with at most d parts, with respect to the dominance order. The Weyl module $\Delta(\lambda)$ has simple head isomorphic to $L(\lambda)$, and any composition factor of its radical is isomorphic to $L(\mu)$ for some $\mu \lhd \lambda$; it is characterized up to isomorphism the largest module by these properties.

If $d \geq \tilde{d}$, there exists a *Green's idempotent* f in $\mathcal{S}_{\Bbbk,q}(d,r)$ and a canonical isomorphism $f\mathcal{S}_{\Bbbk,q}(d,r)f \cong \mathcal{S}_{\Bbbk,q}(\tilde{d},r)$ (see [Gre80]). The exact functor $\mathcal{S}_{\Bbbk,q}(d,r)$-mod $\to \mathcal{S}_{\Bbbk,q}(\tilde{d},r)$-mod : $M \mapsto fM$ sends $L(\lambda)$ to the corresponding simple module $\tilde{L}(\lambda)$ if λ has at most $\tilde{d}$ parts, and to 0 otherwise. Similarly, $f\Delta(\lambda) \cong \tilde{\Delta}(\lambda)$ if λ has at most $\tilde{d}$ parts. In particular, if $\tilde{d} \geq r$, then the functor is an equivalence preserving labels.

Two simple modules $L(\lambda)$ and $L(\mu)$ are in the same block of $\mathcal{S}_{\Bbbk,q}(r) := \mathcal{S}_{\Bbbk,q}(r,r)$ if and only if λ and μ have the same e-core. Note that they are then necessarily of the same e-weight. So, the blocks of $\mathcal{S}_{\Bbbk,q}(r)$, $r \geq 0$ are classified by pairs (τ, w) where τ is an e-core partition, i.e. a partition which is its own e-core, and $w \geq 0$.

We may identify the Fock space $\mathcal{F}$ with the sum of complexified Grothendieck groups of module categories of q-Schur algebras:

$$\begin{aligned} \mathcal{F} &= \bigoplus_{r\geq 0} \mathbb{C} \otimes_{\mathbb{Z}} K(\mathcal{S}_{\Bbbk,q}(r)\text{-mod}) \\ \lambda &\leftrightarrow \qquad [\Delta(\lambda)] \end{aligned}$$

Let B be a block and let $a \in \{0, \ldots, e-1\}$. Then s_a restricts to an isomorphism $K(B\text{-mod}) \xrightarrow{\sim} K(\dot{B}\text{-mod})$ for some block $\dot{B}$ of the same e-weight; we write $s_a B = \dot{B}$ and say that B and $\dot{B}$ form a *Scopes pair*. The induced action of the affine Weyl group on the set of all blocks of $\mathcal{S}_{\Bbbk,q}(r)$, $r \geq 0$ is transitive on blocks of a fixed weight.

We remark that by 'Scopes pair', we mean an arbitrary $[w : k]$ pair of blocks (in the original terminology of Scopes [Sco91]); we do not place the restriction $w \leq k$.

Definition 1. A block B with e-core τ and weight w of $\mathcal{S}_{\Bbbk,q}(r)$ is called a *Rouquier block* if there is some d such that in the d-bead abacus representation of τ, in any pair of adjacent runners there are at least $w-1$ more beads on the righthand runner. The e-core τ is called a *Rouquier core* relative to w.

Clearly for all $w \geq 0$, there exist Rouquier blocks of weight w. So it is convenient to first prove that a statement about blocks is true for the Rouquier blocks, and then use the affine Weyl group action to show that it holds for all blocks.

3 The Main Result

3.1 James–Mathas Construction

To state the main theorem we need to describe a map on partitions, due to James and Mathas. Let $2 \leq e \leq e'$, $d \geq 0$, and $\alpha \in \{0, \ldots, e\}$. Given a partition λ with at most d nonzero parts, add $e' - e$ empty runners between $\rho_{\alpha-1}$ and ρ_α in the abacus representation of λ on an e-runner abacus with d beads. The new configuration on an e'-runner abacus represents a partition which we call λ^+.

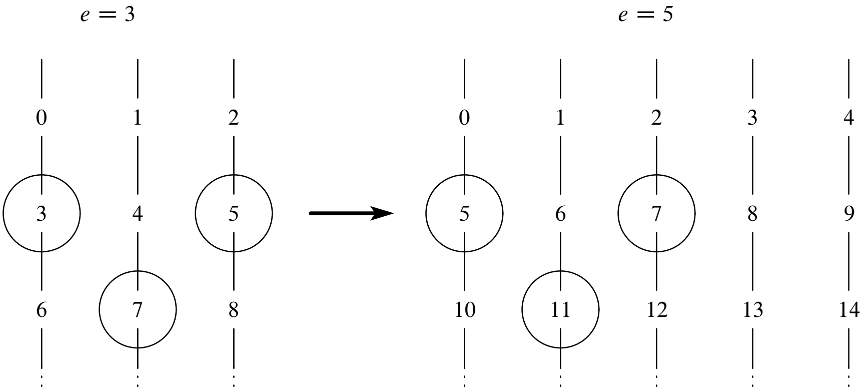

Fig. 1 Adding two runners

To take an example, let $e = 3$, $e' = 5$, $d = 3$ and $\alpha = 3$. Then the operation $\lambda \mapsto \lambda^+$ adds two empty runners on the right of the abacus. So, e.g. if $\lambda = (5, 4, 3)$, then $\lambda^+ = (9, 6, 5)$; see Fig. 1 above.

James and Mathas showed that the map $\lambda \mapsto \lambda^+$ links the representation theory of Schur algebras at complex primitive eth and e'th roots of unity in a precise way, obtaining equalities of decomposition numbers. Our theorem below interprets their result in terms of Morita equivalences of blocks.

3.2 Statement of the Theorem

Fix $q, q' \in \Bbbk^\times$, let $e = e(q)$ and $e' = e(q')$, and assume that $e \leq e'$. Let B be a block ideal of $\mathcal{S} := \mathcal{S}_{\Bbbk,q}(r)$. Let Λ be the set of partitions λ of r such that $B \cdot L(\lambda) \neq 0$.

In what follows we shall be considering a number of blocks which are defined in terms of B; we indicate this relationship notationally using decorations on 'B'. To avoid confusion, we will usually use the same decorations for modules and for the poset of labels of simples.

Fix $d \leq r$, and $\alpha \in \{0, \ldots, e\}$. Let $\underline{\Lambda}$ be the subset of Λ consisting of partitions with at most d parts. We sometimes denote by $l(\lambda)$ the number of nonzero parts of λ. Assume that $\underline{\Lambda}$ is nonempty. Then there exists r' and a block B' of $\mathcal{S}' := \mathcal{S}_{\Bbbk,q'}(r', r')$ such that for all $\lambda \in \underline{\Lambda}$, we have $|\lambda^+| = r'$ and $\lambda^+ \in \Lambda' := \{\mu \mid B' \cdot L'(\mu)\} \neq 0$.

Defining $\underline{\Lambda}'$ to be the set of partitions in Λ' with at most d parts, we have a bijection

$$\underline{\Lambda} \xrightarrow{\sim} \underline{\Lambda}' : \lambda \mapsto \lambda^+$$

preserving the dominance relation.

Now there exists an idempotent $f \in \mathcal{S}$ and an isomorphism $f\mathcal{S}f \cong \underline{\mathcal{S}}$, where $\underline{\mathcal{S}} := \mathcal{S}_{\Bbbk,q}(d, r)$. We define $\underline{B} := fBf$, a sum of blocks of $\underline{\mathcal{S}}$. Then we have a quotient functor B-mod $\to$ $\underline{B}$-mod : $M \mapsto fM$. The simple modules $\underline{L}(\lambda)$ of $\underline{B}$ are indexed by $\underline{\Lambda}$. The Weyl module corresponding to $\lambda \in \underline{B}$ is denoted by $\underline{\Delta}(\lambda)$.

Let $\underline{\mathcal{S}}' := \mathcal{S}_{\Bbbk,q'}(d, r')$, and then define $\underline{B}'$ analogously to $\underline{B}$. We use $\underline{L}'$ and $\underline{\Delta}'$ to denote the simple modules and Weyl modules of $\underline{B}'$.

Theorem 2. *Suppose that one of the following holds:*

- $\Bbbk$ *has characteristic* 0.
- $\Bbbk$ *has characteristic* $\ell > e$ *and* $q, q' \in \mathbb{F}_\ell$ *and the weight of* B *is strictly less than* ℓ.

Then there exists an equivalence

$$\mathsf{F} : \underline{B}\text{-mod} \xrightarrow{\sim} \underline{B}'\text{-mod}$$

such that

$$\mathsf{F}\underline{L}(\lambda) \cong \underline{L}'(\lambda^+)$$

and

$$\mathsf{F}\underline{\Delta}(\lambda) \cong \underline{\Delta}'(\lambda^+)$$

for all $\lambda \in \underline{\Lambda}$.

Remark 3. (1) Since the map $\lambda \mapsto \lambda^+$ preserves the dominance order, the statement concerning Δ-sections is an immediate consequence of the statement about simples.

(2) For a given block B, one can always choose $d = r$, so that $\underline{\Lambda} = \Lambda$, $\underline{\mathcal{S}} = \mathcal{S}$ and $\underline{B} = B$. On the other hand, if $e' > e$, then $\underline{\Lambda}'$ is always strictly smaller than Λ', so that $\underline{B}'$-mod is a proper quotient of B'-mod.

(3) In case $1 \neq q \in \mathbb{F}_\ell$ and the weight of B is strictly less than ℓ, we may always take $q' = 1$, thus obtaining Morita equivalences between blocks of q-Schur algebras and blocks of ordinary Schur algebras. In particular, we have reduced the verification of James's conjecture on decomposition numbers ([Jam90, Sect. 4], [Mat99, p. 117–118, 6.37 and 2nd paragraph of p. 118], [Lus80]) to the case $q = 1$, i.e. to the case of ordinary Schur algebras.

(4) If $q = 1$ and $\Bbbk$ has positive characteristic ℓ, then B is a block of an ordinary Schur algebra, and its weight is strictly less than ℓ if and only if the corresponding block of a symmetric group has abelian defect groups. So the restriction on the weight of B in Theorem 2 should be regarded as an abelian defect condition [Bro90, 6.2. Question], [Bro92, 4.9. Conjecture], [BMM93], which is a blockwise refinement of the assumption of [Mat99, 6.37] and is milder than any known assumptions in any literatures on James's conjecture.

(5) We expect that the hypotheses on q, q' and the weight of B in positive characteristic are not necessary. They just reflect our current state of knowledge on the structure of Rouquier blocks of q-Schur algebras, which constitute the base case of our inductive proof; the inductive step is valid in general. For example, a positive resolution of Turner's remarkable conjectures on Rouquier blocks with nonabelian defect groups [Tur05] would remove the restriction on the weight of B.

If $e = e'$, the map $\lambda \mapsto \lambda^+$ is the identity. We have rigged the statement of the theorem and its proof to include this special case, and obtain the following corollary.

Corollary 4. • *Suppose that* $\Bbbk$ *has characteristic* 0 *and that* q *and* q' *are primitive eth roots of unity in* $\Bbbk$*. Then for all* r *and* d*, we have an isomorphism* $\mathcal{S}_{\Bbbk,q}(d,r) \xrightarrow{\sim} \mathcal{S}_{\Bbbk,q'}(d,r)$ **of** $\Bbbk$**-algebras**.

- *Suppose* $\Bbbk$ *has characteristic* $\ell > e$ *and that* $q, q' \in \mathbb{F}_\ell$*. Then for all* r *and* d*, corresponding blocks of* $\mathcal{S}_{\Bbbk,q}(d,r)$ *and* $\mathcal{S}_{\Bbbk,q'}(d,r)$ *of weight strictly less than* ℓ *are isomorphic* **as** $\Bbbk$**-algebras**.

Proof. Take the notation and hypotheses of Theorem 2. For all $\lambda \in \underline{\Lambda}$, the formal characters of $\Delta(\lambda)$ and $\Delta'(\lambda)$ are equal; hence, $\dim \underline{\Delta}(\lambda) = \dim \underline{\Delta}'(\lambda)$. Since the decomposition numbers of $\underline{B}$ and $\underline{B}'$ are 'the same', we deduce that $\dim \underline{L}(\lambda) = \dim \underline{L}'(\lambda)$. But we already know that $\underline{B}$ and $\underline{B}'$ are Morita equivalent, and so conclude that $\underline{B}$ and $\underline{B}'$ are isomorphic as $\Bbbk$-algebras.

If the characteristic of $\Bbbk$ is 0, we may sum over all blocks to obtain isomorphisms of Schur algebras. □

3.3 Decomposition Numbers

We now formulate the numerical consequences of Theorem 2. In characteristic 0, we recover weak versions of the theorem of James and Mathas [JM02] and the related result of Fayers [Fay07]. They prove equalities of the v-decomposition numbers defined by Lascoux, Leclerc and Thibon (see [Lec02]), which specialize at $v = 1$, via the theorem of Varagnolo–Vasserot [VV99] (and Ariki [Ari96]), to our formulas. Our approach has the advantage of also being valid in positive characteristic, as long as the parameter q is in the ground field and we are in an 'abelian defect' situation.

In comparing out statement with those of [JM02] and [Fay07], it is important to keep in mind that our labelling of modules is conjugate to theirs. In particular, our result is more directly related to that of Fayers.

Theorem 5. *Keep the notation and assumptions of Theorem 2. We have, for all* $\lambda, \mu \in \underline{\Lambda}$*,*

$$[\Delta(\lambda) : L(\mu)] = [\Delta'(\lambda^+) : L'(\mu^+)]$$

and

$$[\Delta(\lambda^{\mathrm{tr}}) : L(\mu^{\mathrm{tr}})] = [\Delta'((\lambda^{\mathrm{tr}})^+) : L'((\mu^{\mathrm{tr}})^+)].$$

Proof. The first equality is equivalent to

$$[\underline{\Delta}(\lambda) : \underline{L}(\mu)] = [\underline{\Delta}'(\lambda^+) : \underline{L}'(\mu^+)],$$

which is an immediate consequence of Theorem 2.

For $\lambda \in \Lambda$, let $T(\lambda)$ be the corresponding indecomposable tilting module of B. If $\lambda \in \underline{\Lambda}$, the image $fT(\lambda)$ under the Schur functor is an indecomposable tilting module of $\underline{B}$. By the conclusion of Theorem 2, we have equalities of filtration multiplicities, for any $\lambda \in \underline{\Lambda}$,

$$\begin{aligned}(T(\mu) : \Delta(\lambda))_{\mathcal{S}(r,r)} &= (fT(\mu) : f\Delta(\lambda))_{\mathcal{S}(d,r)} \\ &= (f'T(\mu^+) : f'\Delta'(\lambda^+))_{\mathcal{S}'(d,r')} \\ &= (T(\mu^+) : \Delta'(\lambda^+))_{\mathcal{S}'(r',r')}.\end{aligned}$$

On the other hand, Donkin's formula [Don98, Proposition 4.1.5] tells us that

$$[\Delta(\lambda^{\mathrm{tr}}) : L(\mu^{\mathrm{tr}})] = (T(\mu) : \nabla(\lambda)),$$

where $\nabla(\lambda)$ is the dual Weyl module associated with λ. Here, we may safely replace $\nabla(\lambda)$ by $\Delta(\lambda)$, since contragradient duality for the q-Schur algebra sends $T(\mu)$ to itself and $\nabla(\lambda)$ to $\Delta(\lambda)$. Thus, the second equality in the statement of the theorem is proved. □

3.4 Truncation Functors

In this subsection, we give an application of truncation functors which will be used as a reduction step in the proof of the main theorem. The main source is [Don98, 4.2].

In this subsection, we denote by $G = G(n)$ the quantum general linear group with simple roots $\Pi = \{\alpha_1, \ldots, \alpha_{n-1}\}$. For any subset Σ of Π, we have an associated standard Levi subgroup G_Σ. We denote by X_Σ the set of dominant weights for G_Σ. To specify the group for standard, costandard, simple, and tilting modules, we again attach subscripts in the notation Δ, ∇, L, T, e.g. $\Delta_\Sigma(\lambda), \nabla_\Sigma(\lambda), L_\Sigma(\lambda), T_\Sigma(\lambda)$, and simply write $\Delta_m(\lambda), \nabla_m(\lambda), L_m(\lambda), T_m(\lambda)$ if for modules over $G(m)$. In case that they are modules over the full parent group G, we sometimes will not attach any subscripts, simply writing $\Delta(\lambda), \nabla(\lambda), L(\lambda), T(\lambda)$, etc.

For a standard Levi subgroup G_Σ of G and a dominant weight λ of G_Σ, we denote by $\mathrm{Tr}_\Sigma^\lambda$ the Harish-Chandra (λ, Σ)-truncation functor (see [Don98, p. 86], or [Jan03, p. 181, 2.11] for its slightly different description); given a G-module V, we define $\mathrm{Tr}_\Sigma^\lambda V = \bigoplus_{\mu \in X, \lambda-\mu \in \mathbb{Z}\Sigma} V_\mu$, a G_Σ-module. The truncation functor satisfies the following properties.

(1) $\mathrm{Tr}_\Sigma^\lambda$ is exact,

(2) $\mathrm{Tr}_\Sigma^\lambda \nabla(\mu) = \begin{cases} \nabla_\Sigma(\mu), & \text{if } \lambda - \mu \in \mathbb{Z}\Sigma, \\ 0, & \text{otherwise} \end{cases}$,

(3) $\mathrm{Tr}_\Sigma^\lambda L(\mu) = \begin{cases} L_\Sigma(\mu), & \text{if } \lambda - \mu \in \mathbb{Z}\Sigma, \\ 0, & \text{otherwise} \end{cases}$.

The following can be found in [Don98, p. 89]: Fix $\lambda \in X_\Pi$. If a module U is filtered by Δ's, a module V is filtered by ∇'s and every weight of U and V is less than or equal to λ, then the map

$$\mathrm{Hom}_G(U, V) \to \mathrm{Hom}_{G_\Sigma}(\mathrm{Tr}^\lambda_\Sigma U, \mathrm{Tr}^\lambda_\Sigma V) \text{ is surjective.} \tag{1}$$

For $\lambda \in X_\Pi$ and $\mu \in X_\Sigma$, we have

$$(T_\Sigma(\lambda) : \nabla_\Sigma(\mu)) = \begin{cases} (T(\lambda) : \nabla(\mu)), & \text{if } \mu \in X_\Pi \text{ and } \lambda - \mu \in \mathbb{Z}\Sigma; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

We have the following application of the truncation functor.

Proposition 6. *Suppose that $\mathcal{T}$ is a cosaturated subset of the set of partitions of r sharing a common e-core, where e is the quantum characteristic of G. Further suppose that there exists a weight σ for $G(k)$ such that $T(\sigma) = L(\sigma)$ and such that for any $\lambda \in \mathcal{T}$ there exists a partition $(\lambda^{\mathrm{tr}})^b$ such that $\lambda^{\mathrm{tr}} = \sigma \cup (\lambda^{\mathrm{tr}})^b$. Here, the notation $\cup$ is taken from page 6 of Macdonald's text book.*[1]

Put $\mathrm{tr}\mathcal{T} := \{\lambda^{\mathrm{tr}} \mid \lambda \in \mathcal{T}\}$ *and fix an integer m such that* $m \geq \max\{l(\lambda^b) \mid \lambda \in \mathrm{tr}\mathcal{T}\}$.

Then,

$$\mathrm{End}_{G(k+m)}\left(\bigoplus_{\lambda \in \mathrm{tr}\mathcal{T}} T_{k+m}(\lambda)\right) \cong \mathrm{End}_{G(m)}\left(\bigoplus_{\lambda \in \mathrm{tr}\mathcal{T}} T_m(\lambda^b)\right). \tag{3}$$

Here, via the isomorphism (3), the idempotent of $\mathrm{End}_{G(k+m)}(T_{k+m}(\lambda))$ *corresponds to the idempotent of* $\mathrm{End}_{G(k+m)}(T_m(\lambda^b))$.

Proof. Since $T(\sigma) = L(\sigma)$, we know that

$$\mathrm{End}_{G(m)}\left(\bigoplus_{\lambda \in \mathrm{tr}\mathcal{T}} T_m(\lambda^b)\right) \cong \mathrm{End}_{G(k)\times G(m)}\left(\bigoplus_{\lambda \in \mathrm{tr}\mathcal{T}} T_k(\sigma) \boxtimes T_m(\lambda^b)\right) \tag{4}$$

Then, by (1) and by taking $\mathrm{Tr}^\tau_{\Sigma(k,m)}$ into account where τ is the maximum of λ^{tr} for all $\lambda \in \mathcal{T}$ and $\Sigma(k,m) = \{\alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_{k+m-1}\}$, we know that there is a surjection of $\mathrm{End}_{G(k+m)}(\bigoplus_{\lambda\in\mathcal{T}} T_{k+m}(\lambda))$ onto the RHS of (4). So, it suffices to count the dimensions of the endomorphism rings in question. The dimension of LHS of (3) is equal to $\sum_{\lambda,\nu\in\mathrm{tr}\mathcal{T}} \sum_{\mu\lhd\lambda} (T_{k+m}(\lambda) : \nabla_{k+m}(\mu))(T_{k+m}(\nu) : \nabla_{k+m}(\mu))$. And the dimension of LHS of (4) is equal to $\sum_{\lambda,\nu\in\mathrm{tr}\mathcal{T}} \sum_{\mu^b\lhd\lambda^b} (T_m(\lambda^b) : \nabla_m(\mu^b))(T_m(\nu^b) : \nabla_m(\mu^b))$. Then, the cosaturation

[1] In our set up, the first entry $(\lambda^{\mathrm{tr}})^b_1$ is at most $\sigma_{l(\sigma)}$. So, it is just a concatenation of two partition. For example, $(9, 8, 7) \cup (3, 2, 1) = (9, 8, 7, 3, 2, 1)$.

condition on $\mathcal{T}$ and the assumption on the unique decomposition $\lambda^{\mathrm{tr}} = \sigma \cup (\lambda^{\mathrm{tr}})^b$, and (2) ensure that each non-zero term in the second summations in the dimension formulas for (3) and (4) match up each other. $\square$

3.5 Proof of the Main Theorem

In this subsection, we reduce the main theorem to the case of large d (relative to the block B) using the results of the following sections, on Scope pairs and Rouquier blocks.

Because the orbit of the block B under the affine Weyl group action contains a Rouquier block, there exists a sequence of blocks $B_0, \ldots, B_s = B$, such that B_0 is a Rouquier block, and any two successive blocks form a Scopes pair.

Now choose c such that all partitions in all blocks B_i have at most $\tilde{d} = d + ec$ parts. Now define blocks $\widetilde{B}'$ and $\underline{\widetilde{B}'}$ analogously to B' and $\underline{B}'$, replacing d by $\tilde{d}$. Then by induction, with base case Sect. 5, which proves the main result for Rouquier blocks, and inductive step Sect. 4.2, we deduce that the main theorem holds for B, as long as we replace d by $\tilde{d}$. In other words, we have an equivalence

$$\tilde{\mathsf{F}} : B\text{-mod} \xrightarrow{\sim} \underline{\widetilde{B}'}\text{-mod}$$

such that

$$\tilde{\mathsf{F}} L(\lambda) \cong \underline{L}'(\widetilde{\lambda^+})$$

for all $\lambda \in \Lambda$. Here $\widetilde{\lambda^+}$ is defined similarly to λ^+, except we use abaci with $\tilde{d}$ beads rather than d beads. Note that $\widetilde{\lambda^+}$ has $ce - \alpha$ more parts than λ^+. (Recall that the extra empty runner in λ^+ is inserted between $\rho_{\alpha-1}$ and ρ_α.)

Let $\sigma = ((d + ce - \alpha)^{e'-e}, \ldots, (d + 2e - \alpha)^{e'-e}, (d + e - \alpha)^{e'-e})$. Then for all $\lambda \in \underline{\Lambda}$, we have

$$(\widetilde{\lambda^+})^{\mathrm{tr}} = \sigma \cup (\lambda^+)^{\mathrm{tr}}. \tag{5}$$

Furthermore, σ is minimal amongst partitions in its block having at most $c(e' - e)$-parts. Hence, we have $T_{c(e'-e)}(\sigma) = L_{c(e'-e)}(\sigma)$.

Let $m := \max\{l((\lambda^+)^{\mathrm{tr}}) \mid \lambda \in \underline{\Lambda}\}$.

For all $\lambda, \mu \in \underline{\Lambda}$, we have

$$\begin{aligned}
\mathrm{Hom}_B(P(\lambda), P(\mu)) &\cong \mathrm{Hom}_{B''}(P(\widetilde{\lambda^+}), P(\widetilde{\mu^+})) \\
&\cong \mathrm{Hom}_{G(c(e'-e)+m)}(T((\widetilde{\lambda^+})^{\mathrm{tr}}), T((\widetilde{\mu^+})^{\mathrm{tr}})) \\
&\cong \mathrm{Hom}_{G(c(e'-e)+m)}(T(\sigma \cup (\lambda^+)^{\mathrm{tr}}), T(\sigma \cup (\mu^+)^{\mathrm{tr}})) \\
&\cong \mathrm{Hom}_{G(m)}(T((\lambda^+)^{\mathrm{tr}}), T((\mu^+)^{\mathrm{tr}})) \\
&\cong \mathrm{Hom}_{B'}(P(\lambda^+), P(\mu^+)),
\end{aligned}$$

where the first isomorphism is deduced from the equivalence $\tilde{\mathsf{F}}$, the second and last isomorphisms by Ringel selfduality [Don98], the third by Proposition 6 and the fourth by (5). Here, the quantum characteristic of the G''s is e'. Now the main theorem follows immediately.

4 Scopes Pairs

In this section, we carry out the inductive step of the proof of the main theorem. Here, we make an additional assumption that d is large, in the sense that all the partitions in the blocks we are considering have at most d parts.

After reviewing the notion of a perverse equivalence [Rou06, Sect. 2.6], we spell out the inductive step. A key point here is that the combinatorics of the perverse equivalences are independent of the 'quantum characteristic' $e(q)$; a proof is relegated to the end of the section.

4.1 Perverse Equivalences

We work in the context of finite-dimensional algebras, sufficient for our application. Roughly speaking, a perverse equivalence is a derived equivalence filtered by shifted Morita equivalences; see [Rou06, Sect. 2.6], [CRa] for more details. Let A and $\dot{A}$ be two finite-dimensional algebras and $\mathcal{S}$ (resp. $\dot{\mathcal{S}}$) the set of isomorphism classes of finite-dimensional simple A-modules (resp. simple $\dot{A}$-modules).

Definition 7. An equivalence $\mathsf{G} : D^b(A\text{-mod}) \xrightarrow{\sim} D^b(\dot{A}\text{-mod})$ is *perverse* if there is

- A filtration $\emptyset = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_r = \mathcal{S}$
- A filtration $\emptyset = \dot{\mathcal{S}}_0 \subset \dot{\mathcal{S}}_1 \subset \cdots \subset \dot{\mathcal{S}}_r = \dot{\mathcal{S}}$
- And a function $p : \{1, \ldots, r\} \to \mathbb{Z}$

such that

- G restricts to equivalences $D^b_{\mathcal{A}_i}(A\text{-mod}) \xrightarrow{\sim} D^b_{\dot{\mathcal{A}}_i}(\dot{A}\text{-mod})$
- $\mathsf{G}[-p(i)]$ induces equivalences $\mathcal{A}_i/\mathcal{A}_{i-1} \xrightarrow{\sim} \dot{\mathcal{A}}_i/\dot{\mathcal{A}}_{i-1}$.

Here $\mathcal{A}_i$ (resp. $\dot{\mathcal{A}}_i$) is the Serre subcategory of A-mod (resp. $\dot{A}$-mod) generated by $\mathcal{S}_i$ (resp. $\dot{\mathcal{S}}_i$), and by $D^b_{\mathcal{A}_i}(A\text{-mod})$ we mean the full subcategory of $D^b(A\text{-mod})$ whose objects are complexes with homology modules belonging to $\mathcal{A}_i$.

The following basic result implies that the filtration $\mathcal{S}_\bullet$ and the perversity p determine $\dot{A}$ (and $\dot{\mathcal{S}}_\bullet$), up to Morita equivalence.

Proposition 8. *Let* $\mathsf{G} : D^b(A\text{-mod}) \xrightarrow{\sim} D^b(\dot{A}\text{-mod})$ *and* $\mathsf{G}' : D^b(A\text{-mod}) \xrightarrow{\sim} D^b(\dot{A}'\text{-mod})$ *be perverse. If* $\mathcal{S}'_\bullet = \mathcal{S}_\bullet$ *and* $p' = p$*, then the composition* $\mathsf{G}'\mathsf{G}^{-1}$ *restricts to an equivalence* $\dot{A}\text{-mod} \xrightarrow{\sim} \dot{A}'\text{-mod}$.

Indeed, the composition $G'G^{-1}$ is also a perverse equivalence with respect to p'', where p'' is identically zero. Then an easy inductive argument shows that $G'G^{-1}$ restricts to an equivalence $\dot{\mathcal{A}}_i \xrightarrow{\sim} \dot{\mathcal{A}}'_i$.

Now let $\mathcal{A} = \bigoplus_{r\geq 0} \mathcal{S}_{\Bbbk,q}(r)$-mod. Let $a \in \{0,\ldots,e-1\}$. Given M, a $\mathcal{S}_{\Bbbk,q}(r-1)$-module lying in a block B, let $\tilde{M}$ be the corresponding $\mathcal{S}_{\Bbbk,q}(r,r-1)$-module, under the canonical equivalence $\mathcal{S}_{\Bbbk,q}(r,r-1)\text{-mod} \xrightarrow{\sim} \mathcal{S}_{\Bbbk,q}(r-1)\text{-mod}$. Tensoring with the natural representation $V = \Bbbk^r$ of $\mathcal{S}_{\Bbbk,q}(r,1) = \mathrm{End}(V)$, we obtain a $\mathcal{S}_{\Bbbk,q}(r)$-module $V \otimes \tilde{M}$. We define $F_a M$ to be the projection of $V \otimes \tilde{M}$ onto the sum of blocks C of $\mathcal{S}_{\Bbbk,q}(r)$ such that $K(C\text{-mod}) \cap \mathfrak{f}_a(K(B\text{-mod})) \neq 0$.

This recipe defines an exact endofunctor F_a on $\bigoplus_{r\geq 0} \mathcal{S}_{\Bbbk,q}(r)$-mod lifting the action of $\mathfrak{f}_a$ on $\mathcal{F}$. Let E_a be a left adjoint to F_a. Then E_a is also a right adjoint to F_a, and E_a lifts the action of $\mathfrak{e}_a$ on $\mathcal{F}$.

The adjoint pair (E_a, F_a) may be used to define a $\mathfrak{sl}_2$-categorification on $\mathcal{A}$ [CR08]. We obtain as a result derived equivalences between blocks in Scopes pairs, which are known to be perverse, by [CRa]. Here are the details: Let B be a block of $\mathcal{S}_{\Bbbk,q}(r)$, and put $\dot{B} = s_a B$, a block of $\mathcal{S}_{\Bbbk,q}(\dot{r})$, for some $\dot{r}$. Let Λ and $\dot{\Lambda}$ be the sets of partitions labelling simple modules in B and $\dot{B}$.

Proposition 9. *There exists an equivalence* $\mathsf{G} : D^b(B\text{-mod}) \xrightarrow{\sim} D^b(\dot{B}\text{-mod})$ *such that*

- G *is perverse with respect to the filtrations*

$$\Lambda_j = \{\lambda \in \Lambda \mid F_a^j L(\lambda) = 0\} \qquad \textit{and} \qquad \dot{\Lambda}_j = \{\lambda \in \dot{\Lambda} \mid E_a^j L(\lambda) = 0\}$$

  and the perversity $p(j) = j - 1$.
- G *induces the isomorphism* $s_a : K(B\text{-mod}) \xrightarrow{\sim} K(\dot{B}\text{-mod})$.
- G (*and* G^{-1}) *are induced by complexes of functors, each of which is a direct summand of a composition of* E_a*'s and* F_a*'s.*

Remark 10. The filtrations above may also be defined by the formulas

$$\Lambda_j = \{\lambda \in \Lambda \mid \widetilde{F}_a^j L(\lambda) = 0\} \qquad \text{and} \qquad \dot{\Lambda}_j = \{\lambda \in \dot{\Lambda} \mid \widetilde{E}_a^j L(\lambda) = 0\},$$

where $\widetilde{F}_a$ and $\widetilde{E}_a$ are the *Kleshchev/Kashiwara operators*:

$$\widetilde{F}_a(M) := \mathrm{Soc}(F_a(M)), \qquad \widetilde{E}_a(M) := \mathrm{Soc}(E_a(M))$$

[Kle95a, Kle95b, Kle96].

In very special cases, we can deduce Morita equivalences of a type first discovered by Scopes for symmetric groups. The following corollary is a slight extension of [EMS94] and [Don94, Sect. 5].

In [EMS94] and [Don94, Sect. 5], they dealt with 1-Schur algebras not for the q-Schur algebras where $q \neq 1$. But, the basic ideas given below are (at least at the level of combinatorics) identical with theirs.

Corollary 11 (Scopes's Morita equivalences). *Fix* $d \geq 0$. *Suppose that for all* $\lambda \in \dot{\Lambda}$, *the* d*-bead abacus representation of* λ *contains no beads in* ρ_i, *where*

$i \equiv a + d$ modulo e. Then there is an equivalence $\underline{B}$-mod $\xrightarrow{\sim}$ $\underline{\dot{B}}$-mod between the blocks $\underline{B}$ and $\underline{\dot{B}}$ of $\mathcal{S}_{\Bbbk,q}(d,r)$ and $\mathcal{S}_{\Bbbk,q}(d,\dot{r})$ corresponding to B and $\dot{B}$, sending $\underline{L}(\lambda)$ to $\underline{L}(\sigma_a\lambda)$ (and $\underline{\Delta}(\lambda)$ to $\underline{\Delta}(\sigma_a\lambda)$).

Proof. Let $\mathsf{G} : D^b(B\text{-mod}) \xrightarrow{\sim} D^b(\dot{B}\text{-mod})$ be the perverse equivalence provided by Proposition 9. Then G restricts to an equivalence $\mathsf{G}_1 : \mathcal{A}_1 \xrightarrow{\sim} \dot{\mathcal{A}}_1$ between the Serre subcategories

$$\mathcal{A}_1 := \{M \in B\text{-mod} \mid F_a M = 0\} \quad \text{and} \quad \dot{\mathcal{A}}_1 := \{M \in \dot{B}\text{-mod} \mid E_a M = 0\}.$$

Let $\lambda \in \underline{\Lambda}$. Then by the assumption that there are no beads in ρ_i we have $\mathfrak{f}_a\lambda = 0$. Thus $\Delta(\lambda) \in \mathcal{A}_1$, and $[\mathsf{G}_1\Delta(\lambda)] = [\Delta(\sigma_a\lambda)]$, since $s_a\lambda = \pm\sigma_a\lambda$ and the sign is determined to be $+$ as G_1 is an equivalence. It follows by induction that for all $\lambda \in \underline{\Lambda}$, we have $\mathsf{G}_1 P(\lambda) \cong P(\sigma_a\lambda)$. This implies the desired statements, since $\underline{B}$ and $\underline{\dot{B}}$ are Morita equivalent to $\operatorname{End}\left(\bigoplus_{\lambda\in\underline{\Lambda}} P(\lambda)\right)$ and $\operatorname{End}\left(\bigoplus_{\lambda\in\underline{\dot{\Lambda}}} P(\lambda)\right)$, respectively. □

4.2 Inductive Step

Consider a Scopes pair of blocks B and $\dot{B} = \sigma_a B$. We assume that Theorem 2 is true for B and aim to verify it for $\dot{B}$.

We *shall also assume* that all partitions in Λ and $\dot{\Lambda}$ have at most d parts. Hence, $\Lambda = \underline{\Lambda}$, $\dot{\Lambda} = \underline{\dot{\Lambda}}$, $B = \underline{B}$ and $\dot{B} = \underline{\dot{B}}$. Let $i \in \{0, \dots, e-1\}$ so that $i \equiv a + d$ mod e. We may assume that $\alpha \neq i \mod e$ and therefore that $\dot{B}' = \sigma_{a'} B'$, where $a' \in \{0, \dots, e'-1\}$ such that $i \equiv a' + d \mod e'$. If $\alpha = i < e$, we reduce to the case $\alpha = i + 1$ by the following observation: $B'_\alpha = \sigma_{\alpha+e'-e} \dots \sigma_{\alpha+1} B'_{\alpha+1}$, and there exists an equivalence $\underline{B}'_{\alpha+1}$-mod $\xrightarrow{\sim}$ $\underline{B}'_\alpha$-mod sending $\underline{L}'(\lambda^{+,\alpha+1})$ to $\underline{L}'(\lambda^{+,\alpha})$ and $\underline{\Delta}'(\lambda^{+,\alpha+1})$ to $\underline{\Delta}'(\lambda^{+,\alpha})$ for all $\lambda \in \underline{\Lambda}$, obtained by $e' - e$ applications of Corollary 11. If $\alpha = e$ and $i = 0$, we can reduce to the case $\alpha = e - 1$ by a similar argument.

By Proposition 9, we have a perverse equivalence $\mathsf{G} : D^b(B\text{-mod}) \xrightarrow{\sim} D^b(\dot{B}\text{-mod})$ specified by the filtration $\Lambda_j = \{\lambda \in \Lambda \mid F_a^j L(\lambda) = 0\}$ and the perversity $p(j) = j - 1$. We have a parallel situation for B' and $\dot{B}'$, a perverse equivalence $\mathsf{G}' : D^b(B'\text{-mod}) \xrightarrow{\sim} D^b(\dot{B}'\text{-mod})$ with respect to the filtration $\Lambda'_j = \{\lambda \in \Lambda' \mid F_{a'}^j L'(\lambda) = 0\}$ and perversity $p(j) = j - 1$. Moreover $[\mathsf{G}\Delta(\lambda)] = \pm[\Delta(\sigma_a\lambda)]$ for all $\lambda \in \Lambda$. An analogous statement holds for G'.

By a result of Cline, Parshall and Scott [CPS82] (see also [PS88]), the exact functor B'-mod $\to \underline{B}'$-mod : $M \mapsto fM$ induces an equivalence of triangulated categories

$$\frac{D^b(B'\text{-mod})}{D^b_{\mathcal{E}}(B'\text{-mod})} \xrightarrow{\sim} D^b(\underline{B}'\text{-mod}),$$

where $\mathcal{E}$ is the Serre subcategory of B'-mod generated by $L(\lambda)$, $\lambda \in \Lambda' \setminus \underline{\Lambda}'$. An analogous statement holds for the dot versions.

We claim that G' restricts to an equivalence $D^b_{\mathcal{E}}(B'\text{-mod}) \xrightarrow{\sim} D^b_{\dot{\mathcal{E}}}(\dot{B}'\text{-mod})$. An equivalent statement is that it restricts to an equivalence between the left perpendicular categories, i.e. the full triangulated subcategories of $D^b(B'\text{-mod})$ and $D^b(\dot{B}'\text{-mod})$ generated by $\{P'(\lambda) \mid \lambda \in \underline{\Lambda}'\}$ and $\{P'(\lambda) \mid \lambda \in \underline{\dot{\Lambda}}'\}$. To see that this latter statement is true, recall (Proposition 9) that G' is induced by a complex of functors, each of which is a direct summand of a composition of powers of $E_{a'}$ and $F_{a'}$. Because of our assumption that all partitions in Λ and $\dot{\Lambda}$ have at most d parts, and that $\alpha \neq i \mod e$, the map between Grothendieck groups induced by any such direct summand functor sends $\sum_{\lambda \in \underline{\Lambda}'} \mathbb{Z}[\Delta'(\lambda)]$ into $\sum_{\lambda \in \underline{\dot{\Lambda}}'} \mathbb{Z}[\Delta'(\lambda)]$, and therefore $\sum_{\lambda \in \underline{\Lambda}'} \mathbb{Z}[P'(\lambda)]$ into $\sum_{\lambda \in \underline{\dot{\Lambda}}'} \mathbb{Z}[P'(\lambda)]$. A similar reasoning applies to G'^{-1}, and the claim follows.

Hence, G' induces a perverse equivalence

$$\underline{\mathsf{G}}' : D^b(\underline{B}'\text{-mod}) \xrightarrow{\sim} D^b(\underline{\dot{B}}'\text{-mod})$$

with respect to the filtration $\underline{\Lambda}'_j = \{\lambda \in \underline{\Lambda}' \mid F_a^j L'(\lambda) = 0\}$ and the perversity $p(j) = j - 1$.

By assumption, we have a Morita equivalence

$$\mathsf{F} : \underline{B}\text{-mod} \xrightarrow{\sim} \underline{B}'\text{-mod}$$

such that $\mathsf{F}\underline{L}(\lambda) \cong \underline{L}'(\lambda^+)$ and $\mathsf{F}\underline{\Delta}(\lambda) \cong \underline{\Delta}'(\lambda^+)$ for all $\lambda \in \Lambda$. Moreover by Lemma 12, proved in the following subsection, the bijection $\underline{\Lambda} \xrightarrow{\sim} \underline{\Lambda}' : \lambda \mapsto \lambda^+$ restricts to bijections $\underline{\Lambda}_j \xrightarrow{\sim} \underline{\Lambda}'_j$ for all j. By Proposition 8, we deduce that the composition

$$\underline{\mathsf{G}}'\mathsf{F}\mathsf{G}^{-1} : D^b(\dot{B}\text{-mod}) \xrightarrow{\sim} D^b(\underline{\dot{B}}'\text{-mod})$$

restricts to an equivalence

$$\dot{\mathsf{F}} : \dot{B}\text{-mod} \xrightarrow{\sim} \underline{\dot{B}}'\text{-mod}\,.$$

Moreover, $[\dot{\mathsf{F}}\Delta(\lambda)] = [\underline{\Delta}'(\lambda^+)]$ for all $\lambda \in \dot{\Lambda}$. By unitriangularity of the decomposition matrices of B and $\underline{B}'$, it follows that $\dot{\mathsf{F}}L(\lambda) \cong \underline{L}'(\lambda^+)$ for all $\lambda \in \dot{\Lambda}$, and therefore that $\dot{\mathsf{F}}\Delta(\lambda) \cong \underline{\Delta}'(\lambda^+)$ for all $\lambda \in \dot{\Lambda}$.

4.3 Comparison of Crystals

The aim of this subsection is to complete the inductive step by proving Lemma 12, which was used above to get a good compatibility between filtrations.

Let $a \in \{0, \ldots, e-1\}$. For any partition λ, we define

$$\begin{aligned} \varphi_a(\lambda) &:= \max\{k \geq 0 \mid (\widetilde{F}_a)^k(L(\lambda)) \neq 0\} \\ &= \max\{k \geq 0 \mid {F_a}^k(L(\lambda)) \neq 0\} \end{aligned}$$

and similarly

$$\begin{aligned}\varphi'_a(\lambda) &:= \max\{k \geq 0 \mid (\widetilde{F}_a)^k(L'(\lambda)) \neq 0\}\\ &= \max\{k \geq 0 \mid {F_a}^k(L'(\lambda)) \neq 0\}\end{aligned}$$

(cf. [HK02, p. 85]), where $\widetilde{F}_a$ is the Kleshchev/Kashiwara operator, defined in Remark 10. Remember that $L(\lambda)$ is a simple module over a Schur algebra with parameter q, while for $L'(\lambda)$ the parameter is q'.

Lemma 12. *Let $i \in \{0, \ldots, e-1\}$ and define a and a' as in Sect. 4.2. Then*

$$\varphi_a(\lambda) = \varphi'_{a'}(\lambda^+)$$

for any $\lambda \in \underline{\Lambda}$.

Proof. We explain below why the statement holds if λ is e-restricted. But first we shall assume the truth of this special case and deduce the statement for arbitrary λ.

We can write any partition λ uniquely as $\lambda = \lambda^{e\text{-res}} + e\tilde{\lambda}$, where $\lambda^{e\text{-res}}$ is e-restricted. In terms of the abacus, $\lambda^{e\text{-res}}$ is obtained from λ by repeatedly moving a bead up a runner, say from position h to $h-e$, where positions $h-e, \ldots, h-1$ are unoccupied. This description makes it clear that $(\lambda^{e\text{-res}})^+ = (\lambda^+)^{e'\text{-res}}$.

By Steinberg's tensor product theorem [Don98, p. 65] (cf. [Lus89, 7.4], [DD91], [PW91, 11.7]), we have $L(\lambda) \cong L(\lambda^{e\text{-res}}) \otimes L(e\tilde{\lambda})$. More generally, tensoring with $L(e\tilde{\lambda})$ sends modules over a block of $\mathcal{S}_{q,\mathbb{k}}(r - e|\tilde{\lambda}|)$ to modules over the block of $\mathcal{S}_{q,\mathbb{k}}(r)$ corresponding to the same e-core. Hence, for any k the isomorphism $V^{\otimes k} \otimes L(\lambda) \cong V^{\otimes k} \otimes L(\lambda^{e\text{-res}}) \otimes L(e\tilde{\lambda})$, after a projection onto appropriate blocks, gives an isomorphism ${F_a}^k(L(\lambda)) \cong {F_a}^k(L(\lambda^{e\text{-res}})) \otimes L(e\tilde{\lambda})$. Thus we deduce that $\varphi_a(\lambda) = \varphi_a(\lambda^{e\text{-res}})$.

Putting together the pieces, we have, for any λ,

$$\varphi_a(\lambda) = \varphi_a(\lambda^{e\text{-res}}) = \varphi'_{a'}((\lambda^{e\text{-res}})^+) = \varphi'_{a'}((\lambda^+)^{e'\text{-res}}) = \varphi'_{a'}(\lambda^+).$$

Now we return to the special case of e-restricted partitions. One can define a summand $F_{\mathcal{H},a} : \mathcal{H}_{\mathbb{k},q}(\mathfrak{S}_{r-1})\text{-mod} \to \mathcal{H}_{\mathbb{k},q}(\mathfrak{S}_r)\text{-mod}$ of the induction functor between Hecke algebras of type A, analogous to F_a. One then has obvious analogues $\widetilde{F}_{\mathcal{H},a}$ and $\varphi_{\mathcal{H},a}$ of $\widetilde{F}_a$ and φ_a.

After Kleshchev's branching rule appeared, Brundan [Bru98] extended Kleshchev's result to Hecke algebras of type A and showed that for any simple $\mathcal{H}_{\mathbb{k},q}(\mathfrak{S}_{r-1})$-module L the $\mathcal{H}_{\mathbb{k},q}(\mathfrak{S}_r)$-module $\widetilde{F}_{\mathcal{H},a}(L)$ is simple. Since there is a commutative diagram of functors

$$\begin{array}{rccc} F_a: & \mathcal{S}_{\mathbb{k},q}(r-1)\text{-mod} & \to & \mathcal{S}_{\mathbb{k},q}(r)\text{-mod} \\ & \downarrow & & \downarrow \\ F_{\mathcal{H},a}: & \mathcal{H}_{\mathbb{k},q}(\mathfrak{S}_{r-1})\text{-mod} & \to & \mathcal{H}_{\mathbb{k},q}(\mathfrak{S}_r)\text{-mod} \end{array},$$

where the horizontal arrows are Schur functors, this implies that $\varphi_a(\lambda) = \varphi_{\mathcal{H},a}(\lambda)$ for any e-restricted λ.

Brundan also showed that for e-restricted partitions, $\varphi_{\mathcal{H},a}(\lambda)$ is the number of conormal indent a-nodes for λ. We will not define this combinatorial notion here; we just observe that his result implies that $\varphi_{\mathcal{H},a}(\lambda)$ depends only on the configuration of beads on runners ρ_{i-1} and ρ_i in the abacus representation of λ, where $i \in \{0, \dots, e-1\}$ is such that $i \equiv a+d \mod e$. We deduce that for any e-restricted partition λ,

$$\varphi_a(\lambda) = \varphi_{\mathcal{H},a}(\lambda) = \varphi'_{\mathcal{H},a'}(\lambda) = \varphi'_{a'}(\lambda),$$

as desired. □

5 Rouquier Blocks

We now complete the proof of Theorem 2 by handling the base case: Rouquier blocks.

5.1 Wreath Product Interpretation of Rouquier Blocks

In this subsection, we assume that $ch(\Bbbk) = \ell > w > 0$, and that q is a prime power not congruent to 0 or 1 modulo ℓ. So $e = e(q)$ is the multiplicative order of $q \cdot 1_{\mathbb{F}_\ell}$. We denote by $\mathbf{L}_\lambda$ the standard Levi subgroup of $\mathbf{GL}_{|\lambda|}(\mathbb{F}_q)$ corresponding to the Young subgroup $\mathfrak{S}_\lambda$. In [Tur02, p. 250, Lemma 1 and p. 249, Theorem 1] and [Miy01, p. 30, Lemma 5.0.6 and p. 31, Theorem 5.0.7] (cf. [Tur05, Theorem 71]), the $\mathbf{GL}_n(\mathbb{F}_q)$ analogue of the main theorem in [CK02, Theorem 2] is proved independently:

Theorem 13. *Suppose γ is a Rouquier e-core with respect to w. Put $r := ew + |\gamma|$ and $\mathbf{G} := \mathbf{GL}_r(\mathbb{F}_q)$. Put $\mathbf{L} := \mathbf{L}_{(e^w,|\gamma|)}$, a Levi subgroup of $\mathbf{G}$. Then $\mathbf{L}$ has a parabolic complement $\mathbf{P} = \mathbf{L}\mathbf{U}_\mathbf{L}$ in $\mathbf{G}$. Put $\mathbf{I} := N_\mathbf{G}(\mathbf{L}) \cong \mathbf{L} \rtimes \mathfrak{S}_w$. So, $\Bbbk[\mathbf{I}] \cong \Bbbk[\mathbf{L}] \rtimes \mathfrak{S}_w$.*

There exists a $(\Bbbk[\mathbf{G}], \Bbbk[\mathbf{I}])$-bimodule M such that

(1) *M is a direct summand of $\Bbbk[\mathbf{G}/\mathbf{U}_\mathbf{L}] \otimes_{\Bbbk[\mathbf{L}]} \Bbbk[\mathbf{I}]$ as a $(\Bbbk[\mathbf{G}], \Bbbk[\mathbf{I}])$-bimodule.*
(2) *$M \otimes_{A^\sharp} M^\vee \cong A$ and $M^\vee \otimes_A M \cong A^\sharp$, where A is the Rouquier unipotent block of $\Bbbk[\mathbf{G}]$ with e-core γ and $A^\sharp$ is the unipotent block of $\Bbbk[\mathbf{I}]$ with $(w+1)$-tuple of e-cores $(\emptyset^w, \gamma)$.*
(3) *M is left projective as well as right projective.*

5.2 Images of Some Modules via the Equivalence

In order to make a connection to q-Schur algebras we need to identify the images of modules under the equivalence in Theorem 13. We retain the assumptions on q and ℓ of the previous section. For $\lambda \vdash n$, we denote by $S(\lambda)$ the Specht $\Bbbk\,\mathbf{GL}_n(\mathbb{F}_q)$-module corresponding to λ (see [Jam84, Jam86]), by $D(\lambda)$ its unique simple quotient, by $P(\lambda)$ the projective cover of $D(\lambda)$ and by $X(\lambda)$ the Young $\Bbbk\,\mathbf{GL}_n(\mathbb{F}_q)$-module corresponding to λ (see [DJ89] for the definition).

The labels of simple modules over the principal block $B_0(\Bbbk[\mathbf{GL}_e(\mathbb{F}_q) \wr \mathfrak{S}_w])$ are given by e-multipartitions of w, which we denote by $\mathcal{MP}(e,w)$. This goes as follows: The principal block $B_0(\mathbf{GL}_e(\mathbb{F}_q))$ has e non-isomorphic simple modules $\{D(\mu) \mid \mu \text{ is a hook partition of } e\}$. For an e-tuple of non-negative integers $\underline{m} = (m_1, \ldots, m_e)$ such that $\sum_i m_i = w$, define $S(\underline{m})$ to be the $\mathbf{GL}_e(\mathbb{F}_q)^w$-module $\boxtimes_{i=1}^{e}(S(e-i+1, 1^{i-1})^{\boxtimes m_i})$. So, the Young subgroup $\mathfrak{S}_{\underline{m}}$ acts on $S(\underline{m})$. For an e-multipartition ($= e$-tuple of partitions) $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(e)})$, put $m_i := |\lambda^{(i)}|$ and then define the Specht $B_0(\Bbbk[\mathbf{GL}_e(\mathbb{F}_q) \wr \mathfrak{S}_w])$-module $S(\underline{\lambda})$ by

$$\operatorname{Ind}_{\mathbf{GL}_e(\mathbb{F}_q)^w \rtimes \mathfrak{S}_{\underline{m}}}^{\mathbf{GL}_e(\mathbb{F}_q) \wr \mathfrak{S}_w} \left(\boxtimes_{i=1}^{e} (S(e-i+1, 1^{i-1})^{\boxtimes m_i}) \otimes_{\Bbbk} S^{\lambda^{(i)}} \right).$$

If we replace $S(\star)$ by $D(\star)$ (resp. $X(\star)$), then we obtain simple modules (resp. Young modules) over $\mathbf{GL}_e(\mathbb{F}_q) \wr \mathfrak{S}_w$.

So, the labels of simple modules over $A^\sharp \cong B_0(\Bbbk[\mathbf{GL}_e(\mathbb{F}_q) \wr \mathfrak{S}_w]) \boxtimes B_\gamma$, where B_γ is the simple block algebra corresponding to the unipotent character indexed by e-core γ, are also given by $\mathcal{MP}(e,w)$.

Define a map $(-)^\natural : \mathcal{MP}(e,w) \to \mathcal{MP}(e,w)$ by

$$((\overline{\lambda})^\natural)_i := \begin{cases} \lambda^{(i)} & \text{if } i+e \text{ is even,} \\ (\lambda^{(i)})^{\mathrm{tr}} & \text{otherwise.} \end{cases}$$

Here, the 0th runner of an e-quotient is treated as the first entry of an e-multipartition.

By Hida–Miyachi [HM00] the images of simple, Specht, Young and projective indecomposable modules over $A^\sharp$ via M are determined explicitly.

Theorem 14 (Hida–Miyachi). *For $K \in \{D, S, X, P\}$ and any $\lambda \in \Lambda$,*

$$K(\lambda) \cong M \otimes_{A^\sharp} K(\overline{\lambda}^\natural).$$

Remark 15. The proof of this result is given in the appendix, thanks to Akihiko Hida's permission. At the level of combinatorics of indices λ for modules $S(\lambda), D(\lambda), X(\lambda), P(\lambda)$ the result above is identical with [CT03] up to replacing p in Chuang–Tan by e in Hida–Miyachi.

For a finite group $\mathbf{H}$ with $|\mathbf{H}|^{-1} \in \Bbbk$, put

$$\mathbf{e_H} := \frac{1}{|\mathbf{H}|} \sum_{h \in \mathbf{H}} h.$$

Let $\mathbf{B}$ (resp. $\mathbf{B_L}$) be a standard Borel subgroup of $\mathbf{G}$ (resp. $\mathbf{L}$).

Put $B := \mathbf{e_B} A \mathbf{e_B}$ and $B^\sharp := \mathbf{e_{B_L}} A^\sharp \mathbf{e_{B_L}}$. Then, by the origin of Iwahori-Hecke algebras we can regard B (resp. $B^\sharp$) as a block of $\mathcal{H}_{\Bbbk,q}(\mathfrak{S}_r)$ (resp. $(\mathcal{H}_{\Bbbk,q}(\mathfrak{S}_e)^{\otimes w} \rtimes \Bbbk[\mathfrak{S}_w]) \otimes_{\Bbbk} \mathcal{H}_{\Bbbk,q}(\mathfrak{S}_{|\gamma|})$).

By applying Schur functors, we obtain the Hecke algebra version of Theorem 14 (cf. [Tur05, Theorem 78]):

Corollary 16. *$Y^\lambda \cong N \otimes_{B^\sharp} Y^{(\overline{\lambda})^\natural}$ for any $\lambda \in \Lambda$ where $Y^\lambda = \mathbf{e_B} X_e(\lambda)$, $Y^{\overline{\nu}} = \mathbf{e_{B_L}} X(\overline{\nu})$ and $N = \mathbf{e_B} M \mathbf{e_{B_L}}$.*

Remark 17. Here, we label Young modules according to Dipper–James's convention. Namely, Y^λ is a unique indecomposable direct summand of the q-permutation module $\mathrm{Ind}_{\mathcal{H}_{\Bbbk,q}(\mathfrak{S}_\lambda)}^{\mathcal{H}_{\Bbbk,q}(\mathfrak{S}_r)}(\mathrm{ind})$ such that Y^λ contains S^λ as a unique submodule where S^λ is Dipper–James's q-Specht module and ind is the index representation.

5.3 *Proof of the Main Theorem: Initial Step*

In this subsection, our assumptions on $\Bbbk$ and q are as in the statement of Theorem 2: $q \in \mathbb{F}_\ell^\times$ and $\ell > w$, or $q \in \mathbb{C}$, i.e. we include the cases $ch(\Bbbk) = 0$ or $e = ch(\Bbbk) > w$. Let γ be the Rouquier e-core with respect to $w > 0$. Put $r = ew + |\gamma|$. We denote by A^w (resp. A_w) the Rouquier block of $\mathcal{H}_{\Bbbk,q}(\mathfrak{S}_r)$ (resp. $\mathcal{S}_{\Bbbk,q}(r)$). We denote $B_0(\mathcal{H}_{\Bbbk,q}(\mathfrak{S}_e))^{\boxtimes w} \boxtimes_{\Bbbk} B^\gamma$ (resp. $B_0(\mathcal{S}_{\Bbbk,q}(e))^{\boxtimes w} \boxtimes_{\Bbbk} B_\gamma$) by B^w (resp. B_w) where B^γ (resp. B_γ) is the defect zero simple block algebra of $\mathcal{H}_{\Bbbk,q}(\mathfrak{S}_{|\gamma|})$ (resp. $\mathcal{S}_{\Bbbk,q}(|\gamma|)$) corresponding to the e-core γ. $\mathfrak{S}_w$ acts on both B^w and B_w by permuting the $\mathcal{H}_{\Bbbk,q}(\mathfrak{S}_e)$-components and the $\mathcal{S}_{\Bbbk,q}(e)$-components, respectively. We denote $B^w \rtimes \Bbbk[\mathfrak{S}_w]$ (resp. $B_w \rtimes \Bbbk[\mathfrak{S}_w]$) by C^w (resp. C_w). Similar to the case $GL_e(\mathbb{F}_q) \wr \mathfrak{S}_w$, by Clifford theory we can construct a standard C_w-module $\Delta(\underline{\lambda})$ for an e-multipartition $\underline{\lambda}$: $\Delta(\underline{\lambda}) := \mathrm{Ind}_{B_w \rtimes \mathfrak{S}_{\underline{m}}}^{B_w \rtimes \mathfrak{S}_w} \left(\boxtimes_{i=1}^e (\Delta(i, 1^{e-i})^{\boxtimes m_i}) \otimes_{\Bbbk} S^{\lambda^{(i)}} \right)$, where $m_i = |\lambda^{(i)}|$ for $i = 1, \ldots, e$.

By replacing Δ by any $K \in \{L, \nabla, P, I, T\}$, we can construct $K(\underline{\lambda})$.

Now, we unify the results in [CT03] on Schur algebras, Theorem 14 on finite general linear groups and a new result in characteristic zero into q-Schur algebras as follows:

Theorem 18. *There exists an (A^w, C^w)-bimodule M^w such that*

(1) *M^w is a direct summand of $A^w \otimes_{B^w} C^w$,*
(2) *$M^w \otimes_{C^w} -$ is an equivalence between A^w-mod and C^w-mod,*
(3) *$Y^\lambda \cong M^w \otimes_{C^w} Y^{(\overline{\lambda})^\natural}$ for any λ such that $A^w S^\lambda \neq 0$.*

So, we have an equivalence $\mathsf{G} : A_w$-mod $\to C_w$-mod such that $\mathsf{G}(K(\lambda)) \cong K((\overline{\lambda})^\natural)$ for any $K \in \{L, \Delta, \nabla, P, I, T\}$ and any λ so that $A_w \cdot \Delta(\lambda) \neq 0$.

Proof. The case $ch(\Bbbk) = e > 0$ is already treated in [CT03], and the statement is true. Next we look at the case $ch(\Bbbk) > e > 0$. However, this is nothing but Theorem 13 and Corollary 16.

The only remaining claim is to prove the statement for the case $ch(\Bbbk) = 0$. We leave the proof of the existence of M^w in characteristic zero to Sect. 6. There, by considering a modular system $(\Bbbk, \mathcal{O}, \mathbb{F})$ where $ch(\Bbbk) = 0$, $\mathcal{O}$ is a complete discrete valuation ring with maximal ideal π, $\mathcal{O}/(\pi) \cong \mathbb{F}, ch(\mathbb{F}) > 0$, and $q \in \Bbbk$ and $\overline{q} = q + (\pi) \in \mathbb{F}$ both have order $e > 0$, we shall realize M^w as a lifting of an $(\mathcal{H}_{\mathbb{F},\overline{q}}(\mathfrak{S}_r), \mathcal{H}_{\mathbb{F},\overline{q}}(\mathfrak{S}_e) \wr \mathfrak{S}_w \boxtimes \mathcal{H}_{\mathbb{F},\overline{q}}(\mathfrak{S}_{|\gamma|}))$-bimodule $M^w_{\mathbb{F}}$ which satisfies statements (1), (2) and (3). Since A^w, B^w and C^w are liftings of the corresponding algebras over $\mathbb{F}$, statement (2) is clear. By statement (1), we can ensure that M^w sends Young modules to Young modules. So, statement (3) follows from a simple chasing of characters, i.e. the weights (partitions) λ. □

Define Λ_w to be the set of partitions whose e-weight is w and e-core is Rouquier with respect to w. Similarly, define Λ'_w for e'.

Define ι to be the embedding $\mathcal{MP}(e, w)$ into $\mathcal{MP}(e', w)$ by $\iota(\overline{\nu})_{i+e'-e} := (\overline{\nu})_i$ for $i = 1, \ldots, e$ and $(\overline{\nu})_j = \emptyset$ for $j \leq e' - e$.

Next, we define an embedding ι of Λ_w into Λ'_w by $\iota(\lambda) := \nu$ where the e'-quotient of ν is $\iota(\overline{\lambda})$.

From now on, we suppose that e' is the quantum characteristic of $q' \cdot 1_{\mathbb{F}_\ell}$ in $\Bbbk$ so that $e' \geq e$, γ' is a Rouquier e'-core with respect to w. Put $r' = e'w + |\gamma'|$.

Lemma 19. *(1) Let f be an idempotent of C'_w such that if $\underline{\lambda}^{(i)} = \emptyset$ for all $i = 1, \ldots, e' - e$, then $fL'(\underline{\lambda}) \neq 0$, otherwise $fL'(\underline{\lambda}) = 0$. Then, $fC'_w f$ is Morita equivalent to C_w.*

(2) There exists an idempotent ξ of A'_w such that $\xi L'(\iota(\lambda)) \neq 0$ for any $\lambda \in \Lambda_w$ and $\xi A'_w \xi$ is Morita equivalent to A_w.

Proof. Recall the definition of $C'_w = B'_w \rtimes \mathfrak{S}_w \cong (B_0(\mathcal{S}_{\Bbbk,q'}(e')) \wr \mathfrak{S}_w) \boxtimes B_\gamma$. Put B' (resp. B) to be $B_0(\mathcal{S}_{\Bbbk,q'}(e'))$ (resp. $B_0(\mathcal{S}_{\Bbbk,q}(e))$). Take an idempotent ξ of B' such that $\xi L'(i, 1^{e'-i}) = 0$ for any $i \leq e' - e$ and $\xi L'(j, 1^{e'-j}) \neq 0$ for any $j > e' - e$. Then, $\xi B' \xi$ is Morita equivalent to B. Indeed, we can show this by the fact that both B and B' are Brauer line tree algebras with no exceptional vertex. Let $f := \xi \boxtimes \cdots \boxtimes \xi \boxtimes 1_\gamma$ be the idempotent of $B^{\boxtimes w} \boxtimes B_\gamma$. So, $fB'_w f$ is Morita equivalent to B_w. Now, by taking wreath products, we know that (1) is clear.

By Theorem 18, we have an idempotent j of C'_w corresponding to ξ so that $j \cdot \mathsf{G}(L(\iota(\lambda))) \neq 0$ for any λ. By definition of ι we know that for $\mu \in \Lambda_w$, $\mu^{(i)} = 0$ for all $0 \leq i \leq e' - e - 1$ if and only if $\mu = \iota(\lambda)$ for some $\lambda \in \Lambda$. So, this means that j satisfies the condition of (1). □

Define $B, B', \underline{B}, \underline{B}', \Lambda, \Lambda', \underline{\Lambda}, \underline{\Lambda}'$ as in Sect. 2, taking α to be 0.

Let f'' be an idempotent of A'_w corresponding to $\bigoplus_{\underline{\Lambda} \in \Lambda} P(\iota(\lambda))$.

By an argument similar to Corollary 11, one can show the following:

Lemma 20. *Suppose $B = A_w$. Then there exists an equivalence*

$$\mathsf{H} : f'' A'_w f''\text{-mod} \to \underline{B}'\text{-mod},$$

such that

$$\mathsf{H}(f''\Delta'(\iota(\mu))) \cong \underline{\Delta}'(\mu^+) \text{ and } \mathsf{H}(f''L'(\iota(\mu))) \cong \underline{L}'(\mu^+) \text{ for any } \mu \in \underline{\Lambda}.$$

Proposition 21. *Suppose that $A_w = B$. There exists an equivalence $\mathsf{F} : \underline{B}$-mod $\to$ $\underline{B}'$-mod, such that $\mathsf{F}(\underline{\Delta}(\lambda)) \cong \underline{\Delta}'(\lambda^+)$ and $\mathsf{F}(\underline{L}(\lambda)) \cong \underline{L}'(\lambda^+)$ for any $\lambda \in \underline{\Lambda}$. Namely, the main theorem is true for Rouquier blocks.*

Proof. By Lemma 19, we know that there exists an idempotent ξ of A'_w such that $\xi L'(\lambda^+) \neq 0$ for any $\lambda \in \Lambda$ and $\xi A'_w \xi$ is Morita equivalent to A_w. Let f be the idempotent of A_w such that $fA_w f$ is a subalgebra of $\mathcal{S}_{\Bbbk,q}(d,r)$, i.e. f kills $L(\mu)$ for $l(\mu) > d$ and $fL(\lambda) \neq 0$ for $l(\lambda) \leq d$. Similarly, we can find that the property of an idempotent f'' of A'_w defined above is that $f''A'_w f''$ is a subalgebra of $\mathcal{S}_{\Bbbk,q}(d,r')$, i.e. f'' kills $L'(\mu)$ for $l(\mu) > d$ and $f''L'(\lambda) \neq 0$ for $l(\lambda) \leq d$ and $\xi = f'' + \xi'$ for some idempotent ξ'. Let T be a functor from $fA_w f$-mod to $f''A'_w f''$-mod such that $\mathsf{T}(f\Delta(\lambda)) \cong f''\Delta'(\iota(\lambda))$, i.e. T is a restriction of the equivalence of Lemma 19.

Since $\max\{l(\lambda) \mid \lambda \in \underline{\Lambda}\} = \max\{l(\lambda^+) \mid \lambda \in \underline{\Lambda}\} = d$ by Theorem 18 and by definition of ι, we know that $fL(\lambda) \neq 0$ if and only if $f'L'(\iota(\lambda)) \neq 0$, and we know that the dominance order of $\underline{\Lambda}$ is preserved by the ι map and the dominance order of $\underline{\Lambda}'$ is preserved by the $+$ map. Therefore, the composition of equivalences T and H in Lemma 20 is an equivalence between $\underline{B}$-mod and $\underline{B}'$-mod, which satisfies the conditions on the images of Δ-sections. $\square$

6 Lifting Morita Equivalences

In this section, we supply a missing argument for the proof of Theorem 18. We will choose an appropriate modular system to work in with the help of the following lemmas.

Let $\zeta_n = \exp(2\pi i/n) \in \mathbb{C}$, and denote by $\Phi_n(x) \in \mathbb{Z}[x]$ the nth cyclotomic polynomial.

Lemma 22. *Suppose that $a \in \mathbb{Z}[\zeta_e]$, $a \neq 0$ and $e > 1$. Then, there exists a prime number ℓ and a homomorphism $\bar{\cdot} : \mathbb{Z}[\zeta_e] \to \mathbb{F}_\ell$ such that $q := \overline{\zeta_e}$ is a primitive eth root of unity in $\mathbb{F}_\ell$ and so that $\overline{a} \neq 0$.*

Proof. Choose $f(x) \in \mathbb{Z}[x]$ such that $f(\zeta_e) = a$. Then, $a \neq 0$ implies $\Phi_e(x) \nmid f(x)$. Since $\Phi_e(x)$ is monic, there exist $Q(x)$ and $r(x) \neq 0$ in $\mathbb{Z}[x]$ so that

(1) $\deg(r) < \deg(\Phi_e)$,
(2) $f(x) = Q(x)\Phi_e(x) + r(x)$.

By Dirichlet's theorem, there exist infinitely many prime numbers $\ell > 0$ such that $e \mid \ell - 1$. Choose one such ℓ so that $\overline{r}(x) \neq 0$. We complete the proof of the lemma by showing that there exists a primitive eth root of unity q in $\mathbb{F}_\ell$ satisfying $\overline{f}(q) \neq 0$. If not, $(x-q)$ divides $\overline{f}(x)$ in $\mathbb{F}_\ell[x]$ for all primitive eth roots of unity q. Hence, $\overline{\Phi_e}(x)$ divides $\overline{f}(x)$ in $\mathbb{F}_\ell[x]$. So, by (2) above, we deduce that $\overline{\Phi_e}(x)$ divides $\overline{r}(x) \neq 0$ in $\mathbb{F}_\ell[x]$, contradicting (1). $\square$

Lemma 23. *Let Γ be a $\mathbb{Z}[\zeta_e]$-algebra, free and of finite rank over $\mathbb{Z}[\zeta_e]$, and let X be a Γ-lattice of finite rank. There exists a prime ℓ and a homomorphism $\overline{\cdot}: \mathbb{Z}[\zeta_e] \to \mathbb{F}_\ell$ such that $q := \overline{\zeta_e}$ is a primitive e-root of unity in $\mathbb{F}_\ell$ and*

$$\dim \mathrm{End}_{\mathbb{Q}(\zeta_e)\otimes_{\mathbb{Z}[\zeta_e]}\Gamma}(\mathbb{Q}(\zeta_e) \otimes_{\mathbb{Z}[\zeta_e]} X) = \dim \mathrm{End}_{\mathbb{F}_\ell\otimes_{\mathbb{Z}[\zeta_e]}\Gamma}(\mathbb{F}_\ell \otimes_{\mathbb{Z}[\zeta_e]} X).$$

Proof. Let $\overline{\cdot}: \mathbb{Z}[\zeta_e] \to \mathbb{F}_\ell$ be a homomorphism, and for $R \in \{\mathbb{Q}(\zeta_e), \mathbb{Z}[\zeta_e], \mathbb{F}_\ell\}$ put $RX = R\otimes_{\mathbb{Z}[\zeta_e]} X$ and $R\Gamma = R\otimes_{\mathbb{Z}[\zeta_e]}\Gamma$. Let I be a finite set of generators for Γ as a $\mathbb{Z}[\zeta_n]$-algebra (we could, for example, take I to be a basis). Then, $\mathrm{End}_R\,\Gamma(RX)$ is the kernel of the R-homomorphism f_R of $\mathrm{End}_R(RX)$ into $\bigoplus_{g\in I} \mathrm{End}_R(RX)$ defined by

$$f_R(x)_g := x \circ g - g \circ x \in \mathrm{End}_R(RX).$$

Let M_f be the matrix representing $f_{\mathbb{Z}[\zeta_e]}$ with respect to some chosen $\mathbb{Z}[\zeta_e]$-bases. Let $a \in \mathbb{Z}[\zeta_e]$ be the product of all nonzero minors of M_f. By Lemma 22, we may choose $\overline{\cdot}: \mathbb{Z}[\zeta_e] \to \mathbb{F}_\ell$ such that $\overline{\zeta_e}$ is a primitive e-root of unity in $\mathbb{F}_\ell$ and $\overline{a} \neq 0$. Then the ranks of M_f as a matrix over $\mathbb{Q}(\zeta_e)$ and over $\mathbb{F}_\ell$ are the same, and it follows that $\mathrm{End}_{\mathbb{F}_\ell\Gamma}(\mathbb{F}_\ell X)$ and $\mathrm{End}_{\mathbb{Q}(\zeta_e)\Gamma}(\mathbb{Q}(\zeta_e)X)$ have the same dimension. $\square$

We are ready to return to Theorem 18. Let γ be a Rouquier e-core with respect to w. Put $r = ew + |\gamma|$. For any domain R and any $\zeta \in R^\times$, let $A_{R,\zeta} = \mathcal{H}_{R,\zeta}(\mathfrak{S}_r)$, $B_{R,\zeta} = \mathcal{H}_{R,\zeta}(\mathfrak{S}_{(e^w,\gamma)})$ and $C_{R,\zeta} = B_{R,\zeta} \rtimes R[\mathfrak{S}_w]$. Furthermore, let $\Gamma_{R,\zeta} = A_{R,\zeta} \otimes_R C_{R,\zeta}$, and let $X_{R,\zeta}$ be the $\Gamma_{R,\zeta}$-module $A_{R,\zeta} \otimes_{B_{R,\zeta}} C_{R,\zeta}$.

Now consider in particular $\Gamma = \Gamma_{\mathbb{Z}[\zeta_e],\zeta_e}$ and $X = X_{\mathbb{Z}[\zeta_e],\zeta_e}$, and choose a prime ℓ and a homomorphism $\overline{\cdot}: \mathbb{Z}[\zeta_e] \to \mathbb{F}_\ell$ according to Lemma 23. Note that X is free over $\mathbb{Z}[\zeta_e]$ since $A = A_{\mathbb{Z}[\zeta_e],\zeta_e}$ and $C = C_{\mathbb{Z}[\zeta_e],\zeta_e}$ are free over $B = B_{\mathbb{Z}[\zeta_e],\zeta_e}$.

Let $\mathcal{O}$ be the completion of $\mathbb{Z}[\zeta_e]$ at the kernel of $\overline{\cdot}$, so that $\mathcal{O}$ is a complete discrete valuation ring. By Lemma 23, the natural embedding $\mathbb{F}_\ell \mathrm{End}_\Gamma(X) \hookrightarrow \mathrm{End}_{\Gamma_{\mathbb{F}_\ell,q}}(X_{\mathbb{F}_\ell,q})$ must be an isomorphism. We saw in the proof of Theorem 18 that there exists a summand $M_{\mathbb{F}_\ell,q}$ of $X_{\mathbb{F}_\ell,q}$ with certain good properties. A projection onto this summand determines an idempotent of $\mathrm{End}_{\Gamma_{\mathbb{F}_\ell,q}}(X_{\mathbb{F}_\ell,q})$, which we may lift to an idempotent of $\mathrm{End}_\Gamma(X)$. We obtain in this way a summand M of X, with the property that $\mathbb{F}_\ell M \cong M_{\mathbb{F}_\ell,q}$. Passing now to the quotient field $\Bbbk$ of $\mathcal{O}$, we obtain a $\Gamma_{\Bbbk,\zeta_e}$-module $\Bbbk M$ that settles Theorem 18 in the characteristic 0 case.

We end this section by remarking that the lifting technique we have used is applicable in some other situations involving 'Rouquier-like' blocks in Hecke algebras of other types. The set up is as follows:

Let W be a finite Weyl group. Let W_L be a parabolic subgroup of W. Take a subgroup $\overline{N} \subset N_W(W_L)/W_L$.

Let $A_{R,\zeta} = \mathcal{H}_{R,\zeta}(W)$, $B_{R,\zeta} = \mathcal{H}_{R,\zeta}(W_L)$ and $C_{R,\zeta} = B_{R,\zeta} \rtimes R[\overline{N}]$. Exactly as above we let $\Gamma_{R,z} = A_{R,\zeta} \otimes_R C_{R,\zeta}$, and let $X_{R,\zeta}$ be the $\Gamma_{R,\zeta}$-module

$A_{R,\zeta} \otimes_{B_{R,\zeta}} C_{R,\zeta}$. Recall that the classification of blocks of $A_{R,\zeta}$, where R is a field, only depends on the multiplicative order of ζ provided that the characteristic of R is either 0 or sufficiently large [FS82, FS89, DJ87, DJ92, GR97].

Proposition 24. *We fix e. Let $\mathfrak{A}$ be a block of $A_{\mathbb{Q}(\zeta_{2e}),\zeta_e}$ and $\mathfrak{C}$ be a block of $C_{\mathbb{Q}(\zeta_{2e}),\zeta_e}$. Suppose that for all sufficiently large primes ℓ there exist a primitive eth root of unity q in $\mathbb{F}_\ell$ and an $(A_{\mathbb{F}_\ell,q}, C_{\mathbb{F}_\ell,q})$-bimodule M_ℓ such that*

(1) *M_ℓ is a direct summand of $X_{\mathbb{F}_\ell,q}$*
(2) *M_ℓ induces a Morita equivalence between the blocks of $A_{\mathbb{F}_\ell,q}$ and $C_{\mathbb{F}_\ell,q}$ that correspond to $\mathfrak{A}$ and $\mathfrak{C}$.*

Then, there exists a direct summand M_0 of $X_{\mathbb{Q}(\zeta_{2e}),\zeta_e}$ inducing a Morita equivalence between $\mathfrak{A}$ and $\mathfrak{C}$.

Example 25. Let $W(X_r)$ be the finite Weyl group of type X_r. Put $\zeta := \sqrt{-1}$, a primitive 4th root of unity. Put $A_{R,\zeta} = \mathcal{H}_{R,\zeta}(W(E_6))$, $B_{R,\zeta} = \mathcal{H}_{R,\zeta}(W(D_4))$, and $C_{R,\zeta} = \mathcal{H}_{R,\zeta,1}(W(F_4)) = B_{R,\zeta} \rtimes R[\mathfrak{S}_3]$ where the parameters of $\mathcal{H}_{R,\zeta,1}(W(F_4))$ are ζ and 1. Define $\Gamma_{R,\zeta}$ and $X_{R,\zeta}$ as above. Suppose that $q \cdot 1_{\mathbb{F}_\ell} \in \mathbb{F}_\ell$, $\ell \mid q^2 + 1$, and ℓ is sufficiently large.[2] Then, by Geck's result on Schur index [Gec03] and the equivalence on blocks for finite Chevalley groups $E_6(q)$ and $D_4(q) \rtimes \mathfrak{S}_3$ ($D_4(q)$ with a triality automorphism group) in [Miy08], there exists an $(A_{\mathbb{F}_\ell,q}, C_{\mathbb{F}_\ell,q})$-bimodule M_ℓ such that

(1) M_ℓ is a direct summand of $X_{\mathbb{F}_\ell,q}$.
(2) M_ℓ induces a Morita equivalence between the principal blocks of $A_{\mathbb{F}_\ell,q}$ and $C_{\mathbb{F}_\ell,q}$.

So, by Proposition 24, we have the corresponding result in characteristic zero.

7 Quantized Enveloping Algebras

7.1 Guessing an Analogue of the Main Theorem

The main theorem suggests an analogous statement for quantized enveloping algebras. Before stating it, we introduce the necessary notation, following Jantzen [Jan03].

Let $\mathfrak{g}$ be a reductive complex Lie algebra. Let k be a commutative ring and q an invertible element of k. Let $U_{q,k} = U_{\mathcal{A}} \otimes_{\mathcal{A}} k$, where $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ and $U_{\mathcal{A}}$ is the divided powers integral form of the quantized enveloping algebra of $\mathfrak{g}$.

[2] Here, the assumption 'ℓ : sufficiently large' is used to make sure that $\mathbb{F}_\ell$ is a splitting field for the principal ℓ-blocks of corresponding finite Chevalley groups $E_6(q)$ and $D_4(q)$. So, we require [Gec03]. The claimed equivalence in finite Chevalley groups does always exist in characteristic $\ell > 3$ such that $\ell \mid q^2 + 1$ by some extension of $\mathbb{F}_\ell$.

Assume that k is a field of characteristic 0, and q is a primitive eth root of unity, where e is odd and 3 does not divide e if $\mathfrak{g}$ has a component of type G_2. Let $U_{q,k}$-mod be the category of finite-dimensional $U_{q,k}$-modules of type I (see [Jan03, p. 523] for definition).

For each dominant weight $\lambda \in X(T)_+$, there is a simple $U_{q,k}$ module $L_q(\lambda)$ of type I with highest weight λ, which is unique up to isomorphism. Every object of $U_{q,k}$-mod has a composition series with factors of the form $L_q(\lambda)$.

Let W_e be the affine Weyl group, acting on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Let

$$\bar{C}_e = \{\lambda \in X(T) \otimes_{\mathbb{Z}} \mathbb{R} \mid 0 \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq e \text{ for all } \alpha \in R^+\}$$

be the closure of the standard e-alcove (see [Jan03, p. 233]); it is a fundamental domain for this action.

Let q' be a primitive e'th root of unity in k, and let $U_{q',k}$ be the corresponding quantum group in obvious fashion. We place the same restrictions on e' as we do on e above, and further we assume $e' \geq e$.

There is an isomorphism $f : W_e \to W_{e'}$ sending $s_{\beta,ne}$ to $s_{\beta,ne'}$ for all $\beta \in R$ and $n \in \mathbb{Z}$. Here, we use the notation $s_{\beta,r}$ of [Jan03, p. 232, 6.1]. Note that the actions of $w \in W_e$ and $f(w) \in W_{e'}$ on $X(T)$ are different!

The inclusion of $\bar{C}_e$ into $\bar{C}_{e'}$ sends some walls of $\bar{C}_e$ into the interior of $\bar{C}_{e'}$, so that affine Weyl group stabilizers are not preserved. To 'correct' this, consider a injective map $\iota : \bar{C}_e \hookrightarrow \bar{C}_{e'}$ with the property that for all $\lambda \in \bar{C}_e$ and $\alpha \in R^+$, we have $\langle \lambda + \rho, \alpha^\vee \rangle = \langle \iota(\lambda) + \rho, \alpha^\vee \rangle$ unless $\langle \lambda + \rho, \alpha^\vee \rangle = e$, in which case $\langle \iota(\lambda) + \rho, \alpha^\vee \rangle = e'$. Such a map always exists and is unique if G is semisimple. Moreover, we can always choose it to be the identity map on the interior of $\bar{C}_e$.

Lemma 26. *Let $\lambda \in X(T) \cap \bar{C}_e$ and $w \in W_e$. Then $w \cdot \lambda \in X(T)_+$ if and only if $f(w) \cdot \iota(\lambda) \in X(T)_+$.*

Guess 27. *There is a full k-linear embedding*

$$F : U_{q,k}\text{-mod} \to U_{q',k}\text{-mod}$$

such that for all $\lambda \in X(T) \cap \bar{C}_e$ and $w \in W_e$ with $w \cdot \lambda \in X(T)_+$, we have

$$F(L_q(w \cdot \lambda)) \cong L_{q'}(f(w) \cdot \iota(\lambda)).$$

The image of F is a sum of blocks of $U_{q',k}$-mod.

7.2 The Case $\mathfrak{gl}_d$

The aim of this section is to confirm that Guess 27 is correct for $\mathfrak{g} = \mathfrak{gl}_d$.

7.2.1 Weights, Abaci and Affine Weyl Group Actions

We begin by making a link, in the $\mathfrak{gl}_d$ case, between the map $w \mapsto f(w)$ appearing above and the James–Mathas operation $\lambda \mapsto \lambda^+$. To this end we describe the dot actions of the affine Weyl group on weights in terms of abaci. We keep the notation in 7.1, taking the usual presentation

$$X(T)_+ = \{\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_d\varepsilon_d \in X(T) \mid \lambda_1 \geq \cdots \geq \lambda_d\}$$

for the set of dominant weights for $\mathfrak{gl}_d$.

The weights lying in the closure of the standard e-alcove are

$$X(T) \cap \bar{C}_e = \{\lambda \in X(T) \mid \ 0 \leq (\lambda_i + d - i) - (\lambda_j + d - j) \leq e \text{ for all } i < j\}.$$

Here, ρ is usually taken to be half the sum of the positive weights. But it does no harm to work with a normalized version that has the same inner products with roots; so we have taken $\rho = (d-1)\varepsilon_1 + (d-2)\varepsilon_2 + \cdots + 0\varepsilon_d$.

We fix a ρ-shifted identification of $X(T)$ with $\mathbb{Z}^d$, sending $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_d\varepsilon_d \in X(T)$ to $(\lambda_1 + d - 1, \lambda_2 + d - 2, \ldots, \lambda_d) \in \mathbb{Z}^d$. This leads to an identification $\bar{C}_e \xrightarrow{\sim} \{\beta \in \mathbb{Z}^d \mid \beta_1 \geq \cdots \geq \beta_d \text{ and } \beta_1 - \beta_d \leq e\}$. In this picture, the action of the affine Weyl group $W_e \cong \mathbb{Z}^{d-1} \rtimes \mathfrak{S}_d$ on $X(T) = \mathbb{Z}^d$ is as follows:

$$\sigma.(\beta_1, \ldots, \beta_d) = (\beta_{\sigma(1)}, \ldots, \beta_{\sigma(d)})$$
$$(m_1, \ldots, m_{d-1}).(\beta_1, \ldots, \beta_d) = (\beta_1 + em_1, \beta_2 + em_2 - em_1, \ldots, \beta_d - em_{d-1})$$

where $\sigma \in \mathfrak{S}_n$, $(m_1, \ldots, m_{d-1}) \in \mathbb{Z}^{d-1}$ and $(\beta_1, \ldots, \beta_d) \in \mathbb{Z}^d$. This is conveniently represented using James's abacus, with e runners and with d beads *labelled* by $1, \ldots, n$. The action of $\mathfrak{S}_d$ is given by permutating the labels, and that of $\mathbb{Z}^{d-1}$ by moving beads up and down runners. From this description, the following lemma, making the combinatorial connection between Theorem 2 and Guess 27, is immediate. We say that a weight $\lambda \in X(T)$ is *polynomial* if λ is dominant and $\lambda_d \geq 0$, i.e. if $(\lambda_1, \ldots, \lambda_d)$ is a partition.

Lemma 28. *Let $\lambda \in X(T) \cap \bar{C}_e$ and $w \in W_e$, and suppose that $w.\lambda = \mu - (me - \lambda_d)(1^d)$ for some polynomial weight μ and some integer m. Then, $f(w).\iota(\lambda) = \mu^{+,e} - (me' - \lambda_d)(1^d)$.*

7.2.2 Proof for $\mathfrak{gl}_d$

Here, we shall confirm Guess 27 for $\mathfrak{g} = \mathfrak{gl}_d$ by appealing to Theorem 2. We denote by $\mathcal{P}_0^e$ the category of polynomial representations over $U_{q,k}(\mathfrak{gl}_d)$, i.e. the full subcategory of $U_{q,k}(\mathfrak{gl}_d)$-mod consisting of modules with composition factors of the form $L_q(\lambda)$ where λ is a polynomial weight.

Tensoring with the representation $\det^{-m} = L(-m,\dots,-m)$ induces a self-equivalence of $U_{q,k}(\mathfrak{gl}_d)$-mod sending $L(\lambda)$ to $L(\lambda - m(1^d))$; denote by $\mathcal{P}^e_m$ the essential image of $\mathcal{P}^e_0$ under this equivalence. Then, we have an exhaustive filtration $\mathcal{P}^e_0 \subset \mathcal{P}^e_1 \subset \cdots$ of $U_{q,k}(\mathfrak{gl}_d)$-mod.

It is known that $\mathcal{P}^e_0$ is equivalent to $\bigoplus_{r=0}^{\infty} \mathcal{S}_{k,q^2}(d,r)$-mod, in a way preserving labels on simple modules. Hence by Theorem 2, there exists a full embedding $G_0 : \mathcal{P}^e_0 \hookrightarrow \mathcal{P}^{e'}_0$ sending $L_q(\mu)$ to $L_{q'}(\mu^+)$ for all polynomial weights μ.

We get for each $m \geq 0$ a corresponding embedding $G_m : \mathcal{P}^e_{me} \to \mathcal{P}^{e'}_{me'}$, such that $G_m(L(\mu - me(1^d))) \cong L(\mu^+ - me'(1^d))$ for any polynomial weight μ. Since $\mu^+ - me'(1^d) = (\mu - me(1^d))^+$, we have a commutative diagram:

$$\begin{array}{rccc} G_m : & \mathcal{P}^e_{me} & \hookrightarrow & \mathcal{P}^{e'}_{me'} \\ & \cup & & \cup \\ G_{m-1} : & \mathcal{P}^e_{(m-1)e} & \hookrightarrow & \mathcal{P}^{e'}_{(m-1)e'}. \end{array}$$

By taking the limit $m \to \infty$, we have a full embedding

$$G := \lim_{m\to\infty} G_m : U_{q,k}(\mathfrak{gl}_d)\text{-mod} = \bigcup_{m=0}^{\infty} \mathcal{P}^e_{me} \to U_{q',k}(\mathfrak{gl}_d)\text{-mod} = \bigcup_{m=0}^{\infty} \mathcal{P}^{e'}_{me'}.$$

This is not quite the functor we want; a slight adjustment is required. By the linkage principle, $U_{q,k}(\mathfrak{gl}_d)\text{-mod} = \bigoplus_{\lambda\in\bar{C}\cap X(T)} \mathcal{M}_{W_e.\lambda}$, where $\mathcal{M}_{W_e.\lambda}$ is the full subcategory consisting of modules with composition factors of the form $L_q(w.\lambda)$. Let Z be the self equivalence of $U_{q,k}(\mathfrak{gl}_d)$-mod whose restriction to $\mathcal{M}_{W_e.\lambda}$ is tensoring with $\det^{\lambda_d}$. One can define a self equivalence Z' of $U_{q',k}(\mathfrak{gl}_d)$-mod analogously.

Let $F := Z'GZ^{-1} : U_{q,k}(\mathfrak{gl}_d)\text{-mod} \to U_{q',k}(\mathfrak{gl}_d)$-mod. Then for each $\lambda \in X(T) \cap \bar{C}_e$, the functor F restricts to an equivalence $\mathcal{M}_{W_e.\lambda} \xrightarrow{\sim} \mathcal{M}_{W_{e'}.\lambda}$ sending $L_q(\mu - (me - \lambda_d)(1^d))$ to $L_{q'}(\mu^+ - (me' - \lambda_d)(1^d))$ for any polynomial weight μ. By Lemma 28, this implies that $F(L_q(w.\lambda)) \cong L_{q'}(f(w).\iota(\lambda))$, as desired.

Appendix:
Module Correspondences in Rouquier Blocks of Finite General Linear Groups[3]

Akihiko Hida and Hyohe Miyachi

Abstract In this chapter we shall consider ℓ-modular represenations of finite general linear groups in non-defining characteristic $\ell > 0$. We focus the nicest ℓ-block algebras in this representation theory, which are also known as unipotent Rouquier blocks.

Let $B_{w,\rho}$ be the unipotent ℓ-block algebra of a general linear group over the finite field with q elements associated with an e-weight $w > 0$ and a Rouquier e-core ρ with respect to w where e is the multiplicative order of q modulo $\ell > 0$. (See the second paragraph of Sect. A for the definition of ρ.)

We assume that $B_{w,\rho}$ has an abelian defect, ie, $\ell > w$. It is known that there exists a Morita equivalence F between $B_{w,\rho}$ and the wreath product block $B_{1,\emptyset} \wr \mathfrak{S}_w$. This result is obtained by W. Turner and the second author independently. The both methods are completely identical each other and are very similar to J. Chuang and R. Kessar's method on symmetric groups.

In this chapter, under the equivalence F we shall determine the explicit correspondences of the simple, the Young and the Specht modules over the Rouquier block $B_{w,\rho}$ and the local subgroup block $B_{1,\emptyset} \wr \mathfrak{S}_w$. The result is used to prove the intial condition of runner removal Morita equivalence theorem.

Keywords Finite general linear groups · Morita equivalence · Green correspondence · Rouquier blocks

Mathematics Subject Classifications (2000): 16D90, 20C05, 20C20, 20G05, 20G40

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[3]The original title of this paper at 7/18/2001 was "Module correspondences in some blocks of finite general linear groups." There were some overlaps between W. Turner's work [Tur02] and the original version of this paper. In this paper, we shall remove the overlaps from the original version of this paper.

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A Introduction

Let ℓ be a prime number. Let $\mathcal{O}$ be a complete discrete valuation ring with fraction field K of characteristic 0 and residue field $\Bbbk = \mathcal{O}/J(\mathcal{O})$ of characteristic ℓ. We assume that K is big enough for all groups in this chapter. Let $GL_n(q)$ be the general linear group over field of q elements, where q is a prime power. We assume that ℓ does not divide q. Let e be the multiplicative order of q in $\Bbbk$. The unipotent blocks of $\mathcal{O}GL_n(q)$ are parametrized by e-weights (semisimple parts) and e-cores (unipotent parts) [FS82]. Let $B_{w,\rho}$ be the unipotent block of $\mathcal{O}GL_n(q)$ with e-weight w and e-core ρ.

Let ρ be an e-core satisfying the following property: ρ has an (e-runner) abacus representation such that $\Gamma_{i-1} + w - 1 \leq \Gamma_i$ where Γ_i is the number of beads on the ith runner [Rou98, CK02]. Let $m = ew + |\rho|$. Let $N_w = (GL_e(q) \wr \mathfrak{S}_w) \times GL_{|\rho|}(q)$. Let f be the block idempotent of $B_0(\mathcal{O}(GL_e(q) \wr \mathfrak{S}_w)) \otimes B_{0,\rho}$, where $B_0(\mathcal{O}(GL_e(q) \wr \mathfrak{S}_w))$ is the principal block of the wreath product $GL_e(q) \wr \mathfrak{S}_w$. Let D be a Sylow ℓ-subgroup of $GL_e(q) \wr \mathfrak{S}_w$. Then, as an $\mathcal{O}(GL_m(q) \times N_w)$-module, $B_{w,\rho} f$ has a unique indecomposable direct summand X_w with vertex $\Delta D = \{(d,d) \mid d \in D\}$. Turner [Tur02] proved the following which is analogous to the result of Chuang and Kesser for symmetric groups [CK02].

Theorem A.1. *If $w < \ell$, then a $(B_{w,\rho}, \mathcal{O}N_w f)$-bimodule X_w induces a Morita equivalence between $\mathcal{O}N_w f$ and $B_{w,\rho}$.*

This result was proved by the second author independently [Miy01].

Note that using this result, combining with [Mar96, Rou95] and [Chu99], one can prove Broué's conjecture [Bro90, Bro92] for weight two unipotent blocks of finite general linear groups very easily.

In this chapter, we consider the correspondences of various modules under the equivalence above. In Sect. 3, we consider simple modules. In Sect. 4, we treat Young modules and Specht type modules. Throughout the chapter, we assume $w < \ell$. The main results of this chapter, namely, Corollary C.3 and Theorem D.9, enable us to calculate not only decomposition numbers of $B_{w,\rho}$ but also the radical series of Specht type modules lying in $B_{w,\rho}$. These graded decomposition numbers are calculated in [Miy01] explicitly in terms of the Littlewood–Richardson coefficients. (Moreover, we can also calculate the Loewy layers for Young modules lying in $B_{w,\rho}$.) The second author conjectured that under the condition "$w < \ell$" these graded decomposition numbers lying in $B_{w,\rho}$ are coincident with crystallized decomposition numbers introduced in [LT96] (see, also [LLT96]). This conjecture is proved by [LM02].

B Modules for the Wreath Product

In this chapter, *modules* always mean finitely generated right modules, unless stated otherwise. Let $\lambda \vdash n$. In [Jam86], James defined a $K[GL_n(q)]$-module (resp. $\Bbbk GL_n(q)$-module) $S_K(\lambda)$ (resp. $S_\Bbbk(\lambda)$). $S_K(\lambda)$ is simple and affords an irreducible unipotent character. On the other hand, $S_\Bbbk(\lambda)$ has a unique simple quotient $D(\lambda)$. Moreover, $\{D(\lambda) \mid \lambda \vdash m,\ e\text{-core of } \lambda \text{ is } \rho\}$ is a complete set of representatives of isomorphic classes of simple $\Bbbk \otimes B_{w,\rho}$-modules [DJ86]. Let $X_\mathcal{O}(\lambda)$ be the Young $\mathcal{O}GL_n(q)$-module corresponding to λ [DJ89]. Then, $X_\mathcal{O}(\lambda)$ is an indecomposable direct summand of a permutation $\mathcal{O}GL_n(q)$-module induced from a parabolic subgroup. For $R \in \{K, \Bbbk\}$, let $X_R(\lambda)$ be $R \otimes_\mathcal{O} X_\mathcal{O}(\lambda)$. Let $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_{e-1})$ be a multipartition of w. Let $R \in \{K, \mathcal{O}, \Bbbk\}$. For each $\boldsymbol{\lambda}_i$, we write $S_R^{\boldsymbol{\lambda}_i}$ for the Specht $R\mathfrak{S}_{|\boldsymbol{\lambda}_i|}$-module corresponding to $\boldsymbol{\lambda}_i$. Note that, since $w < \ell$, $S_\Bbbk^{\boldsymbol{\lambda}_i}$ is simple projective. Hence, there is a unique projective $\mathcal{O}\mathfrak{S}_{|\boldsymbol{\lambda}_i|}$-module $S_\mathcal{O}^{\boldsymbol{\lambda}_i}$ which is a lift of $S_\Bbbk^{\boldsymbol{\lambda}_i}$. Moreover, $S_K^{\boldsymbol{\lambda}_i} = K \otimes S_\mathcal{O}^{\boldsymbol{\lambda}_i}$. Let $\nu_i = (i+1, 1^{e-1-i}) \vdash e$ for $0 \le i \le e-1$. Let T_R be one of X_K, $X_\Bbbk$, S_K, $S_\Bbbk$ and D. Then $T_R(\nu_i)^{\otimes|\boldsymbol{\lambda}_i|} \otimes S_R^{\boldsymbol{\lambda}_i}$ is an $R[GL_e(q) \wr \mathfrak{S}_{|\boldsymbol{\lambda}_i|}]$-module [15]. We set,

$$\tilde{T}_R(\boldsymbol{\lambda}) = \mathrm{Ind}_{GL_e(q)\wr\mathfrak{S}_{\boldsymbol{\lambda}}}^{GL_e(q)\wr\mathfrak{S}_w} \left(\bigotimes_{i=0}^{e-1} T_R(\nu_i)^{\otimes|\boldsymbol{\lambda}_i|} \otimes S_R^{\boldsymbol{\lambda}_i} \right) \otimes S_R(\rho),$$

where $S_\mathcal{O}(\rho)$ is a projective indecomposable $B_{0,\rho}$-module (note that $B_{0,\rho}$ has defect zero), $S_R(\rho) = R \otimes S_\mathcal{O}(\rho)$ is a simple $R[N_w]f$-module for $R = K, \Bbbk$. Then

$$\{\tilde{D}(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} : e\text{-tuple partition of } w\}$$

is a complete set of isomorphism classes of simple $\Bbbk N_w f$-modules.

Let $\mathcal{P}(\rho, w)$ be the set of all partitions of m with e-core ρ. Let $\lambda \in \mathcal{P}(\rho, w)$. Using the abacus representation of ρ mentioned in Introduction, we can consider the e-quotient of λ [15]. We denote the e-quotient of λ by

$$\bar{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(e-1)}).$$

The correspondence

$$\lambda \leftrightarrow \bar{\lambda}$$

gives a bijection between $\mathcal{P}(\rho, w)$ and the set of all e-tuples partitions of w. Hence,

$$\{\tilde{D}(\bar{\lambda}) \mid \lambda \in \mathcal{P}(\rho, w)\}$$

is a complete set of representatives of isomorphism classes of simple $\Bbbk N_w f$-modules.

For $\alpha \in P(\rho, w-1)$, let

$$\Gamma(\alpha, i) = \{\lambda \in \mathcal{P}(\rho, w) \mid \alpha^{(i)} \subseteq \lambda^{(i)},\ \lambda^{(j)} = \alpha^{(j)}\ (i \neq j)\}.$$

On the other hand, let $\mathcal{P}(w)$ be the set of all partitions of w and let $\mathcal{P}_\alpha(w) = \{\lambda \in \mathcal{P}(w) \mid \alpha \subset \lambda\}$ for $\alpha \vdash w-1$. We need the following combinatorial lemma.

Lemma B.1. (1) *Let σ and τ be bijections from $\mathcal{P}(w)$ to $\mathcal{P}(w)$. Suppose $w > 2$. If $\sigma(\mathcal{P}_\alpha(w)) = \tau(\mathcal{P}_\alpha(w))$ for any $\alpha \vdash w-1$, then $\sigma = \tau$.*
(2) *Let σ and τ be bijections from $\mathcal{P}(\rho, w)$ to $\mathcal{P}(\rho, w)$. Suppose $w > 2$. If $\sigma(\Gamma(\alpha, i)) = \tau(\Gamma(\alpha, i))$ for any α, i, then $\sigma = \tau$.*

Proof. (1) Let $\lambda = (\lambda_1, \lambda_2, \dots) \vdash w$. Suppose that $\lambda_i > \lambda_{i+1} \geq \cdots \geq \lambda_j > 0$ for some $i \neq j$. We define two partitions α, β as follows:

$$\alpha = (\alpha_1, \alpha_2, \dots),\ \alpha_i = \lambda_i - 1,\ \alpha_k = \lambda_k\ (k \neq i)$$

$$\beta = (\beta_1, \beta_2, \dots),\ \beta_j = \lambda_j - 1,\ \beta_k = \lambda_k\ (k \neq j).$$

Then $\mathcal{P}_\alpha(w) \cap \mathcal{P}_\beta(w) = \{\lambda\}$, and $\sigma(\lambda) = \tau(\lambda)$.
So we may assume that $\lambda = (\lambda_1^m)$. Let $\gamma = (\lambda_1^{m-1}, \lambda_1 - 1)$. Then $\sigma(\mu) = \tau(\mu)$ for $\mu \in \mathcal{P}_\gamma(w)$ other than λ by the above argument. Hence, we have $\sigma(\lambda) = \tau(\lambda)$.

(2) Let $\lambda \in \mathcal{P}(\rho, w)$. If there exists i such that $\lambda^{(j)} = \emptyset$ for any $j \neq i$, then $\sigma(\lambda) = \tau(\lambda)$ by (1). So we may assume that $\lambda^{(i)} \neq \emptyset$, $\lambda^{(j)} \neq \emptyset$ for some $i \neq j$.
Take $\alpha, \beta \in \mathcal{P}(\rho, w-1)$ satisfying:

$$\alpha^{(i)} \subset \lambda^{(i)},\ \beta^{(j)} \subset \lambda^{(j)}.$$

Then $\Gamma(\alpha, i) \cap \Gamma(\beta, j) = \{\lambda\}$, and $\sigma(\lambda) = \tau(\lambda)$. □

C Simple Modules

Definition

Groups: Let G be $GL_n(q)$ and let G' be a Levi subgroup of G corresponding to a composition $(e, n-e)$, which is isomorphic to $GL_e(q) \times GL_{n-e}(q)$. Let $N = N_w$ (resp. N') be the normalizer of a Levi subgroup corresponding to a composition $(e^w, |\rho|)$ in G (resp. G'), which is isomorphic to $GL_e(q) \wr \mathfrak{S}_w \times GL_{|\rho|}(q)$ (resp. $GL_e(q) \times N_{w-1}$).

Blocks: Let B_0 be the principal block of $\mathcal{O}GL_e(q)$. Let b be the block idempotent of $B_{w,\rho}$. Let f (resp. f') be the the block idempotent of $B_0(\mathcal{O}[GL_e(q) \wr \mathfrak{S}_w]) \otimes B_{0,\rho}$ (resp. $B_0 \otimes B_0(\mathcal{O}[GL_e(q) \wr \mathfrak{S}_{w-1}]) \otimes B_{0,\rho}$). For a block ideal B of $\mathcal{O}H$ for some subgroup H of G, we denote $\Bbbk \otimes B$ by $\bar{B}$.

Functors: For a finite dimensional algebra A, we denote by mod-A the category of finitely generated right A-modules. For any Levi subgroup L of G, we denote by $\mathcal{R}_L^G$ (resp. ${}^*\mathcal{R}_L^G$) the Harish-Chandra induction (resp. restriction) functor for G and L. Let $X_w^\vee$ be $\mathrm{Hom}_{B_{w,\rho}}(X_w, B_{w,\rho})$. Define functors F_w, $(F_w)_K$ and $\bar{F}_w$ as follows:

$$F_w = - \otimes_{\mathcal{O}Nf} X_w^\vee : \text{mod-}\mathcal{O}Nf \longrightarrow \text{mod-}B_{w,\rho}$$

$$(F_w)_K = - \otimes_{K[N]f} (K \otimes X_w^\vee) : \text{mod-}K[N]f \longrightarrow \text{mod-}K \otimes B_{w,\rho}$$

$$\bar{F}_w = - \otimes_{\Bbbk N\bar{f}} \bar{X}_w^\vee : \text{mod-}\Bbbk N\bar{f} \longrightarrow \text{mod-}\bar{B}_{w,\rho}$$

For $F \in \{F_w, (F_w)_K, \bar{F}_w\}$, we denote the inverse of F by F^*.

Lemma C.1. $\mathrm{Res}^{G\times N}_{G'\times N'}(X_w) \cong b\,\mathcal{R}^{G\times N}_{G'\times N'}(B_0 \otimes X_{w-1})$.

Proof. First,

$$\mathrm{Ind}^{G\times N'}_{N'\times N'}(\mathcal{O}N'f') \cong \mathcal{R}^{G\times N'}_{G'\times N'}(B_0 \otimes X_{w-1}) \oplus V,$$

where V is a direct sum of indecomposable modules with vertex not conjugate of ΔD. On the other hand, $\mathrm{Res}^{G\times N}_{G\times N'}(X_w)$ is a direct summand of $\mathrm{Ind}^{G\times N'}_{N'\times N'}(\mathcal{O}N'f')$. Since $\mathrm{Res}^{G\times N}_{G\times N'}(X_w)$ is indecomposable by [Tur02], $\mathrm{Res}^{G\times N}_{G\times N'}(X_w)$ is a direct summand of $\mathcal{R}^{G\times N'}_{G'\times N'}(B_0 \otimes X_{w-1})$. Since

$$(F_w)_K(\mathrm{Ind}^N_{N'}(K[N']f')) \cong b(\mathcal{R}^G_{G'})(\mathrm{id} \otimes (F_{w-1})_K)(K[N']f')$$

by [Tur02],

$$K \otimes X_w \cong b\,\mathcal{R}^{G\times N'}_{G'\times N'}(K \otimes (B_0 \otimes X_{w-1}))$$

as $K[G]$-modules. Hence,

$$\mathrm{Res}^{G\times N}_{G'\times N'}(X_w) \cong b\,\mathcal{R}^{G\times N}_{G'\times N'}(B_0 \otimes X_{w-1}).$$

□

In order to compare the labels of some $\bar{B}_{w,\rho}$-modules with that of some $\Bbbk Nf$-modules, the following definition will be important.

Definition Let $\check{\lambda} \in \mathcal{P}(\rho, w)$ be the partition such that

$$\check{\lambda}^{(i)} = \begin{cases} \lambda^{(i)} & \text{if } e+i \ : \ \text{odd}, \\ \lambda^{(i)\prime} & \text{if } e+i \ : \ \text{even}. \end{cases}$$

Theorem C.2. $(F_w)_K(\tilde{S}_K(\bar{\lambda})) \cong S_K(\check{\lambda})$.

Proof. We proceed by induction on w. The case $w = 2$ is proved in [Tur02]. So we assume $w > 2$. Define a bijection $\sigma : \mathcal{P}(\rho, w) \longrightarrow \mathcal{P}(\rho, w)$ by

$$(F_w)_K(\tilde{S}_K(\bar{\lambda})) \cong S_K(\sigma(\lambda)).$$

It suffices to show that $\sigma(\lambda) = \check{\lambda}$. By the Littlewood–Richardson rule,

$$(F_w)_K(\mathrm{Ind}_{N'}^{N} S_K(\nu_i) \otimes \tilde{S}_K(\bar{\alpha})) = (F_w)_K\left(\bigoplus_{\mu\in\Gamma(\alpha,i)} \tilde{S}_K(\bar{\mu})\right) = \bigoplus_{\mu\in\Gamma(\alpha,i)} S_K(\sigma(\mu))$$

for $\alpha \in \mathcal{P}(\rho, w-1)$ and $0 \le i \le e-1$. On the other hand,

$$\begin{aligned} b\,\mathcal{R}_{G'}^{G}(\mathrm{id} \otimes (F_{w-1})_K)(S_K(\nu_i) \otimes \tilde{S}_K(\bar{\alpha})) &= b\,\mathcal{R}_{G'}^{G}(S_K(\nu_i) \otimes S_K(\check{\alpha})) \\ &= \bigoplus_{\mu\in\Gamma(\check{\alpha},i)} S_K(\mu) = \bigoplus_{\mu\in\Gamma(\alpha,i)} S_K(\check{\mu}) \end{aligned}$$

by induction and [Tur02]. Since these two modules are isomorphic byLemma C.1, we have $\sigma(\Gamma(\alpha,i)) = (\Gamma(\alpha,i))^{\vee}$. Hence $\sigma(\lambda) = \check{\lambda}$ by Lemma B.1. $\square$

Corollary C.3. $\bar{F}_w(\tilde{D}(\bar{\lambda})) \cong D(\check{\lambda})$ *for* $\lambda \in \mathcal{P}(\rho, w)$.

Proof. Let $\lambda \in \mathcal{P}(\rho, w)$. We will show that $\bar{F}_w(\tilde{D}(\bar{\check{\lambda}})) \cong D(\lambda)$ by induction on λ. Suppose that $\bar{F}_w(\tilde{D}(\bar{\check{\mu}})) \cong D(\mu)$ for any $\mu \in \mathcal{P}(\rho, w)$, $\mu > \lambda$. Suppose that $\bar{F}_w(\tilde{D}(\bar{\check{\nu}})) \cong D(\lambda)$ for $\nu \in \mathcal{P}(\rho, w)$. We write $V \leftrightarrow W$ if modules V and W have the same composition factors. Then

$$S_{\Bbbk}(\lambda) \leftrightarrow \left(\bigoplus_{\mu>\lambda} d_{\lambda\mu} D(\mu)\right) \oplus D(\lambda)$$

for some nonnegative integers $d_{\lambda\mu}$. By Theorem C.2,

$$\tilde{S}_{\Bbbk}(\bar{\check{\lambda}}) \leftrightarrow \left(\bigoplus_{\mu>\lambda} d_{\lambda\mu} \tilde{D}(\bar{\check{\mu}})\right) \oplus \tilde{D}(\bar{\check{\nu}}).$$

Since $\tilde{S}_{\Bbbk}(\bar{\check{\lambda}})$ has $\tilde{D}(\bar{\check{\lambda}})$ as a composition factor by definition, we have $\nu = \lambda$. $\square$

D Young Modules

Recall that

$$\tilde{X}_R(\boldsymbol{\lambda}) = \mathrm{Ind}_{GL_e(q)\wr\mathfrak{S}_{\boldsymbol{\lambda}}}^{GL_e(q)\wr\mathfrak{S}_w}\left(\bigotimes_{i=0}^{e-1} X_R(\nu_i)^{\otimes|\lambda_i|} \otimes S_R^{\lambda_i}\right) \otimes S_R(\rho)$$

and $\tilde{S}_R(\boldsymbol{\lambda})$ is a submodule of $\tilde{X}_R(\boldsymbol{\lambda})$ by definition.

Lemma D.1. $[\tilde{X}_{\Bbbk}(\boldsymbol{\lambda}) : \tilde{D}(\boldsymbol{\lambda})] = 1$.

Proof. By Mackey's decomposition theorem, we know

$$\operatorname{Res}^{GL_e(q)\wr\mathfrak{S}_w}_{GL_e(q)^{\times w}} \operatorname{Ind}^{GL_e(q)\wr\mathfrak{S}_w}_{GL_e(q)\wr\mathfrak{S}_{\boldsymbol{\lambda}}}(V) \cong \bigoplus_{g\in[\mathfrak{S}_{\boldsymbol{\lambda}}\backslash\mathfrak{S}_w]} (\operatorname{Res}^{GL_e(q)\wr\mathfrak{S}_{\boldsymbol{\lambda}}}_{GL_e(q)^{\times w}}(V))^g$$

for any $R[GL_e(q)\wr\mathfrak{S}_{\boldsymbol{\lambda}}]$-module V. Here, $[\mathfrak{S}_{\boldsymbol{\lambda}}\backslash\mathfrak{S}_w]$ is a set of the right coset representatives of $\mathfrak{S}_{\boldsymbol{\lambda}}\backslash\mathfrak{S}_w$.

In particular,

$$\operatorname{Res}^{GL_e(q)\wr\mathfrak{S}_w}_{GL_e(q)^{\times w}}(D(\boldsymbol{\lambda}))$$

$$\cong \left(\prod_{i=0}^{e-1}\dim_R S_{\Bbbk}^{\boldsymbol{\lambda}_i}\right)\cdot \bigoplus_{g\in[\mathfrak{S}_{\boldsymbol{\lambda}}\backslash\mathfrak{S}_w]} \left(\bigotimes_{i=0}^{e-1}(D(\nu_i))^{\otimes|\boldsymbol{\lambda}_i|}\right)^g \otimes S_{\Bbbk}(\rho). \tag{1}$$

Comparing the composition factors of $\operatorname{Res}^{GL_e(q)\wr\mathfrak{S}_{\boldsymbol{\lambda}}}_{GL_e(q)^{\times w}} \bigotimes_{i=0}^{e-1} X_{\Bbbk}(\nu_i)^{\otimes|\boldsymbol{\lambda}_i|}\otimes S_{\Bbbk}^{\boldsymbol{\lambda}_i}$ with $\bigotimes_{i=0}^{e-1} D(\nu_i)^{\otimes|\boldsymbol{\lambda}_i|}$, we get

$$\left[\operatorname{Res}^{GL_e(q)\wr\mathfrak{S}_{\boldsymbol{\lambda}}}_{GL_e(q)^{\times w}} \bigotimes_{i=0}^{e-1} X_{\Bbbk}(\nu_i)^{\otimes|\boldsymbol{\lambda}_i|}\otimes S_{\Bbbk}^{\boldsymbol{\lambda}_i} \ : \ \bigotimes_{i=0}^{e-1} D(\nu_i)^{\otimes|\boldsymbol{\lambda}_i|}\right] = \prod_{i=0}^{e-1}\dim_{\Bbbk} S_{\Bbbk}^{\boldsymbol{\lambda}_i}.$$

Therefore, by (1) we have $[\tilde{X}_{\Bbbk}(\boldsymbol{\lambda}) : \tilde{D}(\boldsymbol{\lambda})] = 1$. □

Let $\eta_{\boldsymbol{\lambda}}$ (resp. $\chi_{\boldsymbol{\lambda}}$) be the character afforded by $\tilde{X}_{\mathcal{O}}(\boldsymbol{\lambda})$ (resp. $\tilde{S}_K(\boldsymbol{\lambda})$).

For e-tuple multipartitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, we define a total order $\boldsymbol{\lambda}\succ\boldsymbol{\mu}$ by

$$\boldsymbol{\lambda}\succ\boldsymbol{\mu}, \text{ if there exists } m \text{ such that } \boldsymbol{\lambda}_m > \boldsymbol{\mu}_m \text{ and } \boldsymbol{\lambda}_j = \boldsymbol{\mu}_j \text{ for } j > m.$$

Then, by definition clearly the following holds:

Proposition D.2. $\langle\eta_{\boldsymbol{\lambda}},\chi_{\boldsymbol{\lambda}}\rangle = 1$ *and* $\langle\eta_{\boldsymbol{\lambda}},\chi_{\boldsymbol{\mu}}\rangle = 0$ *if* $\boldsymbol{\lambda}\succ\boldsymbol{\mu}$.

In particular, we know the following:

Corollary D.3. *If* $\boldsymbol{\lambda}\neq\boldsymbol{\mu}$, *then* $\tilde{X}_{\Bbbk}(\boldsymbol{\lambda})\ncong\tilde{X}_{\Bbbk}(\boldsymbol{\mu})$.

Lemma D.4. $\operatorname{Soc}(\tilde{X}_{\Bbbk}(\boldsymbol{\lambda})) = \operatorname{Soc}(\tilde{S}_{\Bbbk}(\boldsymbol{\lambda}))$

Proof. Since the functor $\operatorname{Ind}^{GL_e(q)\wr\mathfrak{S}_w}_{GL_e(q)\wr\mathfrak{S}_{\boldsymbol{\lambda}}}$ preserves the Loewy layers of modules,

$$\operatorname{Soc}(\tilde{X}_{\Bbbk}(\boldsymbol{\lambda})) = \operatorname{Ind}^{GL_e(q)\wr\mathfrak{S}_w}_{GL_e(q)\wr\mathfrak{S}_{\boldsymbol{\lambda}}}\left(\bigotimes_{i=0}^{e-1}(\operatorname{Soc}(X_{\Bbbk}(\nu_i)))^{\otimes|\boldsymbol{\lambda}_i|}\otimes S_{\Bbbk}^{\boldsymbol{\lambda}_i}\right)\otimes S_{\Bbbk}(\rho).$$

Similarly, we have

$$\mathrm{Soc}(\tilde{S}_{\Bbbk}(\boldsymbol{\lambda})) = \mathrm{Ind}_{GL_e(q)\wr\mathfrak{S}_{\boldsymbol{\lambda}}}^{GL_e(q)\wr\mathfrak{S}_w}\left(\bigotimes_{i=0}^{e-1}(\mathrm{Soc}(S_{\Bbbk}(\nu_i)))^{\otimes|\boldsymbol{\lambda}_i|}\otimes S_{\Bbbk}^{\boldsymbol{\lambda}_i}\right)\otimes S_{\Bbbk}(\rho).$$

However, $\mathrm{Soc}(X_{\Bbbk}(\nu_i)) = \mathrm{Soc}(S_{\Bbbk}(\nu_i))$ for $i = 0, \ldots, e-1$. Hence, we are done. □

Since $\tilde{S}_{\Bbbk}(\boldsymbol{\lambda})$ is indecomposable, by Lemma D.1 we immediately know

Corollary D.5. *$\tilde{X}_{\Bbbk}(\boldsymbol{\lambda})$ is indecomposable.*

Lemma D.6. *$\bar{F}_w^*(X_{\Bbbk}(\lambda)) \cong \tilde{X}_{\Bbbk}(\bar{\nu})$ for some $\nu \in \mathcal{P}(\rho, w)$.*

Proof. Let $L := L_{(e^w,\rho)}$ be the Levi subgroup of G corresponding to the composition $(e^w, \rho) \models m$. $\mathrm{Res}_L^N(\bar{F}_w^*(V))$ is a direct summand of $b_w \cdot {}^*\mathcal{R}_L^G(V)$ for any $\bar{B}_{w,\rho}$-module V. Any indecomposable direct summand of $\mathrm{Res}_L^N(\bar{F}_w^*(X_{\Bbbk}(\lambda)))$ has the following shape:

$$\bigotimes_{i=1}^{w} X(\nu_{m(i)}) \otimes S_{\Bbbk}(\rho).$$

Here, $m(i)$ is an element of $\{0, 1, 2, \ldots, e-1\}$ for any $i \in \{1, 2, \ldots, w\}$. Hence, any indecomposable direct summand of $\bar{F}_w^*(X_{\Bbbk}(\lambda))$ is a direct summand of

$$\left(\mathrm{Ind}_L^N\left(\bigotimes_{i=0}^{e-1} X(\nu_i)^{\otimes n_i}\right)\right)\otimes S_{\Bbbk}(\rho)$$

for some $(n_0, n_1, \ldots, n_{e-1}) \models w$. Since $\bar{F}_w^*$ is an equivalence, we get

$$\bar{F}_w^*(X_{\Bbbk}(\lambda)) \cong \tilde{X}_{\Bbbk}(\nu) \text{ for some } \nu.$$

□

Theorem D.7. *$\bar{F}_w(\tilde{X}_{\Bbbk}(\bar{\lambda})) \cong X_{\Bbbk}(\check{\lambda})$ for any $\lambda \in \mathcal{P}(\rho, w)$.*

Proof. It suffices to show that

$$\bar{F}_w^*(X_{\Bbbk}(\lambda)) \cong \tilde{X}_{\Bbbk}(\bar{\check{\lambda}}) \tag{2}$$

for any $\lambda \in \mathcal{P}(\rho, w)$. We proceed by induction on $(>, \mathcal{P}(\rho, w))$. By Corollary C.3, we have already shown (2) for the maximal element of $\mathcal{P}(\rho, w)$. Suppose that the claim (2) holds for any $\mu \gneq \lambda$. By Lemma D.6, there exists $\nu \in \mathcal{P}(\rho, w)$ such that

$$\bar{F}_w^*(X_{\Bbbk}(\lambda)) \cong \tilde{X}_{\Bbbk}(\bar{\check{\nu}}).$$

In particular, $\tilde{X}_{\Bbbk}(\bar{\check{\nu}})$ must have $\tilde{D}(\bar{\check{\nu}})$ as a composition factor. So, we deduce by Corollary C.3 that $X_{\Bbbk}(\lambda) \cong \bar{F}_w(\tilde{X}_{\Bbbk}(\bar{\check{\nu}}))$ must have $D(\nu) \cong \bar{F}_w(\tilde{D}(\bar{\check{\nu}}))$ as a composition factor. Hence, by [Jam84, 16.3] we deduce $\nu \trianglerighteq \lambda$. Suppose that $\nu \gneqq \lambda$. By the assumption of induction, we have

$$\bar{F}_w^*(X_{\Bbbk}(\nu)) \cong \tilde{X}_{\Bbbk}(\check{\bar{\nu}}).$$

In other words, we have

$$\bar{F}_w^*(X_{\Bbbk}(\nu)) \cong \bar{F}_w^*(X_{\Bbbk}(\lambda)).$$

Therefore, we have $\nu = \lambda$ and get a contradiction. □

Let $P(\lambda)$ (resp. $\tilde{P}(\bar{\lambda})$) be the projective indecomposable module corresponding to $D(\lambda)$ (resp. $\tilde{D}(\bar{\lambda})$). Then, Specht type modules $S_{\Bbbk}(\lambda)$, $\tilde{S}_{\Bbbk}(\bar{\lambda})$ and Young modules $X_{\Bbbk}(\lambda)$, $\tilde{X}_{\Bbbk}(\bar{\lambda})$ enjoy the following properties:

Lemma D.8. *(1) If* $\tau \in \mathrm{Hom}_{\Bbbk G}(P(\lambda), X_{\Bbbk}(\lambda))$ *satisfies* $\mathrm{Top}(\tau(P(\lambda))) \cong D(\lambda)$, *then* $\tau(P(\lambda)) \cong S_{\Bbbk}(\lambda)$.
(2) If $\psi \in \mathrm{Hom}_{\Bbbk N}(\tilde{P}(\bar{\lambda}), \tilde{X}_{\Bbbk}(\bar{\lambda}))$ *satisfies* $\mathrm{Top}(\psi(\tilde{P}(\bar{\lambda}))) \cong \tilde{D}(\bar{\lambda})$, *then* $\psi(\tilde{P}(\bar{\lambda})) \cong \tilde{S}_{\Bbbk}(\bar{\lambda})$.

Proof. (1) is clear by [DJ89]. (2): By Lemma D.1, we have already known $[\tilde{X}_{\Bbbk}(\bar{\lambda}) : \tilde{D}(\bar{\lambda})] = 1$. Moreover, $\tilde{S}_{\Bbbk}(\bar{\lambda})$ is a submodule of $\tilde{X}_{\Bbbk}(\bar{\lambda})$ and have its unique top $\tilde{D}(\bar{\lambda})$. So, $\psi(\tilde{P}(\bar{\lambda}))$ must be isomorphic to $\tilde{S}_{\Bbbk}(\bar{\lambda})$. □

By Corollary C.3, Theorem D.7 and Lemma D.8, we have

Theorem D.9. $\bar{F}_w(\tilde{S}_{\Bbbk}(\bar{\lambda})) \cong S_{\Bbbk}(\check{\lambda})$ *for any* $\lambda \in \mathcal{P}(\rho, w)$.

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Quantum $\mathfrak{gl}_n$, q-Schur Algebras and Their Infinite/Infinitesimal Counterparts

Jie Du and Qiang Fu

Abstract We present a survey of recent developments of the Beilinson–Lusztig–MacPherson approach in the study of quantum $\mathfrak{gl}_n$, infinitesimal quantum $\mathfrak{gl}_n$, quantum $\mathfrak{gl}_\infty$ and their associated q-Schur algebras, little q-Schur algebras and infinite q-Schur algebras. We also use the relationship between quantum $\mathfrak{gl}_\infty$ and infinite q-Schur algebras to discuss their representations.

Keywords Quantum group · Quantum Schur algebras · Infinite q-Schur algebras · Representations

Mathematics Subject Classifications (2000): 17B20, 17B35, 20G15

1 Introduction

Almost at the same time when Ringel discovered the Hall algebra realization [24] of the positive part of the quantum enveloping algebras associated with finite dimensional semisimple complex Lie algebras, Beilinson, Lusztig and MacPherson discovered a realization [1] for the entire quantum $\mathfrak{gl}_n$ via a geometric setting of quantum Schur algebras (or q-Schur algebras). This remarkable work has many applications. For example, it provides a crude model for the introduction of modified quantum groups as introduced in [21], it leads to the settlement of the integral Schur–Weyl reciprocity and, hence, the reciprocity at any root of unity [6, 10], and it has also provided a geometric approach to study quantum affine $\mathfrak{gl}_n$ [14, 22]. The BLM work has also been used to investigate the presentations of q-Schur algebras [9], infinitesimal quantum $\mathfrak{gl}_n$ and their associated little q-Schur algebras [11], and quantum $\mathfrak{gl}_\infty$, infinite q-Schur algebras and their representations [7]. This chapter presents a brief account of these developments.

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We organize the chapter as follows. Section 2 collects the definitions of quantum $\mathfrak{gl}_\eta$ for any consecutive (finite or infinite) segment η of $\mathbb{Z}$, the q-Schur algebras and the infinitesimal quantum $\mathfrak{gl}_\eta$. In Sect. 3, we generalize the geometric setting in [1] for q-Schur algebras to introduce the algebras $\mathcal{K}(\eta, r)$, for any η, and discuss the stabilization property for $\mathcal{K}(\eta, r)$. This property allows us to define a new algebra $\boldsymbol{\mathcal{K}}(\eta)$ over $\mathbb{Q}(v)$ of which a certain completion $\widehat{\boldsymbol{\mathcal{K}}}(\eta)$ of $\boldsymbol{\mathcal{K}}(\eta)$ contains a subalgebra isomorphic to the quantum group $\mathbf{U}(\eta) = \mathbf{U}_v(\mathfrak{gl}_\eta)$. This is the BLM realization which is discussed in Sect. 4. When η is an infinite segment, the completion $\widehat{\boldsymbol{\mathcal{K}}}(\eta, r)$ of $\boldsymbol{\mathcal{K}}(\eta, r)$ contains a subalgebra $\mathbf{V}(\eta, r)$ which is isomorphic to a homomorphic image $\mathbf{U}(\eta, r)$ of $\mathbf{U}(\eta)$. In Sect. 5, we use the integral version $\mathcal{K}(\eta)$ of $\boldsymbol{\mathcal{K}}(\eta)$ to obtain its specialization $\mathcal{K}(\eta)_k$ at a root of unity, and then to construct a subalgebra $\mathcal{W}$ of the completion $\widehat{\mathcal{K}}(\eta)_k$ of $\mathcal{K}(\eta)_k$. The algebra $\mathcal{W}$ is isomorphic to the infinitesimal quantum $\mathfrak{gl}_\eta$. From Sect. 6 onwards, we present various applications. In Sect. 6, q-Schur algebras are investigated via their presentations. In particular, we display several bases and mention a nice application to Hecke algebras. Parallel to the infinitesimal theory for quantum $\mathfrak{gl}_n$, little q-Schur algebras are discussed in Sect. 7. For the rest of the chapter, we focus on the case when $\eta = \mathbb{Z}$. In Sect. 8, we discuss presentations of the algebras $\mathbf{U}(\infty, r)$. Infinite q-Schur algebras $\boldsymbol{\mathcal{S}}(\infty, r)$ are introduced and the relations between $\boldsymbol{\mathcal{S}}(\infty, r)$ and quantum $\mathfrak{gl}_\infty$ are discussed in Sect. 9. In particular, we show that $\mathbf{U}(\infty, r)$ is a proper subalgebra of $\boldsymbol{\mathcal{S}}(\infty, r)$. We end the chapter by discussions on the representation theory of quantum $\mathfrak{gl}_\infty$. We discuss the highest weight representations in Sect. 10 and their polynomial type representations in Sect. 11.

2 Quantum $\mathfrak{gl}_\eta$, Infinitesimal Quantum $\mathfrak{gl}_\eta$ and q-Schur Algebras

Let $\eta \subseteq \mathbb{Z}$ be a consecutive segment of $\mathbb{Z}$. In other words, η is a subset of the form $[m, n] := \{i \in \mathbb{Z} \mid m \leqslant i \leqslant n\}$ or one of the sets $(-\infty, m], [n, \infty)$ and $\mathbb{Z}$, where $m, n \in \mathbb{Z}$. Let $\eta^{\dashv} = \eta \setminus \{\max(\eta)\}$ if η has an upper bound, and $\eta^{\dashv} = \eta$ otherwise. We also denote by $M_\eta(R)$ (resp. R^η) the set of all matrices $(a_{i,j})_{i,j\in\eta}$ (resp. all sequences $(a_i)_{i\in\eta}$) with all entries in a set R of numbers, and will always abbreviate the sub-/supscript η by n if $\eta = [1, n]$, and by ∞ if $\eta = \mathbb{Z}$. Moreover, we also assume that if η is infinite, then the elements in $M_\eta(R)$ (resp. R^η) have *finite support*, i.e. all $a_{i,j} = 0$ (resp. $a_i = 0$) but finitely many of them.

Let $\mathbf{U}(\eta) := \mathbf{U}_v(\mathfrak{gl}_\eta)$ be quantum $\mathfrak{gl}_\eta$ defined over $\mathbb{Q}(v)$. Then, $\mathbf{U}(\eta)$ is the algebra over $\mathbb{Q}(\upsilon)$ presented by generators

$$E_i,\ F_i \quad (i \in \eta^{\dashv}),\ K_j,\ K_j^{-1} \quad (j \in \eta)$$

and relations

(a) $K_i K_j = K_j K_i,\ K_i K_i^{-1} = 1;$
(b) $K_i E_j = \upsilon^{\delta_{i,j} - \delta_{i,j+1}} E_j K_i;$

(c) $K_i F_j = \upsilon^{\delta_{i,j+1}-\delta_{i,j}} F_j K_i$;
(d) $E_i E_j = E_j E_i$, $F_i F_j = F_j F_i$ when $|i-j|>1$;
(e) $E_i F_j - F_j E_i = \delta_{i,j} \frac{\widetilde{K}_i - \widetilde{K}_i^{-1}}{\upsilon - \upsilon^{-1}}$, where $\widetilde{K}_i = K_i K_{i+1}^{-1}$;
(f) $E_i^2 E_j - (\upsilon+\upsilon^{-1}) E_i E_j E_i + E_j E_i^2 = 0$ when $|i-j|=1$;
(g) $F_i^2 F_j - (\upsilon+\upsilon^{-1}) F_i F_j F_i + F_j F_i^2 = 0$ when $|i-j|=1$.

When η is an infinite segment, $\mathbf{U}(\eta)$ can be regarded as the quantum enveloping algebra associated with an $\infty\times\infty$ Cartan matrix.

We also set

$$\mathbf{U}(\eta) = \begin{cases} \mathbf{U}(n), & \text{if } \eta = [1,n]; \\ \mathbf{U}(\infty), & \text{if } \eta = \mathbb{Z}. \end{cases}$$

Clearly, we have natural algebra embedding $\mathbf{U}([-n,n]) \subseteq \mathbf{U}([-n-1,n+1])$ for all $n \geqslant 0$. Hence, obtain an algebra isomorphism

$$\mathbf{U}(\infty) \cong \varinjlim_n \mathbf{U}([-n,n]). \tag{2.0.1}$$

We will see in Sect. 4 that $\mathbf{U}(\eta)$ can be reconstructed as a vector space together with a given basis and certain explicit multiplication formulas between basis elements and generators.

Let $\mathcal{Z} = \mathbb{Z}[v,v^{-1}]$ and let $U(\eta)^0$ (resp. $U(\eta)^+$, $U(\eta)^-$) be the $\mathcal{Z}$-subalgebra of $\mathbf{U}(\eta)$ generated by all K_j, K_j^{-1}, $\begin{bmatrix} K_j;c \\ t \end{bmatrix}$ (resp. $E_i^{(m)}$, $F_i^{(m)}$), where

$$E_i^{(m)} = \frac{E_i^m}{[m]^!},\ F_i^{(m)} = \frac{F_i^m}{[m]^!},\ \text{and} \begin{bmatrix} K_j;c \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{K_j \upsilon^{c-s+1} - K_j^{-1}\upsilon^{-c+s-1}}{\upsilon^s - \upsilon^{-s}}$$

with $[m]^! = [1][2]\cdots[m]$ and $[t] = \frac{\upsilon^t - \upsilon^{-t}}{\upsilon - \upsilon^{-1}}$. Also, let $U(\eta) = U(\eta)^+ U(\eta)^0 U(\eta)^-$. By [25, Sect. 3] and [20, 2.3 (g9)(g10)], all $\begin{bmatrix} \tilde{K}_j;c \\ t \end{bmatrix} \in U(\eta)$. Thus, $U(\eta)$ is a $\mathcal{Z}$-subalgebra of $\mathbf{U}(\eta)$ and there is a triangular decomposition

$$U(\eta) = U(\eta)^+ \otimes U(\eta)^0 \otimes U(\eta)^-.$$

Let Ω_η be a free $\mathcal{Z}$-module with basis $\{\omega_i\}_{i\in\eta}$. Let $\mathbf{\Omega}_\eta = \Omega_\eta \otimes \mathbb{Q}(v)$. Then $\mathbf{U}(\eta)$ acts on $\mathbf{\Omega}_\eta$ naturally defined by

$$K_a\omega_b = \upsilon^{\delta_{a,b}}\omega_b\ (a,b\in\eta),\ E_a\omega_b = \delta_{a+1,b}\omega_a,\ F_a\omega_b = \delta_{a,b}\omega_{a+1}\ (a\in\eta^{\dashv}, b\in\eta).$$

This action extends to the tensor space $\mathbf{\Omega}_\eta^{\otimes r}$ ($r\geqslant 1$) via the coalgebra structure Δ on $\mathbf{U}(\eta)$ defined by

$$\Delta(E_i) = E_i \otimes \tilde{K}_i + 1 \otimes E_i,\ \Delta(F_i) = F_i \otimes 1 + \tilde{K}_i^{-1} \otimes F_i,\ \Delta(K_j) = K_j \otimes K_j.$$

Thus, we have a $\mathbb{Q}(v)$-algebra homomorphism

$$\zeta_r : \mathbf{U}(\eta) \to \mathrm{End}(\boldsymbol{\Omega}_\eta^{\otimes r}). \tag{2.0.2}$$

Restriction induces a $\mathcal{Z}$-algebra homomorphism

$$\zeta_r|_{U(\eta)} : U(\eta) \to \mathrm{End}(\Omega_\eta^{\otimes r}). \tag{2.0.3}$$

Let

$$\mathbf{U}(\eta, r) = \mathrm{im}(\zeta_r) \quad \text{and} \quad U(\eta, r) = \mathrm{im}(\zeta_r|_{U(\eta)}). \tag{2.0.4}$$

If η is *finite*, both $\mathbf{U}(\eta, r)$ and $U(\eta, r)$ are called *q-Schur algebras at η*; see [16]. Note that q-Schur algebras can be described as the endomorphism of the tensor space regarded as a module of the Hecke algebra associated with the symmetric groups on r letters (see Sect. 9 for more details).

Let $U(\eta)_k = U(\eta) \otimes_{\mathcal{Z}} k$, where k is a field containing an l'th primitive root ε of 1 ($l' \geqslant 3$), and is regarded as an $\mathcal{Z}$-module via the specialization of υ to ε. We will denote the images of $E_i^{(m)} \otimes 1$, etc. in $U(\eta)_k$ by the same letters. Clearly, (2.0.1) induces a k-algebra isomorphism

$$U(\infty)_k = \varinjlim_n U([-n, n])_k \tag{2.0.5}$$

We now follow [20] to introduce infinitesimal quantum $\mathfrak{gl}_\eta$. Let $\tilde{u}_k(\eta)$ be the k-subalgebra of $U(\eta)_k$ generated by the elements E_i, F_i, $K_j^{\pm 1}$ for all i, j, and define $\tilde{u}_k(\eta)^+$, $\tilde{u}_k(\eta)^0$ and $\tilde{u}_k(\eta)^-$. Similarly, we have an inherited triangular decomposition

$$\tilde{u}_k(\eta) = \tilde{u}_k(\eta)^+ \tilde{u}_k(\eta)^0 \tilde{u}_k(\eta)^- \cong \tilde{u}_k(\eta)^+ \otimes \tilde{u}_k(\eta)^0 \otimes \tilde{u}_k(\eta)^-.$$

Moreover, we have

$$\tilde{u}_k(\infty) = \varinjlim_n \tilde{u}_k([-n, n]) \tag{2.0.6}$$

We may further introduce the so-called *infinitesimal quantum group $u_k(\eta)$* when l' is odd. Let

$$l = \begin{cases} l', & \text{if } l' \text{ is odd} \\ l'/2, & \text{if } l' \text{ is even} \end{cases}. \tag{2.0.7}$$

Thus, $\varepsilon^l - \varepsilon^{-l} = 0$ and further $E_i^l = 0 = F_i^l$ and $K_j^{2l} = 1$ in $U(\eta)_k$. *If l' is odd*, then the elements $K_i^l - 1$ ($i \in \eta$) are central in $\tilde{u}_k(\eta)$ (and in $U(\eta)_k$). These elements generate an ideal $\langle K_i^l - 1 \mid i \in \eta \rangle$ of $\tilde{u}_k(\eta)$. Define

$$u_k(\eta) = \tilde{u}_k(\eta) / \langle K_i^l - 1 \mid i \in \eta \rangle.$$

Call $u_k(\eta)$ the *infinitesimal quantum* $\mathfrak{gl}_\eta$.

An alternative way to see the algebra $\tilde{u}_k(\eta)$ and $u_k(\eta)$ is using the De Concini–Kac quantum group (see [3]). Let $\mathcal{U}_k(\eta)$ be the algebra over k with generators E_i, F_i and $K_j^{\pm 1}$ $(i \in \eta^{\dashv}, j \in \eta)$ and the relations (a)–(g) where υ is replaced by ε (noting our assumption on l' in the definition of $u_k(\eta)$). This algebra is a $\mathfrak{gl}_\eta$ version of a De Concini–Kac quantum group. Clearly, there is an algebra homomorphism $\delta : \mathcal{U}_k(\eta) \to U(\eta)_k$ mapping the generators of $\mathcal{U}_k(\eta)$ to their counterparts in $U(\eta)_k$. The image of δ is the algebra $\tilde{u}_k(\eta)$.

The following result is given in [11, 2.5] and [13, 2.1] (the proof for an infinite η is entirely similar).

Proposition 2.1. (1) *Let I be the* (*two-sided*) *ideal of $\mathcal{U}_k(\eta)$ generated by $E_i^l,\ F_i^l,\ K_j^{2l} - 1,\ i \in \eta^{\dashv},\ j \in \eta$. Then there is an algebra isomorphism*

$$\mathcal{U}_k(\eta)/I \ \xrightarrow{\sim} \tilde{u}_k(\eta).$$

In other words, $\tilde{u}_k(\eta)$ is the k-algebra defined by generators

$$E_i,\ F_i,\ K_j\ (i \in \eta^{\dashv},\ j \in \eta)$$

and relations (a)–(g) *with υ replaced by ε, together with the relations:*
($\tilde{\mathrm{h}}$) $E_i^l = 0,\ F_i^l = 0$ *and* $K_j^{2l} = 1$.

(2) *If l' be odd, then $u_k(\eta)$ is the k-algebra defined by generators*

$$E_i,\ F_i,\ K_j\ (i \in \eta^{\dashv},\ j \in \eta)$$

and relations (a)–(g) *with υ replaced by ε, together with the relations:*
(h) $E_i^l = 0,\ F_i^l = 0$ *and* $K_j^l = 1$.

In Sect. 5, we will discuss the realizations of $\tilde{u}_k(\eta)$ and $u_k(\eta)$.

3 The Algebra $\mathcal{K}(\eta, r)$ and the Stabilization Property

For a matrix $A = (a_{ij}) \in M_\eta(\mathbb{N})$ (resp. $\mathbf{j} = (j_i) \in \mathbb{N}^\eta$), let $|A| = \sum_{i,j} a_{ij}$ (resp. $|\mathbf{j}| = \sum_i j_i$). Let $\Xi(\eta) = M_\eta(\mathbb{N})$, $\Xi(\eta, r) = \{A \in \Xi(\eta) \mid r = |A|\}$ and $\widetilde{\Xi}(\eta) = \{(a_{ij}) \in M_\eta(\mathbb{Z}) \mid a_{ij} \geqslant 0\ \forall i \neq j\}$.

Let V be a vector space of dimension r over a field k. An η-step flag is a collection $\mathfrak{f} = (V_i)_{i \in \eta}$ of subspaces of V such that $V = \bigcup_{i \in \eta} V_i$, $V_i \subseteq V_{i+1}$ for all $i \in \eta^{\dashv}$ and $V_i = 0$ for $i \ll 0$ if η has no lower bound. Let $\mathcal{F}$ be the set of η-step flags. The group $G = G(k) := GL(V)$ acts naturally on $\mathcal{F}$ with orbits being the fibres of the map $\mathcal{F} \to \Lambda(\eta, r) := \{\lambda \in \mathbb{N}^\eta \mid r = |\lambda|\}$ given by

$$(V_i)_{i \in \eta} \mapsto (\dim V_i / V_{i-1})_{i \in \eta},$$

where $V_{\min(\eta)-1} = 0$ if η is bounded below. If $\mathcal{F}_\lambda$ denotes the inverse image of $\lambda \in \Lambda(\eta, r)$, then $\mathcal{F} = \bigcup_{\lambda\in\Lambda(\eta,r)} \mathcal{F}_\lambda$, a disjoint union of G-orbits. If $\mathfrak{f} \in \mathcal{F}_\lambda$ and P_λ is the stabilizer of $\mathfrak{f}$ in G, then $\mathcal{F}_\lambda \cong G/P_\lambda$.

Let $\mathcal{V} = \mathcal{V}(k) = \mathcal{F} \times \mathcal{F}$ and let G act on $\mathcal{V}$ diagonally. For $(\mathfrak{f}, \mathfrak{f}') \in \mathcal{V}$ where $\mathfrak{f} = (V_i)_{i\in\eta}$ and $\mathfrak{f}' = (V_i')_{i\in\eta}$, the subspaces

$$X_{i,j} = X_{i,j}(\mathfrak{f}, \mathfrak{f}') = V_{i-1} + (V_i \cap V_j')$$

(where $i, j \in \eta$) form an $(\eta \times \eta)$-flag:

$$\cdots \subseteq X_{i,j} \subseteq X_{i,j+1} \subseteq \cdots \subseteq X_{i+1,j} \subseteq X_{i+1,j+1} \subseteq \cdots \subseteq V.$$

Let $a_{i,j} = \dim X_{i,j}/X_{i,j-1}$. Setting $\Psi(\mathfrak{f}, \mathfrak{f}') = (a_{i,j})$ defines a map $\Psi : \mathcal{V} \to \Xi(\eta)$. Then $\Xi(\eta, r) = \operatorname{im}\Psi$. The G-orbits $\mathcal{O}_A$, $A \in \Xi(\eta, r)$, in $\mathcal{V}$ are the fibres of Ψ.

When $k = \mathbb{F}_q$ is a finite field of q elements, then the action of $G(q) := G(k)$ on $\mathcal{V}(q) := \mathcal{V}(k)$ induces a permutation module $\mathbb{Z}\mathcal{V}(q)$. Let $\mathcal{E}_{\eta,r}(q)$ be the $\mathbb{Z}$-algebra with basis $\{e_A\}_{A\in\Xi(\eta,r)}$ and multiplication: for $A, A' \in \Xi(\eta, r)$,

$$e_A \cdot e_{A'} = \sum_{A''\in\Xi(\eta,r)} c_{A,A',A''} e_{A''}$$

where, for fixed $(\mathfrak{f}_1, \mathfrak{f}_2) \in \mathcal{O}_{A''}$,

$$c_{A,A',A''} = \{\mathfrak{f} \in \mathcal{V}(q) \mid (\mathfrak{f}_1, \mathfrak{f}) \in \mathcal{O}_A, (\mathfrak{f}, \mathfrak{f}_2) \in \mathcal{O}_{A'}\}.$$

Note that if η is finite, then

$$\mathcal{E}_{\eta,r}(q) = \operatorname{End}_{\mathbb{Z}G(q)}(\mathbb{Z}\mathcal{V}(q))^{\mathrm{op}}.$$

Thus, $\mathcal{E}_{\eta,r}(q)$ has an identity. This is not the case if η is infinite.

It is well known that there are polynomials $f_{A,A',A''}$ such that $f_{A,A',A''}(q) = c_{A,A',A''}$ for all $A, A', A'' \in \Xi(\eta, r)$. Thus, we define a $\mathcal{Z}$-algebra $\mathcal{K}(\eta, r)$ with basis $\{\phi_B\}_{B\in\Xi(\eta,r)}$ and multiplication

$$\phi_A \cdot \phi_{A'} = \sum_{A''} f_{A,A',A''}(v^2)\phi_{A''}.$$

For any $n \geqslant 1$, we may embed $\Xi([-n, n], r)$ into $\Xi([-n-1, n+1], r)$ by adding zeros in rows and columns labelled with $-n-1$ or $n+1$. This induces an algebra embedding $\mathcal{K}([-n, n], r)$ into $\mathcal{K}([-n-1, n+1], r)$ as a centralizer subalgebra $e\mathcal{K}([-n-1, n+1], r)e$ for an idempotent e. Thus, we obtain a direct system $\{\mathcal{K}([-n, n], r)\}_{n\geqslant 1}$.

Lemma 3.1. (1) *Let* $\mathbb{Z}[\sqrt{q}, \sqrt{q}^{-1}]$ *be the subring of* $\mathbb{C}$ *generated by* $\mathbb{Z}$ *and* $\sqrt{q}, \sqrt{q}^{-1}$, *and let* $\mathcal{K}(\eta, r)|_{\sqrt{q}}$ *be the specialization of* $\mathcal{K}(\eta, r)$ *at* $v = \sqrt{q}$. *Then we have an algebra isomorphism*

$$\mathcal{K}(\eta, r)|_{\sqrt{q}} \cong \mathcal{E}_{\eta,r}(q) \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{q}, \sqrt{q}^{-1}].$$

(2) *We have an algebra isomorphism:*

$$\mathcal{K}(\infty, r) = \varinjlim_{n} \mathcal{K}([-n, n], r).$$

For any $A = (a_{i,j}) \in \widetilde{\Xi}(\eta)$, let $\mathrm{row}\,(A) = \left(\sum_{j\in\eta} a_{i,j}\right)_{i\in\eta}$ and $\mathrm{col}\,(A) = \left(\sum_{i\in\eta} a_{i,j}\right)_{j\in\eta}$ be the sequences of row and column sums of A. Note that $\mathrm{row}\,(A)$ and $\mathrm{col}\,(A)$ are well defined for any $A \in M_\infty(\mathbb{Z})$ since A has a finite support.

Let $\mathcal{Z}_1$ be the subring of $\mathbb{Q}(v)[v']$ generated by v^j $(j \in \mathbb{Z})$ and the elements

$$\prod_{i\in[1,t]} \frac{v^{-2(a-i)}v'^2 - 1}{v^{-2i} - 1}$$

for all $a \in \mathbb{Z}$ and $t \geqslant 1$. For $n \geqslant n_0$ and $A \in \Xi([-n_0, n_0])$, define $A^{[-n,n]} \in \Xi([-n, n])$ by adding zeros at the (i, j)-position for all $i < -n_0$ or $i > n_0$ or $j < -n_0$ or $j > n_0$.

The following stabilization property is a slight modification of the result [1, 4.2] for q-Schur algebras; see [7, Sect. 3] for more details.

Theorem 3.2. *For* $B \in \Xi(\eta, r)$ *let* $d(B) = \sum_{i\geqslant k, j<l} b_{ij}b_{kl}$. *The basis* $\{[B] := v^{-d_B}\phi_B\}_{B\in\Xi(\eta,r)}$ *for* $\mathcal{K}(\eta, r)$ *satisfies the following property: if* $A, B \in \widetilde{\Xi}([-n_0, n_0])$ *for some* $n_0 \geqslant 1$ *with* $\mathrm{col}\,(A) = \mathrm{row}\,(B)$, *then there exist* $g_{A,B,C}(v, v') \in \mathcal{Z}_1$ *with* $C \in \widetilde{\Xi}([-n_0, n_0])$ *such that*

$$[{}_pA^{[-n,n]}] \cdot [{}_pB^{[-n,n]}] = \sum_{C\in\widetilde{\Xi}([-n_0,n_0])} g_{A,B,C}(v, v^{-p})[{}_pC^{[-n,n]}]$$

for all large p *and* $n \geqslant n_0$, *where* ${}_pC^{[-n,n]} = pI_{[-n,n]} + C^{[-n,n]}$ *and* $I_{[-n,n]}$ *is the identity matrix in* $\Xi[-n, n]$.

4 The BLM Construction of $\mathbf{U}(\eta)$

Beilinson–Lusztig–MacPherson [1] used the stabilization property of the q-Schur algebras to define an algebra over $\mathcal{Z}_1$ with a basis indexed by $\widetilde{\Xi}(n)$, and then, by specializing v' to 1, to obtain an algebra $\mathcal{K}$ free over $\mathcal{Z}$ with a basis $\{[A]\}_{A\in\widetilde{\Xi}(n)}$. The quantum group $\mathbf{U}(n)$ is realized as a subalgebra of a certain completion of $\mathbf{K} := \mathcal{K} \otimes \mathbb{Q}(v)$. In this section, we state the result with respect to a consecutive segment η of $\mathbb{Z}$. We refer [7] for the treatment of the case when $\eta = \mathbb{Z}$.

Let $\mathcal{K}_\eta(v,v')$ be the free $\mathcal{Z}_1$-module with basis $\{A \mid A \in \tilde{\Xi}(\eta)\}$. Define a multiplication $\cdot$ on $\mathcal{K}_\eta(v,v')$ by linearly extending the products on basis elements:

$$A \cdot A' := \begin{cases} \sum\limits_{A'' \in \Xi(\eta)} g_{A,A',A''} A'', & \text{if } \mathrm{col}\,(A) = \mathrm{row}\,(A') \\ 0, & \text{otherwise} \end{cases}$$

for all $A, A' \in \tilde{\Xi}(\eta)$. $\mathcal{K}_\eta(v,v')$ is an associative algebra without 1 by Theorem 3.2.

Let $\mathcal{K}(\eta) = \mathcal{K}_\eta(v,v') \otimes_{\mathcal{Z}_1} \mathcal{Z}$ and $\boldsymbol{\mathcal{K}}(\eta) = \mathcal{K}_\eta(v,v') \otimes_{\mathcal{Z}_1} \mathbb{Q}(v)$ where $v' \mapsto 1$. Then $\mathcal{K}(\eta)$ and $\boldsymbol{\mathcal{K}}(\eta)$ are associative algebras with basis $\{[A] := A \otimes 1 \mid A \in \tilde{\Xi}(\eta)\}$. We also have

$$\mathcal{K}(\infty) = \varinjlim_n \mathcal{K}([-n,n]), \qquad \boldsymbol{\mathcal{K}}(\infty) = \varinjlim_n \boldsymbol{\mathcal{K}}([-n,n]). \tag{4.0.1}$$

Let $\widehat{\boldsymbol{\mathcal{K}}}(\eta)$ be the vector space of all formal (possibly infinite) $\mathbb{Q}(v)$-linear combinations $\sum_{A \in \tilde{\Xi}(\eta)} \beta_A[A]$ which have the following properties: for any $\mathbf{x} \in \mathbb{Z}^\eta$,

$$\text{the sets } \begin{matrix} \{A \in \widetilde{\Xi}(\eta) \mid \beta_A \neq 0,\ \mathrm{row}\,(A) = \mathbf{x}\} \\ \{A \in \widetilde{\Xi}(\eta) \mid \beta_A \neq 0,\ \mathrm{col}\,(A) = \mathbf{x}\} \end{matrix} \text{ are finite.} \tag{4.0.2}$$

In other words, for any $\lambda, \mu \in \mathbb{Z}^\eta$, the sums $\sum_{A \in \tilde{\Xi}(\eta)} \beta_A[\mathrm{diag}(\lambda)] \cdot [A]$ and $\sum_{A \in \tilde{\Xi}(\eta)} \beta_A[A] \cdot [\mathrm{diag}(\mu)]$ are finite. We can define the product of two elements $\sum_{A \in \tilde{\Xi}(\eta)} \beta_A[A]$, $\sum_{B \in \tilde{\Xi}(\eta)} \gamma_B[B]$ in $\widehat{\boldsymbol{\mathcal{K}}}(\eta)$ to be $\sum_{A,B} \beta_A\gamma_B[A] \cdot [B]$. This defines an associative algebra structure on $\widehat{\boldsymbol{\mathcal{K}}}(\eta)$. This algebra has an identity element $\sum_{\lambda \in \mathbb{Z}^\eta}[\mathrm{diag}(\lambda)]$, the sum of all $[D]$ with D a diagonal matrix in $\tilde{\Xi}(\eta)$. $\boldsymbol{\mathcal{K}}(\eta)$ is naturally a subalgebra of $\widehat{\boldsymbol{\mathcal{K}}}(\eta)$ (without 1).

Note that one may define $\widehat{\boldsymbol{\mathcal{K}}}(\eta, r)$ similarly. However, if $|\eta| < \infty$, then $\boldsymbol{\mathcal{K}}(\eta, r)$ is finite dimensional and $\widehat{\boldsymbol{\mathcal{K}}}(\eta, r) = \boldsymbol{\mathcal{K}}(\eta, r)$. If $\eta = \mathbb{Z}$, then $\boldsymbol{\mathcal{K}}(\infty, r)$ is infinite dimensional. We will see in Sect. 9 that the algebra $\mathbf{U}(\infty, r)$ defined in (2.0.4) is embedded in the completion algebra $\widehat{\boldsymbol{\mathcal{K}}}(\infty, r)$.

Let $\Xi^\pm(\eta)$ be the set of all $A \in \Xi(\eta)$ whose diagonal entries are zero, and let $\Xi^+(\eta)$ (resp. $\Xi^-(\eta)$, $\tilde{\Xi}(\eta)^0$) denote the subset of $\tilde{\Xi}(\eta)$ consisting of those matrices $(a_{i,j})$ with $a_{i,j} = 0$ for all $i \geqslant j$ (resp. $i \leqslant j, i \neq j$). For any $A \in \Xi(\eta)$, there exist unique $A^+ \in \Xi^+(\eta)$, $A^- \in \Xi^-(\eta)$ and $A^0 \in \Xi(\eta)^0$ such that $A = A^+ + A^0 + A^-$.

Given $r > 0$, $A \in \Xi^\pm(\eta)$ and $\mathbf{j} = (j_i)_{i \in \eta} \in \mathbb{Z}^\eta$, we define

$$A(\mathbf{j}) = A(\mathbf{j})_\eta = \sum_{D \in \widetilde{\Xi}(\eta)^0} v^{\sum_i d_i j_i}[A + D] \in \widehat{\boldsymbol{\mathcal{K}}}(\eta),$$

$$A(\mathbf{j}, r) = A(\mathbf{j}, r)_\eta = \sum_{\substack{D \in \Xi(\eta)^0 \\ |A+D| = r}} v^{\sum_i d_i j_i}[A + D] \in \widehat{\boldsymbol{\mathcal{K}}}(\eta, r),$$

where $\Xi(\eta)^0$ denotes the subset of diagonal matrices in $\Xi(\eta)$ and d_i are diagonal entries of D.

Let $\mathbf{V}(\eta)$ (resp. $\mathbf{V}(\eta, r)$) be the subspace of $\widehat{\mathcal{K}}(\eta)$ (resp. $\widehat{\mathcal{K}}(\eta, r)$) spanned by

$$\mathfrak{B}(\eta) = \{A(\mathbf{j}) \mid A \in \Xi^{\pm}(\eta),\ \mathbf{j} \in \mathbb{Z}^{\eta}\}$$

(resp., $\mathfrak{B}(\eta, r) = \{A(\mathbf{j}, r) \mid A \in \Xi^{\pm}(\eta, r),\ \mathbf{j} \in \mathbb{Z}^{\eta}\}$). Note that $\mathbf{V}(\eta, r) = \widehat{\mathcal{K}}(\eta, r)\mathcal{K}(\eta, r)$ if $|\eta| < \infty$.

The following results are proved using the multiplication formulas given in [1, Lemma 5.3]. The details in the infinite case can be found in [7, 3.3]. For $1 \leqslant i, j \leqslant n$, let $E_{i,j} \in \Xi(\eta)$ be the matrix $(a_{k,l})$ with $a_{k,l} = \delta_{i,k}\delta_{j,l}$.

Theorem 4.1. *Let $\eta \subseteq \mathbb{Z}$ be a consecutive segment of $\mathbb{Z}$.*

(1) $\mathbf{V}(\eta)$ *is a subalgebra of $\widehat{\mathcal{K}}(\eta)$ with $\mathbb{Q}(\upsilon)$-basis $\mathfrak{B}(\eta)$. It is generated by $E_{h,h+1}(\mathbf{0})$, $E_{h+1,h}(\mathbf{0})$ and $0(\mathbf{j})$ for all $h \in \eta^{\dashv}$ and $\mathbf{j} \in \mathbb{Z}^{\eta}$.*
(2) *There is an algebra monomorphism $\iota : \mathbf{U}(\eta) \to \widehat{\mathcal{K}}(\eta)$ satisfying*

$$E_h \mapsto E_{h,h+1}(\mathbf{0}),\ K_1^{j_1} K_2^{j_2} \cdots K_n^{j_n} \mapsto 0(\mathbf{j}),\ F_h \mapsto E_{h+1,h}(\mathbf{0}).$$

Hence, $\mathrm{im}(\iota) = \mathbf{V}(\eta)$.
(3) *For any positive integer r, there is an algebra homomorphism $\xi_r : \mathbf{U}(\eta) \to \widehat{\mathcal{K}}(\eta, r)$ satisfying*

$$E_h \mapsto E_{h,h+1}(\mathbf{0}, r),\ K_1^{j_1} K_2^{j_2} \cdots K_n^{j_n} \mapsto 0(\mathbf{j}, r),\ F_h \mapsto E_{h+1,h}(\mathbf{0}, r)$$

such that $\mathbf{V}(\eta, r) = \mathrm{im}(\xi_r)$ is a subalgebra of $\widehat{\mathcal{K}}(\eta, r)$. In particular, $\mathbf{V}(\eta, r)$ is generated by the elements $E_{h,h+1}(\mathbf{0}, r)$, $E_{h+1,h}(\mathbf{0}, r)$, and $0(\mathbf{j}, r)$ for all $h \in \eta^{\dashv}$ and $\mathbf{j} \in \mathbb{N}^{\eta}$.
(4) *By identifying $\mathbf{U}(\eta)$ with $\mathbf{V}(\eta)$ via ι, the algebra homomorphism ξ_r in* (3) *satisfies $\xi_r(A(\mathbf{j})) = A(\mathbf{j}, r)$, for any $A \in \Xi^{\pm}(\eta)$, $\mathbf{j} \in \mathbb{Z}^{\eta}$.*

Remarks 4.2. (1) From Theorem 4.1 (2), $\mathbf{U}(\eta) \cong \mathbf{V}(\eta)$ gives a realization of quantum $\mathfrak{gl}_{\eta}$. This realization provides explicit multiplication formulas between the generators and basis elements $A(\mathbf{j})$; see [1, 5.3].
(2) It is known $\mathbf{U}(\eta, r) \cong \mathbf{V}(\eta, r)$ if η is finite. We shall see in Sect. 9 that $\mathbf{U}(\infty, r) \cong \mathbf{V}(\infty, r)$. In particular, $\mathbf{V}(\infty, r)$ is a realization of $\mathbf{U}(\infty, r)$.

5 Constructions of $u_k(\eta)$

In this section, we discuss the realizations of $\tilde{u}_k(\eta)$ and $u_k(\eta)$. As set in Sect. 2, k is a field and $\varepsilon \in k$ is an l'th primitive root of 1.

Let $\tilde{\Xi}_l(\eta)$ be the set of all $A = (a_{i,j}) \in \tilde{\Xi}(\eta)$ such that $a_{i,j} < l$ for all $i \neq j$. Let $\Xi_l^{\pm}(\eta)$ be the set of all $A \in \tilde{\Xi}_l(\eta)$ whose diagonal entries are zero. In other words, $\Xi_l^{\pm}(\eta) = \Xi^{\pm}(\eta) \cap \widetilde{\Xi}_l(\eta)$. Define $\Xi_l^{+}(\eta)$ and $\Xi_l^{-}(\eta)$ similarly. Let $\Xi_{l,l'}(\eta)$ be the set of all $n \times n$ matrices $A = (a_{i,j})$ with $a_{i,j} \in \mathbb{N}$, $a_{i,j} < l$ for all $i \neq j$ and $a_{i,i} \in \mathbb{Z}_{l'} := \mathbb{Z}/l'\mathbb{Z}$ for all i. We have an obvious map $pr : \tilde{\Xi}_l(\eta) \to \Xi_{l,l'}(\eta)$ defined by reducing the diagonal entries modulo l'.

Let $\mathcal{K}(\eta)_k = \mathcal{K}(\eta) \otimes_{\mathcal{Z}} k$. Mimicking the construction of $\widehat{\mathcal{K}}(\eta)$, we define $\widehat{\mathcal{K}}(\eta)_k$ to be the k-vector space of all formal (possibly infinite) k-linear combinations $\sum_{A\in\widetilde{\Xi}(\eta)} \beta_A[A]$ satisfying the property (4.0.2) with a similar multiplication. This is an associative algebra with an identity: the sum of all $[D]$ with D a diagonal matrix in $\widetilde{\Xi}(\eta)$. The elements $A(\mathbf{j})$ defined earlier becomes

$$A(\mathbf{j}) = \sum_{\mathbf{z}\in\mathbb{Z}^\eta} \varepsilon^{\mathbf{j}\cdot\mathbf{z}}[A + \mathrm{diag}(\mathbf{z})] \in \widehat{\mathcal{K}}(\eta)_k,$$

where $\mathbf{j}\cdot\mathbf{z} = \Sigma_i j_i z_i$. Clearly, $A(\mathbf{j}) = A(\mathbf{j}')$ whenever $\bar{\mathbf{j}} = \bar{\mathbf{j}}'$. Here $^{-} : \mathbb{Z}^\eta \to (\mathbb{Z}_{l'})^\eta$ is the map defined by $\overline{(j_1, j_2, \ldots, j_n)} = (\bar{j}_1, \bar{j}_2, \ldots, \bar{j}_n)$. Thus, we shall write $A(\bar{\mathbf{j}}) := A(\mathbf{j})$. Similarly, we shall use $A(\bar{\mathbf{j}}, r) := A(\mathbf{j}, r)$ to denote the element defined earlier with υ replaced by ε for the algebra $\widehat{\mathcal{K}}(\eta, r)_k = \widehat{\mathcal{K}}(\eta, r) \otimes k$ over k.

Let $\mathcal{W}$ be the subspace of $\widehat{\mathcal{K}}(\eta)_k$ spanned by

$$\mathfrak{B}_k = \{A(\bar{\mathbf{j}}) \mid A \in \Xi_l^{\pm}(\eta),\ \bar{\mathbf{j}} \in (\mathbb{Z}_{l'})^\eta\}.$$

We have clearly $\dim \mathcal{W} \leqslant (l')^n l^{n^2-n}$ if $|\eta| = n$.

We remark that all the results given below are stated for an arbitrary η. Their proofs for an infinite η is entirely similar to the proofs given in [11] (when l' is odd) and in [13] (when l' is even).

The following result is [11, 4.4] and [13, 4.1].

Lemma 5.1. (1) *$\mathcal{W}$ is a subalgebra of $\widehat{\mathcal{K}}(\eta)_k$.*
(2) *The elements $E_{h,h+1}(\bar{\mathbf{0}})$, $E_{h+1,h}(\bar{\mathbf{0}})$, $0(\bar{\boldsymbol{e}}_i)$ (for $h \in \eta^{\dashv}$ and $i \in \eta$) generate $\mathcal{W}$ as an algebra.*

Given $A \in \Xi_l^{\pm}(\eta)$ and $\mathbf{j} = (j_i)_{i\in\eta} \in \mathbb{Z}^\eta$, we rewrite

$$\begin{aligned} A(\bar{\mathbf{j}}) &:= \sum_{\mathbf{z}\in\mathbb{Z}^\eta} \varepsilon^{\mathbf{j}\cdot\mathbf{z}}[A + \mathrm{diag}(\mathbf{z})] \in \widehat{\mathcal{K}}(\eta)_k \\ &= \sum_{\bar{\mathbf{z}}\in(\mathbb{Z}_{l'})^\eta} \varepsilon^{\mathbf{j}\cdot\bar{\mathbf{z}}} \sum_{\substack{\mathbf{x}\in\mathbb{Z}^\eta\\ \bar{\mathbf{x}}=\bar{\mathbf{z}}}} [A + \mathrm{diag}(\mathbf{x})] \\ &= \sum_{\bar{\mathbf{z}}\in(\mathbb{Z}_{l'})^\eta} \varepsilon^{\mathbf{j}\cdot\bar{\mathbf{z}}} [\![A + \mathrm{diag}(\bar{\mathbf{z}})]\!], \end{aligned}$$

where $\varepsilon^{\mathbf{j}\cdot\bar{\mathbf{z}}} = \varepsilon^{\mathbf{j}\cdot\mathbf{z}}$ and

$$[\![A + \mathrm{diag}(\bar{\mathbf{z}})]\!] = \sum_{\substack{\mathbf{x}\in\mathbb{Z}^\eta\\ \bar{\mathbf{x}}=\bar{\mathbf{z}}}} [A + \mathrm{diag}(\mathbf{x})] \in \widehat{\mathcal{K}}(\eta)_k.$$

Note that $A + \mathrm{diag}(\bar{\mathbf{z}}) \in \Xi_{l,l'}(\eta)$ and the elements $[\![A]\!]$, $A \in \Xi_{l,l'}(\eta)$, are linearly independent. Thus, we have the following; see [11, 13].

Lemma 5.2. (1) *The set* $\{A(\bar{\mathbf{j}}) \mid A \in \Xi_l^{\pm}(\eta),\ \bar{\mathbf{j}} \in (\mathbb{Z}_{l'})^{\eta}\}$ *forms a* k*-basis for* $\mathcal{W}$.
(2) *If* $n = |\eta|$ *is finite, then the set* $\{[\![A]\!] \mid A \in \Xi_{l,l'}(\eta)\}$ *forms a* k*-basis for* $\mathcal{W}$ *and* $\dim \mathcal{W} = (l')^n l^{n^2-n}$.

Assume that η is a finite segment of $\mathbb{Z}$. Following [1, 6.3], let $\mathrm{K}(\eta)$ be the k-subspace of $\mathcal{K}(\eta)_k$ spanned by the elements $[A]$ with $A \in \tilde{\Xi}_l(\eta)$. It is proved that $\mathrm{K}(\eta)$ is a subalgebra of $\mathcal{K}(\eta)_k$.

Let $\mathrm{K}'(\eta)$ be the free k-module with basis elements $[A]$ in bijection with the elements $A \in \Xi_{l,l'}(\eta)$. There is an algebra structure on $\mathrm{K}'(\eta)$ given, for $A, A' \in \Xi_{l,l'}(\eta)$, by

$$[A] \cdot [A'] = \begin{cases} 0, & \text{if } \mathrm{col}\,(A) \neq \mathrm{row}\,(A') \text{ in } \mathbb{Z}_{l'}, \\ \sum \rho_{\tilde{A}''}[pr(\tilde{A}'')], & \text{otherwise,} \end{cases}$$

where $\rho_{\tilde{A}''}$ and $\tilde{A}''$ are determined by a product in $\mathrm{K}(\eta)$: $[\tilde{A}] \cdot [\tilde{A}'] = \sum \rho_{\tilde{A}''}[\tilde{A}'']$ for any $\tilde{A}$, $\tilde{A}' \in \tilde{\Xi}_l(\eta)$ satisfying $\mathrm{col}\,(\tilde{A}) = \mathrm{row}\,(\tilde{A}')$ (in $\mathbb{Z}$), $pr(\tilde{A}) = A$ and $pr(\tilde{A}') = A'$. Unlike $\mathcal{K}(\eta)$ or $\mathrm{K}(\eta)$, the algebra $\mathrm{K}'(\eta)$ has an identity element: the sum $\sum_{\lambda \in \mathbb{Z}_{l'}^{\eta}} [\mathrm{diag}(\lambda)]$.

We now have the following results.

Theorem 5.3 ([11], [13]). (1) *Assume that* η *is finite. There is an algebra isomorphism* $\psi : \mathcal{W} \xrightarrow{\sim} \mathrm{K}'(\eta)$ *satisfying* $[\![A]\!] \mapsto [A]$ *for* $A \in \Xi_{l,l'}(\eta)$.
(2) *There is an algebra epimorphism* $\varphi : \tilde{u}_k(\eta) \twoheadrightarrow \mathcal{W}$ *satisfying* $E_h \mapsto E_{h,h+1}(\bar{\mathbf{0}})$, $F_h \mapsto E_{h+1,h}(\bar{\mathbf{0}})$, $K_j \mapsto 0(\bar{\boldsymbol{e}}_j)$.
(3) *If* l' *is even,* φ *is an isomorphism, and if* l' *is odd, then* φ *induces isomorphism* $\bar{\varphi} : u_k(\eta) \xrightarrow{\sim} \mathcal{W}$.

Corollary 5.4 ([1]). *Assume that* η *is finite. If* $l' = l$ *is odd, then we have an algebra isomorphism* $u_k(\eta) \cong \mathrm{K}'(\eta)$.

6 Bases for q-Schur Algebras

In this section, we first describe the Drinfeld–Jimbo type presentations for q-Schur algebras at $\eta = [1, n]$, and then we introduce several resulting bases.

Theorem 6.1. *The* q*-Schur algebra* $\mathbf{U}(n, r)$ *over* $\mathbb{Q}(\upsilon)$ *has the following presentations with*

(1) [4] *generators*

$$\mathtt{e}_i,\ \mathtt{f}_i\ (1 \leqslant i \leqslant n-1),\ \mathtt{k}_j\ (1 \leqslant j \leqslant n)$$

and relations:

(S1) $\mathtt{k}_i\mathtt{k}_j = \mathtt{k}_j\mathtt{k}_i$;
(S2) $\mathtt{k}_i\mathtt{e}_j = \upsilon^{\delta_{i,j}-\delta_{i,j+1}}\mathtt{e}_j\mathtt{k}_i$, $\mathtt{k}_i\mathtt{f}_j = \upsilon^{\delta_{i,j+1}-\delta_{i,j}}\mathtt{f}_j\mathtt{k}_i$;
(S3) $\mathtt{e}_i\mathtt{e}_j = \mathtt{e}_j\mathtt{e}_i$, $\mathtt{f}_i\mathtt{f}_j = \mathtt{f}_j\mathtt{f}_i$, *when* $|i-j|>1$;
(S4) $\mathtt{e}_i^2\mathtt{e}_j - (\upsilon+\upsilon^{-1})\mathtt{e}_i\mathtt{e}_j\mathtt{e}_i + \mathtt{e}_j\mathtt{e}_i^2 = 0$, *when* $|i-j|=1$;
(S5) $\mathtt{f}_i^2\mathtt{f}_j - (\upsilon+\upsilon^{-1})\mathtt{f}_i\mathtt{f}_j\mathtt{f}_i + \mathtt{f}_j\mathtt{f}_i^2 = 0$, *when* $|i-j|=1$;
(S6) $\mathtt{e}_i\mathtt{f}_j - \mathtt{f}_j\mathtt{e}_i = \delta_{ij}\frac{\widetilde{\mathtt{k}}_i-\widetilde{\mathtt{k}}_i^{-1}}{\upsilon-\upsilon^{-1}}$, *where* $\widetilde{\mathtt{k}}_i = \mathtt{k}_i\mathtt{k}_{i+1}^{-1}$;
(S7) $\mathtt{k}_1\cdots\mathtt{k}_n = v^r$ *and, for all* $1\leqslant i\leqslant n$, $(\mathtt{k}_i-1)(\mathtt{k}_i-v)\cdots(\mathtt{k}_i-v^r)=0$.

(2) [9] *generators*

$$\mathtt{e}_i,\ \mathtt{f}_i,\ \mathtt{k}_j\ (1\leqslant i\leqslant n-1)$$

and relations (S1)–(S6) *together with*

(S0) $[\mathtt{k}_1;t_1]^![\mathtt{k}_2;t_2]^!\cdots[\mathtt{k}_{n-1};t_{n-1}]^! = 0$ *for* $t_i\in\mathbb{N}$ *such that* $t_1+\cdots+t_{n-1}=r+1$,

where $[X;t]^! = (X-1)(X-v)\cdots(X-v^{t-1})$, *and* $\mathtt{k}_n = \upsilon^r\mathtt{k}_1^{-1}\cdots\mathtt{k}_{n-1}^{-1}$ *in* (S6).

Proof. Let $\widetilde{\mathbf{U}}(n,r)$ be the algebra with either of the described presentations. Then there exist an algebra epimorphism

$$\tilde{\zeta}_r : \mathbf{U}(n) \twoheadrightarrow \widetilde{\mathbf{U}}(n,r).$$

The algebra epimorphism $\zeta_r : \mathbf{U}(n) \twoheadrightarrow \mathbf{V}(n,r) = \mathbf{U}(n,r)$ given by

$$E_h \mapsto E_{h,h+1}(\mathbf{0},r),\ K_i \mapsto 0(\boldsymbol{e}_i,r),\ F_h \mapsto E_{h+1,h}(\mathbf{0},r),$$

where $\boldsymbol{e}_i = (\ldots,0,\underset{i}{1},0,\ldots)$, induces an epimorphism $\widetilde{\mathbf{U}}(n,r) \twoheadrightarrow \mathbf{U}(n,r)$. Thus, the result follows if we could prove that $\dim\widetilde{\mathbf{U}}(n,r)\leqslant\dim\mathbf{U}(n,r)$.

We prove this inequality by constructing some integral monomial bases below. □

We first display an integral monomial basis for $U(n)$. For $A = A^+ + A^- \in \Xi^\pm(n)$, where $A^+\in\Xi^+(n)$ and $A^-\in\Xi^-(n)$, and $\mathbf{j}\in\mathbb{Z}^n$, let

$$M^{(A,\mathbf{j})} = E^{(A^+)}\cdot K(\mathbf{j})\cdot F^{(A^-)} \in \mathbf{U}(n),$$

where $E^{(A^+)} = \prod_{1\leqslant i\leqslant h<j\leqslant n} E_h^{(a_{i,j})}$, $F^{(A^-)} = \prod_{1\leqslant j\leqslant h<i\leqslant n} F_h^{(a_{i,j})}$ and

$$K(\mathbf{j}) = K_1^{\delta_1}\cdots K_n^{\delta_n}\begin{bmatrix}K_1;0\\|j_1|\end{bmatrix}\cdots\begin{bmatrix}K_n;0\\|j_n|\end{bmatrix},$$

where $\delta_i = 0$ (resp. 1) if $j_i\geqslant 0$ (resp. $j_i<0$). The order in the products $E^{(A^+)}$ and $F^{(A^-)}$ is taken as follows: For the jth column (reading upwards) $a_{j-1,j},\ldots,a_{1,j}$

$(2 \leqslant j \leqslant n)$ of A^+, fix the reduced expression for the longest word of the symmetric group $\mathfrak{S}_j$ on j letters:

$$w_{0,j} := s_{j-1}(s_{j-2}s_{j-1})\cdots(s_1 s_2 \cdots s_{j-1})$$

and put

$$M_j = E_{j-1}^{(a_{j-1,j})}(E_{j-2}^{(a_{j-2,j})}E_{j-1}^{(a_{j-2,j})})\cdots(E_1^{(a_{1,j})}E_2^{(a_{1,j})}\cdots E_{j-1}^{(a_{1,j})}).$$

Similarly, for the jth row (reading to the right) $a_{j,1},\ldots,a_{j,j-1}$ $(2 \leqslant j \leqslant n)$ of A^-, put

$$M'_j = (F_{j-1}^{(a_{j,1})}\cdots F_2^{(a_{j,1})}F_1^{(a_{j,1})})\cdot(F_{j-1}^{(a_{j,j-2})}F_{j-2}^{(a_{j,j-2})})F_{j-1}^{(a_{j,j-1})}.$$

Then $E^{(A^+)} = M_n M_{n-1}\cdots M_2$ and $F^{(A^-)} = M'_2 M'_3 \cdots M'_n$.

Lemma 6.2. *([19]) The set*

$$\{M^{(A,\mathbf{j})} \mid A \in \Xi^{\pm}(n),\ \mathbf{j} \in \mathbb{Z}^n\}$$

forms a $\mathcal{Z}$-basis for $U(n)$.

We want to construct a similar basis for $\widetilde{\mathbf{U}}(n,r)$. Let

$$\mathtt{e}^{(A^+)} = \tilde{\zeta}_r(E^{(A^+)}), \quad \mathtt{f}^{(A^-)} = \tilde{\zeta}_r(F^{(A^-)}) \text{ and } \begin{bmatrix} \mathtt{k}_i;0 \\ t \end{bmatrix} = \tilde{\zeta}_r\left(\begin{bmatrix} K_i;0 \\ t \end{bmatrix}\right),$$

and put, for $\lambda \in \mathbb{N}^n$,

$$\mathtt{k}_\lambda = \prod_{i=1}^{n} \begin{bmatrix} \mathtt{k}_i;0 \\ \lambda_i \end{bmatrix}. \tag{6.2.1}$$

Lemma 6.3. *We have, for any $\lambda \in \mathbb{N}^n$,*

$$\mathtt{k}_\lambda = \begin{cases} [\mathrm{diag}(\lambda)], & \text{if } |\lambda| = r, \\ 0, & \text{otherwise.} \end{cases}$$

The following result is proved in [9].

Theorem 6.4. *Let*

$$\mathcal{M} = \{\mathfrak{m}^{(A)} := \mathtt{e}^{(A^+)}\mathtt{k}_{\sigma(A)}\mathtt{f}^{(A^-)} \mid A \in \Xi(n,r)\},$$

where $\boldsymbol{\sigma}(A) = (\sigma_1(A),\ldots,\sigma_n(A))$ with $\sigma_i(A) = a_{i,i} + \sum_{1\leqslant j<i}(a_{i,j}+a_{j,i})$. Then $\mathcal{M}$ is a spanning set and, hence, forms a $\mathbb{Q}(v)$-basis for $\widetilde{\mathbf{U}}(n,r)$. (So we may identify $\widetilde{\mathbf{U}}(n,r)$ with $\mathbf{U}(n,r)$, and ζ_r with $\tilde{\zeta}_r$.) Moreover, $\mathcal{M}$ gives a $\mathcal{Z}$-basis for the integral q-Schur algebra $U(n,r)$.

We call $\mathcal{M}$ an integral monomial basis. This basis leads to several other bases for $\mathbf{U}(n,r)$ which we now describe.

Fix the reduced expression

$$\mathbf{i} = (i_1, i_2, \ldots, i_\nu) = (n-1, \ldots, 2, 1, \ldots, n-1, n-2, n-1) \tag{6.4.1}$$

of the longest word w_0 of the symmetric group $\mathfrak{S}_n$, that is, $w_0 = s_{i_1}s_{i_2}\cdots s_{i_\nu}$. For any $\mathbf{c} = (c_1, \ldots, c_\nu) \in \mathbb{N}^\nu$, define monomials in root vectors $E_{\mathbf{i}}^{\mathbf{c}}$ and $F_{\mathbf{i}}^{\mathbf{c}}$ as in [19, 2.2] (using braid group actions). Let $\mathtt{e}_{\mathbf{i}}^{\mathbf{c}} = \zeta_r(E_{\mathbf{i}}^{\mathbf{c}})$ and $\mathtt{f}_{\mathbf{i}}^{\mathbf{c}} = \zeta_r(F_{\mathbf{i}}^{\mathbf{c}})$.

For any $A \in \Xi^\pm(n)$, let $\mathbf{c}(A^+) = (c_1, \ldots, c_\nu) \in \mathbb{N}^\nu$, where the first $n-1$ components of $\mathbf{c}(A^+)$ is the nth column of A^+ reading upwards, and the next $n-2$ components is the $(n-1)$st column and so on, i.e.

$$c_1 = a_{n-1,n}, \ldots, c_{n-1} = a_{1,n},\ c_n = a_{n-2,n-1}, \ldots .$$

Define $\mathbf{c}(A^-)$ symmetrically.

Theorem 6.5 ([11, 13]). *We list the following $\mathbb{Q}(v)$-bases for the q-Schur algebra $\mathbf{U}(n,r)$. Let i_0 be a fixed integer with $1 \leqslant i_0 \leqslant n$.*

(1) *The monomial basis:*

$$\mathcal{N}_{i_0} = \left\{ \mathtt{e}^{(A^+)}\mathtt{k}^{\mathbf{j}}\mathtt{f}^{(A^-)} \;\middle|\; A \in \Xi^\pm(n),\ \mathbf{j} \in \mathbb{N}^n,\ j_{i_0} = 0,\ |\mathbf{j}| + |A| \leqslant r \right\};$$

(2) *The BLM basis:*

$$\mathcal{B}_{i_0} = \left\{ A(\mathbf{j}, r) \;\middle|\; A \in \Xi^\pm(n),\ \mathbf{j} \in \mathbb{N}^n,\ j_{i_0} = 0,\ |\mathbf{j}| + |A| \leqslant r \right\};$$

(3) *The PBW basis:*

$$\mathcal{P}_{i_0} = \left\{ \mathtt{e}_{\mathbf{i}}^{\mathbf{c}(A^+)}\mathtt{k}^{\mathbf{j}}\mathtt{f}_{\mathbf{i}}^{\mathbf{c}(A^-)} \;\middle|\; A \in \Xi^\pm(n),\ \mathbf{j} \in \mathbb{N}^n,\ j_{i_0} = 0,\ |\mathbf{j}| + |A| \leqslant r \right\}.$$

We end this section with an application. We identify the Hecke algebra part of the monomial basis $\mathcal{M}$. We assume now $n = r$ though all results hold for $n \geqslant r$. Let

$$\varpi = (1, 1, \ldots, 1) \in \mathbb{N}^n,$$

and let $\boldsymbol{\mathcal{H}} = \mathtt{k}_\varpi \mathbf{U}(n,n)\mathtt{k}_\varpi$ and $\mathcal{H} = \mathtt{k}_\varpi U(n,n)\mathtt{k}_\varpi$. For any $w \in \mathfrak{S}_n$, let $A_w \in \Xi(n,n)$ be the permutation matrix defined inductively by $A_w = A_y A_{s_i}$, where $w = ys_i$ with $y < w$ and $s_i = (i, i+1)$ for some i.

Lemma 6.6 ([9, 9.1]). *The algebra $\mathcal{H}$ is free over $\mathcal{Z}$ with basis*

$$\mathcal{M}_\varpi = \{\mathfrak{m}^{(A_w)} \mid w \in \mathfrak{S}_n\}.$$

Let $\mathtt{c}_i = \mathfrak{m}^{(A_{s_i})} = \mathtt{e}^{(A_{s_i}^+)}\mathtt{k}_{\sigma(A_{s_i})}\mathtt{f}^{(A_{s_i}^-)}$, and let $\mathcal{T}_i = \mathtt{c}_i - v^{-1}$.

Theorem 6.7 ([9, 9.3–9.4]). (1) *The elements* $\mathcal{T}_i$, $1 \leqslant i \leqslant n-1$, *satisfy the following relations:*

$$\begin{cases} \text{(a)} & (\mathcal{T}_i - v)(\mathcal{T}_i + v^{-1}) = 0; \\ \text{(b)} & \mathcal{T}_i\mathcal{T}_j = \mathcal{T}_j\mathcal{T}_i, \quad |i-j| > 1; \\ \text{(c)} & \mathcal{T}_i\mathcal{T}_{i+1}\mathcal{T}_i = \mathcal{T}_{i+1}\mathcal{T}_i\mathcal{T}_{i+1}, \quad 1 \leqslant i \leqslant n-2. \end{cases}$$

In particular, $\mathcal{T}_i$ *is invertible and* $\mathcal{T}_i^{-1} = \mathtt{c}_i - v$.

(2) *For any* $w \in \mathfrak{S}_n$, *there is a reduced expression* $w = s_{i_1} \cdots s_{i_l}$ *satisfying*

$$\mathfrak{m}^{(A_w)} = \mathtt{c}_{i_1} \cdots \mathtt{c}_{i_l}.$$

Therefore, $\mathcal{H}$ *is isomorphic to the Hecke algebra over* $\mathcal{Z}$ *associated with* $\mathfrak{S}_n$.

Remark 6.8. (1) Some commuting relations between the $\mathtt{e}_i$ and $\mathtt{f}_i$ and the quantum Serre relations give rise to the braid relations (b) and (c).

(2) Using the notation in [17], $\mathtt{c}_i$ corresponds to C'_{s_i}, and $\mathcal{T}_s = v^{-1}T_{s_i}$.

7 Little q-Schur Algebras

In this section, we continue to assume $\eta = [1, n]$. By restricting the map $\zeta_r : \mathbf{U}(n) \twoheadrightarrow \mathbf{U}(n, r)$ defined in (2.0.2) to the $\mathcal{Z}$-form $U(n)$, we obtain a surjective map $\zeta_r : U(n) \twoheadrightarrow U(n, r)$ (see [6]).

As in Sect. 2, let k be a field which is a $\mathcal{Z}$-module via $v \mapsto \varepsilon$, where ε is an l'th primitive root of 1. Then base change induces a surjective homomorphism

$$\zeta_{r,k} := \zeta_r \otimes 1 : U(n)_k \twoheadrightarrow U(n, r)_k$$

and hence, a map

$$\zeta_{r,k} : \tilde{u}_k(n) \to U(n, r)_k$$

by restriction. The image $\zeta_{r,k}(\tilde{u}_k(n))$, denoted $u_k(n, r)$, is called a *little q-Schur algebra*. Note that if l' is odd, then $\zeta_{r,k}(K_i^l - 1) = 0$ for all i. Hence, $\zeta_{r,k}$ induces a surjective map, the version of ζ_r over k,

$$\zeta_{r,k} : u_k(n) \twoheadrightarrow u_k(n, r).$$

Let $u_k(n, r)^+ = \zeta_{r,k}(\tilde{u}_k(n)^+)$, $u_k(n, r)^- = \zeta_{r,k}(\tilde{u}_k(n)^-)$, $u_k(n, r)^0 = \zeta_{r,k}(\tilde{u}_k(n)^0)$. By abuse of notation, we shall continue to denote the images of the generators E_i, F_i, K_j for $\tilde{u}_k(n)$ by the same letters $\mathtt{e}_i$, $\mathtt{f}_i$, $\mathtt{k}_j$ used for $\mathbf{U}(n, r)$.

We first have the following.

Theorem 7.1 ([11, 7.6]). *The set* $\{\mathtt{e}^{(A)} \mid A \in \Xi_l^+(n),\ |A| \leqslant r\}$ (*resp.* $\{\mathtt{f}^{(A)} \mid A \in \Xi_l^-(n),\ |A| \leqslant r\}$) *forms a k-basis of* $u_k(n, r)^+$ (*resp.* $u_k(n, r)^-$).

For any $\lambda \in \Lambda(n,r)$, let $\overline{\lambda}$ be the image in $(\mathbb{Z}_{l'})^n$ obtained by reducing every entry modulo l'. Let, for $A \in \Xi_l^{\pm}(n)$ with $|A| \leqslant r$ and $\lambda \in \Lambda(n, r-|A|)$,

$$[\![A + \mathrm{diag}(\overline{\lambda}), r]\!] = \sum_{\substack{\mu\in\Lambda(n,r-|A|)\\ \overline{\mu}=\overline{\lambda}}} [A + \mathrm{diag}(\mu)] \in U_k(n,r).$$

We also set $\overline{\Lambda(n,r)_{l'}} = \{\overline{\lambda} \in (\mathbb{Z}_{l'})^n \mid \lambda \in \Lambda(n,r)\}$. For $\nu \in (\mathbb{Z}_{l'})^n$, define

$$\mathtt{p}_\nu = \begin{cases} \displaystyle\sum_{\mu\in\Lambda(n,r),\overline{\mu}=\nu} \mathtt{k}_\mu & \text{if } \nu \in \overline{\Lambda(n,r)_{l'}},\\ 0 & \text{otherwise.}\end{cases}$$

Let $\mathbb{N}_{l'}^n = \{\lambda \in \mathbb{N}^n \mid \lambda_i < l' \ \forall i\}$. In contrast with Theorems 6.4 and 6.5, we have the following bases for $u_k(n,r)$.

Theorem 7.2 ([11, 8.2, 8.5]). *Fix any integer i_0 with $1 \leqslant i_0 \leqslant n$. Each of the following sets forms a k-basis for $u_k(n,r)$.*

(1) $\mathcal{L}_k := \{[\![A + \mathrm{diag}(\overline{\lambda}), r]\!] \mid A \in \Xi_l^{\pm}(n),\ |A| \leqslant r,\ \lambda \in \Lambda(n, r-|A|)\}$;
(2) $\mathcal{M}_k := \{\mathtt{e}^{(A^+)}\mathtt{p}_{\overline{\lambda}}\mathtt{f}^{(A^-)} \mid A \in \Xi_l^{\pm}(n),\ \lambda \in \Lambda(n,r),\ \lambda_i \geqslant \sigma_i(A)\ \forall i\}$;
(3) $\mathcal{N}_{i_0,k} = \big\{\mathtt{e}^{(A^+)}\mathtt{k}^{\mathbf{j}}\mathtt{f}^{(A^-)} \mid A \in \Xi_l^{\pm}(n),\ \mathbf{j} \in \mathbb{N}_{l'}^n,\ j_{i_0} = 0,\ |\mathbf{j}| + |A| \leqslant r\big\}$;
(4) $\mathcal{B}_{i_0,k} = \big\{A(\mathbf{j}, r) \mid A \in \Xi_l^{\pm}(n),\ \mathbf{j} \in \mathbb{N}_{l'}^n,\ j_{i_0} = 0,\ |\mathbf{j}| + |A| \leqslant r\big\}$;
(5) $\mathcal{P}_{i_0,k} = \big\{\mathtt{e}_{\mathbf{i}}^{\mathbf{c}(A^+)}\mathtt{k}^{\mathbf{j}}\mathtt{f}_{\mathbf{i}}^{\mathbf{c}(A^-)} \mid A \in \Xi_l^{\pm}(n),\ \mathbf{j} \in \mathbb{N}_{l'}^n,\ j_{i_0} = 0,\ |\mathbf{j}| + |A| \leqslant r\big\}$.

We may use the bases to derive certain dimension formulas. Let, for $n \geqslant m \geqslant 1$,

$$\Lambda(n,r;m) = \{\lambda \in \Lambda(n,r) \mid 0 \leqslant \lambda_2, \ldots, \lambda_m < l', 0 \leqslant \lambda_{m+1}, \ldots, \lambda_n < l\}.$$

Then $\#\Lambda(n,r;1) = \#\overline{\Lambda(n,r)_l}$ and $\#\Lambda(n,r;n) = \#\overline{\Lambda(n,r)_{l'}}$. By Theorem 7.1, we have $\dim u_k(n,r)^{\pm} = \#\Lambda(N,r;1)$, where $N = \binom{n}{2} + 1$, while, by Theorem 7.2, $\dim u_k(n,r) = \#\Lambda(n^2,r;n)$. More explicitly, we have the following dimension formulas.

Theorem 7.3 ([11, 13]). *We have, for $1 \leqslant m \leqslant n$,*

$$\#\Lambda(n,r;m) = \sum_{s\geqslant 0}(-1)^{s+t}\binom{m-1}{s}\binom{n-m}{t}\binom{n+r-tl-sl'-1}{n-1}.$$

In particular, we have

(1) $\dim u_k(n,r)^0 = \sum_{s\geqslant 0}(-1)^s\binom{n-1}{s}\binom{n+r-sl'-1}{n-1}$*;*
(2) $\dim u_k(n,r)^+ = \sum_{s\geqslant 0}(-1)^s\binom{N-1}{s}\binom{N+r-sl-1}{N-1}$ *with* $N = \binom{n}{2} + 1$*;*
(3) $\dim u_k(n,r) = \sum_{s,t\geqslant 0}(-1)^{s+t}\binom{n-1}{s}\binom{n^2-n}{t}\binom{n^2+r-tl-sl'-1}{n^2-1}$.

There is a simpler description of the number $\dim u_k(n,r)$ if $l' = l$ is odd. For any $A = (a_{i,j}) \in M_n(\mathbb{Z})$, let $\bar{A} = \overline{(a_{i,j})} = (\bar{a}_{i,j})$ where $\bar{a}_{i,j} \in \mathbb{Z}_l$, and let $\overline{\Xi(n,r)} = \{\bar{A} \mid A \in \Xi(n,r)\}$. Then it is easy to see that all the sets $\overline{\Xi(n,r)}$, $\Lambda(n^2, r; 1)$ and $\overline{\Lambda(n^2,r)_l}$ have the sane cardinality. Hence, we have

$$\dim u_k(n,r) = \#\overline{\Xi(n,r)} = \sum_{s\geqslant 0}(-1)^s \binom{n^2-1}{s}\binom{n^2+r-sl-1}{n^2-1}.$$

Remark 7.4. Infinitesimal Schur/q-Schur algebras were introduced in [2, 5] as the dual algebras of the homogeneous components of the infinitesimal thickening (by the torus) of the Frobenius kernel of the quantum coordinate algebra of GL_n. It is proved in [12] that the subalgebra of $U(n,r)_k$ generated by the little q-Schur algebra $u_k(n,r)$ and $\left[\begin{smallmatrix}\mathsf{k}_i;0\\ t\end{smallmatrix}\right]$ $(1 \leqslant i \leqslant n,\ t \in \mathbb{N})$ is isomorphic to the infinitesimal q-Schur algebra investigated in [2].

8 Presenting $\mathbf{V}(\infty, r)$

Recall from Sect. 4 the $\mathbb{Q}(v)$-algebra $\mathbf{V}(\infty,r)$. By Theorem 4.1 (3) for $\eta = \mathbb{Z}$, it is generated by the elements

$$\mathsf{e}_i = E_{i,i+1}(\mathbf{0},r),\ \mathsf{f}_i = E_{i+1,i}(\mathbf{0},r),\ \text{and}\ \mathsf{k}_i = 0(\boldsymbol{e}_i, r)$$

for all $i \in \mathbb{Z}$, where $\boldsymbol{e}_i \in \mathbb{Z}^\infty$ has 1 as the ith component and 0 elsewhere.

For any $\lambda \in \mathbb{N}^\infty$, since λ has finite support, the product (cf. (6.2.1))

$$\mathsf{k}_\lambda = \prod_{i=-\infty}^{\infty} \left[\begin{matrix}\mathsf{k}_i;0\\ \lambda_i\end{matrix}\right],$$

is well defined, and a result similar to Lemma 6.3 holds. We also introduce the products $\mathsf{e}^{(A^+)}$ and $\mathsf{f}^{(A^-)}$. Bases similar to those given in Theorem 6.5 exist for $\mathbf{V}(\infty,r)$.

Lemma 8.1 ([7, 4.8–4.9]). *Each of the following sets forms a basis for* $\mathbf{V}(\infty,r)$*:*

(1) $\mathcal{N}_\infty = \left\{\mathsf{e}^{(A^+)}\mathsf{k}^{\mathbf{j}}\mathsf{f}^{(A^-)} \;\middle|\; A \in \Xi^\pm(\infty),\ \mathbf{j} \in \mathbb{N}^\infty,\ |\mathbf{j}| + |A| \leqslant r\right\}$;

(2) $\mathcal{B}_\infty = \left\{A(\mathbf{j},r) \;\middle|\; A \in \Xi^\pm(\infty),\ \mathbf{j} \in \mathbb{N}^\infty,\ |\mathbf{j}| + |A| \leqslant r\right\}$.

We remark that a basis similar to the PBW basis 6.5 (3) can also be constructed. In order to avoid the dependence of the sequence $\mathbf{i}$ defined in (6.4.1), we may simply use the PBW basis constructed in [8, 4.6].

We now present $\mathbf{V}(\infty,r)$ by generators and relations; cf. Theorem 6.1. The proof of the following theorem relies on the basis $\mathcal{N}_\infty$.

Theorem 8.2 ([7, 4.7]). *The algebra* $\mathbf{V}(\infty, r)$ *over* $\mathbb{Q}(\upsilon)$ *has the following presentations with generators*

$$\mathsf{e}_i,\ \mathsf{f}_i,\ \mathsf{k}_i,\ i \in \mathbb{Z}$$

and relations: for $i, j \in \mathbb{Z}$,

(1) $\mathsf{k}_i\mathsf{k}_j = \mathsf{k}_j\mathsf{k}_i$;
(2) $\prod_{i\in\mathbb{Z}}[\mathsf{k}_i; t_i]^! = 0$ *for all* $\mathbf{t} = (t_i) \in \mathbb{N}^\infty$ *with* $|\mathbf{t}| = \sum_{i\in\mathbb{Z}} t_i = r+1$;
(3) $\mathsf{k}_i\mathsf{e}_j = \upsilon^{\delta_{i,j}-\delta_{i,j+1}}\mathsf{e}_j\mathsf{k}_i$, $\mathsf{k}_i\mathsf{f}_j = \upsilon^{\delta_{i,j+1}-\delta_{i,j}}\mathsf{f}_j\mathsf{k}_i$;
(4) $\mathsf{e}_i\mathsf{e}_j = \mathsf{e}_j\mathsf{e}_i$, $\mathsf{f}_i\mathsf{f}_j = \mathsf{f}_j\mathsf{f}_i$ *when* $|i-j| > 1$;
(5) $\mathsf{e}_i^2\mathsf{e}_j - (\upsilon+\upsilon^{-1})\mathsf{e}_i\mathsf{e}_j\mathsf{e}_i + \mathsf{e}_j\mathsf{e}_i^2 = 0$ *when* $|i-j| = 1$;
(6) $\mathsf{f}_i^2\mathsf{f}_j - (\upsilon+\upsilon^{-1})\mathsf{f}_i\mathsf{f}_j\mathsf{f}_i + \mathsf{f}_j\mathsf{f}_i^2 = 0$ *when* $|i-j| = 1$;
(7) $\mathsf{e}_i\mathsf{f}_j - \mathsf{f}_j\mathsf{e}_i = \delta_{ij}\frac{\widetilde{\mathsf{k}}_i - \widetilde{\mathsf{k}}_i^{-1}}{\upsilon-\upsilon^{-1}}$, *where* $\widetilde{\mathsf{k}_i} = \mathsf{k}_i\mathsf{k}_{i+1}^{-1}$.

We rewrite e_i etc. as $\mathsf{e}_{i,r}$ etc. if different r's are under consideration.

Corollary 8.3. *For* $r \in \mathbb{N}$, *there is an epimorphism from* $\mathbf{V}(\infty, r+1)$ *to* $\mathbf{V}(\infty, r)$ *by sending* $\mathsf{e}_{i,r+1}$ *to* $\mathsf{e}_{i,r}$, $\mathsf{f}_{i,r+1}$ *to* $\mathsf{f}_{i,r}$ *and* $\mathsf{k}_{i,r+1}$ *to* $\mathsf{k}_{i,r}$ *for* $i \in \mathbb{Z}$.

We shall see in the next section that $\mathbf{V}(\infty, r)$ is isomorphic to $\mathbf{U}(\infty, r)$. Thus, we obtain the so-called transfer maps from $\mathbf{U}(\infty, r+1)$ to $\mathbf{U}(\infty, r)$ (cf. [23]) which is useful in the study of the polynomial representation category $\mathcal{C}_r$ in Sect. 11.

Let $V(\infty, r) = \xi_r(U(\infty))$ be the integral $\mathcal{Z}$-form of $\mathbf{V}(\infty, r)$, where ξ_r is defined in Theorem 4.1 (3). Then we have the following integral basis for $V(\infty, r)$; cf. Theorem 6.4.

Proposition 8.4 ([7, 4.12]). *The set*

$$\mathcal{M}_\infty = \{\mathsf{e}^{(A^+)}\mathsf{k}_\lambda\mathsf{f}^{(A^-)} \mid A \in \Xi^\pm(\infty), \lambda \in \mathbb{N}^\infty, \lambda \geqslant \boldsymbol{\sigma}(A), |\lambda| \leqslant r\}.$$

forms a basis for $V(\infty, r)$, *where* $\lambda \geqslant \boldsymbol{\sigma}(A)$ *means* $\lambda_i \geqslant \sigma_i(A)$ *for all* i.

Moreover, the subset $\{\mathsf{e}^{(A^+)}\mathsf{k}_{\boldsymbol{\sigma}(A)}\mathsf{f}^{(A^-)} \mid A \in \Xi(\infty, r)\}$ *of* $\mathcal{M}_\infty$ *forms a basis for* $\mathcal{K}(\infty, r)$.

Note that the last assertion is seen from [7, (3.6.1)].

Remarks 8.5. (1) Since the relation (S7) from Theorem 6.1 does not hold in $\mathbf{V}(\infty, r)$, there is no natural homomorphism from the q-Schur algebra $\mathbf{U}([-n, n], r)$ to $\mathbf{V}(\infty, r)$. However, there is a natural homomorphism from a Borel subalgebra $\mathbf{U}([-n, n], r)^{\geqslant 0}$ or $\mathbf{U}([-n, n], r)^{\leqslant 0}$ into $\mathbf{V}(\infty, r)$ (sending 1 to 1). Thus, the identity element 1 of $\mathbf{V}(\infty, r)$ is a finite sum of orthogonal idempotent elements labelled by the elements in the set $\Lambda([-n, n], r)$. This fact is useful in the proof of Theorem 8.2.

(2) It is interesting to point out that the Doty–Giaquinto type presentation described in Theorem 6.1 (1) does not exist for $\mathbf{V}(\infty, r)$.

9 Infinite q-Schur Algebras and Their Elementary Structure

In this section, we first introduce the q-Schur algebra $\mathcal{S}(\eta, r)$ at η. We will mainly focus on the structure of infinite q-Schur algebras.

Let $\mathcal{H}$ be the Hecke algebra over $\mathcal{Z}$ associated with the symmetric group $\mathfrak{S}_r$ on r letters. Then $\mathcal{H}$ as a $\mathcal{Z}$-algebra has a basis $\{\mathcal{T}_w\}_{w\in\mathfrak{S}_r}$ subject the relations (cf. Theorem 6.7): for all $w \in \mathfrak{S}_r$ and $s \in S := \{(i, i+1) \mid 1 \leqslant i \leqslant n-1\}$,

$$\mathcal{T}_s\mathcal{T}_w = \begin{cases} \mathcal{T}_{sw}, & \text{if } sw > w, \\ (v - v^{-1})\mathcal{T}_w + \mathcal{T}_{sw}, & \text{if } sw < w. \end{cases}$$

Let $\boldsymbol{\mathcal{H}} = \mathcal{H} \otimes \mathbb{Q}(v)$.

Let η be a consecutive segment of $\mathbb{Z}$. Let Ω_η be a free $\mathcal{Z}$-module with basis $\{\omega_i\}_{i\in\eta}$ and let $\boldsymbol{\Omega}_\eta = \Omega_\eta \otimes \mathbb{Q}(v)$. $\mathcal{H}$ acts on $\Omega_\eta^{\otimes r}$ from the right by "place permutations":

$$(\omega_{i_1}\cdots\omega_{i_r})\mathcal{T}_{(j,j+1)} = \begin{cases} \omega_{i_1}\cdots\omega_{i_{j+1}}\omega_{i_j}\cdots\omega_{i_r}, & \text{if } i_j < i_{j+1}; \\ v\omega_{i_1}\cdots\omega_{i_r}, & \text{if } i_j = i_{j+1}; \\ (v-v^{-1})\omega_{i_1}\cdots\omega_{i_r} + \omega_{i_1}\cdots\omega_{i_{j+1}}\omega_{i_j}\cdots\omega_{i_r}, & \text{if } i_j > i_{j+1}. \end{cases}$$

The algebras

$$\boldsymbol{\mathcal{S}}(\eta, r) := \mathrm{End}_{\boldsymbol{\mathcal{H}}}(\boldsymbol{\Omega}_\eta^{\otimes r}), \qquad \mathcal{S}(\eta, r) := \mathrm{End}_{\mathcal{H}}(\Omega_\eta^{\otimes r})$$

are called *q-Schur algebras at η*. Note that $\boldsymbol{\mathcal{S}}(n, r) := \boldsymbol{\mathcal{S}}(\eta, r)$ for $\eta = [1, n]$ is the q-Schur algebra we discussed before. Clearly, if η is finite, $\boldsymbol{\mathcal{S}}(\eta, r) \cong \boldsymbol{\mathcal{S}}(|\eta|, r)$ and $\boldsymbol{\mathcal{S}}(\eta, r) \cong \mathcal{S}(\eta, r) \otimes_{\mathcal{Z}} \mathbb{Q}(v)$. When $\eta = \mathbb{Z}$, the algebras $\mathcal{S}(\infty, r) := \mathcal{S}(\mathbb{Z}, r)$ and $\boldsymbol{\mathcal{S}}(\infty, r)$ are called *infinite q-Schur algebras*.

Since the $\boldsymbol{\mathcal{H}}$-action commutes with the action of $\mathbf{U}(\eta)$, it follows that $\mathbf{U}(\eta, r) \subseteq \boldsymbol{\mathcal{S}}(\eta, r)$. Hence, we obtain algebra homomorphisms

$$\zeta_r : \mathbf{U}(\eta) \to \boldsymbol{\mathcal{S}}(\eta, r), \qquad \zeta_r|_{U(\eta)} : U(\eta) \to \mathcal{S}(\eta, r). \tag{9.0.1}$$

These homomorphisms are surjective whenever η is finite. In particular, we have $U(n, r) = \mathcal{S}(n, r)$ and $\mathbf{U}(n, r) = \boldsymbol{\mathcal{S}}(n, r)$ for all $n, r \geqslant 1$.

Since $\Omega_{[-n,n]}^{\otimes r}$ is a direct summand of the $\mathcal{H}$-submodule $\Omega_{[-n-1,n+1]}^{\otimes r}$ for all $n \geqslant 1$, it follows that we may embed $\mathcal{S}([-n, n], r)$ as a centralizer subalgebra of $\mathcal{S}([-(n+1), n+1], r)$. Hence, we obtain a direct limit system $\{\mathcal{S}([-n, n], r)\}_{n\geqslant 1}$. By Lemma 3.1 (2), we have

$$\mathcal{K}(\infty, r) \cong \varinjlim_n \mathcal{S}([-n, n], r), \qquad \boldsymbol{\mathcal{K}}(\infty, r) \cong \varinjlim_n \boldsymbol{\mathcal{S}}([-n, n], r).$$

This isomorphism implies immediately the second isomorphism in the following.

Proposition 9.1. *The $\mathcal{H}$-module $\Omega_\infty^{\otimes r}$ is isomorphic to the $\mathcal{H}$-module $\bigoplus_{\lambda\in\Lambda(\infty,r)} x_\lambda\mathcal{H}$. Thus, we have the following algebra isomorphisms*

$$\mathcal{S}(\infty,r)\cong\prod_{\mu\in\Lambda(\infty,r)}\bigoplus_{\lambda\in\Lambda(\infty,r)}\mathrm{Hom}_{\mathcal{H}}(x_\mu\mathcal{H},x_\lambda\mathcal{H}),$$
$$\mathcal{K}(\infty,r)\overset{\theta}{\cong}\bigoplus_{\mu\in\Lambda(\infty,r)}\bigoplus_{\lambda\in\Lambda(\infty,r)}\mathrm{Hom}_{\mathcal{H}}(x_\mu\mathcal{H},x_\lambda\mathcal{H}).$$

Here the multiplication in $\prod_{\mu\in\Lambda(\infty,r)}\bigoplus_{\lambda\in\Lambda(\infty,r)}\mathrm{Hom}_{\mathcal{H}}(x_\mu\mathcal{H},x_\lambda\mathcal{H})$ is given as follows:

$$\big(\textstyle\sum_\nu g_{\nu,\tau}\big)_\tau\big(\sum_\lambda f_{\lambda,\mu}\big)_\mu=\big(\sum_\nu\sum_\lambda g_{\nu,\lambda}f_{\lambda,\mu}\big)_\mu,$$
$$\text{where } f_{\lambda,\mu}, g_{\lambda,\mu}\in\mathrm{Hom}_{\mathcal{H}}(x_\mu\mathcal{H},x_\lambda\mathcal{H}) \text{ for all } \lambda,\mu.$$

Similar results hold with $\mathcal{S}$, Ω, $\mathcal{H}$, $\mathcal{K}$ replaced by $\boldsymbol{\mathcal{S}}$, $\boldsymbol{\Omega}$, $\boldsymbol{\mathcal{H}}$, $\boldsymbol{\mathcal{K}}$, respectively.

The second isomorphism can also be made more explicit. For any $\lambda\in\Lambda(\infty,r)$, let $\mathfrak{S}_\lambda$ be the corresponding Young subgroup, and let $\mathfrak{D}_\lambda$ be the set of distinguished right $\mathfrak{S}_r$-coset representatives. We also put $\mathfrak{D}_{\lambda\mu}=\mathfrak{D}_\lambda\cap\mathfrak{D}_\mu^{-1}$. This is the set of distinguished double $(\mathfrak{S}_\lambda,\mathfrak{S}_\mu)$-coset representatives. We also define, for any $w\in\mathfrak{D}_{\lambda,\mu}$, a map

$$\phi_{\lambda\mu}^w:\bigoplus_{\lambda\in\Lambda(\infty,r)}x_\lambda\mathcal{H}\to\bigoplus_{\lambda\in\Lambda(\infty,r)}x_\lambda\mathcal{H}$$

by setting $\phi_{\lambda\mu}^w(x_\nu h)=\delta_{\mu,\nu}\sum_{x\in\mathfrak{S}_\lambda w\mathfrak{S}_\mu}h$. It is well known (see, e.g., [15, (1.3.10)]) that there is a bijection

$$\jmath:\{(\lambda,w,\mu)\mid\lambda,\mu\in\Lambda(\infty,r),w\in\mathfrak{D}_{\lambda,\mu}\}\longrightarrow\Xi(\infty,r).$$

Corollary 9.2. *The isomorphism θ is induced by $\jmath$. In other words, if $\jmath(\lambda,w,\mu)=A$, then $\theta(\phi_A)=\phi_{\lambda\mu}^w$.*

We will identify the two bases under θ.

Let $\varpi\in\Lambda(\infty,r)$ such that $\varpi_i=1$ if $1\leqslant i\leqslant r$ and $\varpi_i=0$ otherwise. The following describe some elementary structure of $\mathcal{S}(\infty,r)$. Part (1) follows from Lemma 6.3, and Part (4) shows that the Hecke algebra $\mathcal{H}$ is always a centralizer subalgebra of $\mathcal{S}(\infty,r)$.

Lemma 9.3. *Let $\lambda,\mu\in\Lambda(\infty,r)$.*

(1) $\mathsf{k}_\lambda=[\mathrm{diag}(\lambda)]=\phi_{\mathrm{diag}(\lambda)}=\phi_{\lambda\lambda}^1$ *for all $\lambda\in\Lambda(\infty,r)$;*
(2) $\mathcal{S}(\infty,r)\mathsf{k}_\lambda=\mathcal{K}(\infty,r)\mathsf{k}_\lambda$*;*
(3) $\mathsf{k}_\lambda\mathcal{S}(\infty,r)\mathsf{k}_\mu=\mathsf{k}_\lambda\mathcal{K}(\infty,r)\mathsf{k}_\mu=\mathrm{Hom}_{\mathcal{H}}(x_\mu\mathcal{H},x_\lambda\mathcal{H})$*;*
(4) *We have $\mathcal{H}\cong\mathsf{k}_\varpi\mathcal{S}(\infty,r)\mathsf{k}_\varpi\cong\mathrm{End}_{\mathcal{S}(\infty,r)}(\Omega_\infty^{\otimes r})\cong\mathrm{End}_{U(\infty)}(\Omega_\infty^{\otimes r})$.*

Similar results hold for the algebras over $\mathbb{Q}(v)$.

We continue to use boldface fonts for objects over the field $\mathbb{Q}(v)$. The identification of $\mathrm{End}_{\mathcal{H}}(\mathbf{\Omega}_\infty^{\otimes r})$ with $\prod_{\mu\in\Lambda(\infty,r)}\bigoplus_{\lambda\in\Lambda(\infty,r)}\mathrm{Hom}_{\mathcal{H}}(x_\mu\mathcal{H}, x_\lambda\mathcal{H})$ gives rise to a natural injective algebra homomorphism $\bar{\zeta}_r : \mathcal{K}(\infty,r)\to\mathrm{End}_{\mathcal{H}}(\mathbf{\Omega}_\infty^{\otimes r})$. By 9.1 and the definition of $\widehat{\mathcal{K}}(\infty,r)$, we know that $\bar{\zeta}_r$ induces naturally an injective algebra homomorphism

$$\bar{\zeta}_r : \widehat{\mathcal{K}}(\infty,r)\to\mathrm{End}_{\mathcal{H}}(\mathbf{\Omega}_\infty^{\otimes r}),$$

sending $\sum_{A\in\Xi(\infty,r)}\beta_A[A] = \sum_{\mu\in\Lambda(\infty,r)}(\sum_{\substack{\mathrm{col}(A)=\mu\\ A\in\Xi(\infty,r)}}\beta_A[A])$ to $(\sum_{\substack{\mathrm{col}(A)=\mu\\ A\in\Xi(\infty,r)}}\beta_A[A])_{\mu\in\Lambda(\infty,r)}$. We will view $\widehat{\mathcal{K}}(\infty,r)$ as a subalgebra of $\mathrm{End}_{\mathcal{H}}(\mathbf{\Omega}_\infty^{\otimes r})$.

Recall from Theorem 4.1 (3) the algebra homomorphism $\xi_r : \mathbf{U}(\infty)\to\mathbf{V}(\infty,r)$. We are now ready to identify $\mathbf{U}(\infty,r)$ with $\mathbf{V}(\infty,r)$.

Theorem 9.4 ([7, 5.4]). *The map $\zeta_r : \mathbf{U}(\infty)\to\mathrm{End}_{\mathcal{H}}(\mathbf{\Omega}_\infty^{\otimes r})$ factors through $\bar{\zeta}_r : \mathbf{V}(\infty,r)\to\mathrm{End}_{\mathcal{H}}(\mathbf{\Omega}_\infty^{\otimes r})$. In other words, we have $\zeta_r = \bar{\zeta}_r\circ\xi_r$. Hence, $\mathbf{U}(\infty,r)=\mathbf{V}(\infty,r)$.*

Remark 9.5. (1) The injective map $\bar{\zeta}_r : \widehat{\mathcal{K}}(\infty,r)\to\mathrm{End}_{\mathcal{H}}(\mathbf{\Omega}_\infty^{\otimes r})$ is not surjective; see [7, 5.4]. Thus, unlike the (finite) q-Schur algebra case, the homomorphism ζ_r from $\mathbf{U}(\infty)$ to the infinite q-Schur algebra $\boldsymbol{\mathcal{S}}(\infty,r)$ is not surjective, i.e. the algebra $\mathbf{U}(\infty,r)$ is a proper subalgebra of $\boldsymbol{\mathcal{S}}(\infty,r)$.

(2) It can be proved (see [7, 6.9]) that the epimorphism $\zeta_r : \mathbf{U}(\infty)\to\mathbf{U}(\infty,r)$ can be extended to an algebra epimorphism $\hat{\zeta}_r : \widehat{\mathcal{K}}(\infty)\to\widehat{\mathcal{K}}(\infty,r)$ by sending $\sum_{A\in\widetilde{\Xi}(\infty)}\beta_A[A]$ to $\sum_{A\in\Xi(\infty,r)}\beta_A[A]$. Thus, we obtain an algebra homomorphism $\zeta_r : \widehat{\mathcal{K}}(\infty)\to\mathrm{End}_{\mathcal{H}}(\mathbf{\Omega}_\infty^{\otimes r})$.

(3) It is possible to introduce a new algebra $\widehat{\mathcal{K}}^\dagger(\infty)$ which has the infinite q-Schur algebra as a homomorphic image. Let $\widehat{\mathcal{K}}^\dagger(\infty)$ be the vector space of all formal (possibly infinite) $\mathbb{Q}(v)$-linear combinations $\sum_{A\in\widetilde{\Xi}(\infty)}\beta_A[A]$ which have the following properties: for any $\mathbf{x}\in\mathbb{Z}^\infty$,

$$\text{the set } \{A\in\widetilde{\Xi}(\infty)\mid \beta_A\neq 0,\ \mathrm{col}\,(A)=\mathbf{x}\} \text{ is finite.} \tag{9.5.1}$$

In other words, for any $\mu\in\mathbb{Z}^n$, the sum $\sum_{A\in\widetilde{\Xi}(\infty)}\beta_A[A]\cdot[\mathrm{diag}(\mu)]$ is finite. We can define the product of two elements $\sum_{A\in\widetilde{\Xi}(\infty)}\beta_A[A]$, $\sum_{B\in\widetilde{\Xi}(\infty)}\gamma_B[B]$ in $\widehat{\mathcal{K}}^\dagger(\infty)$ to be $\sum_{A,B}\beta_A\gamma_B[A]\cdot[B]$. This defines an associative algebra structure on $\widehat{\mathcal{K}}^\dagger(\infty)$. This algebra has an identity element $\sum_{\lambda\in\mathbb{Z}^\infty}[\mathrm{diag}(\lambda)]$: the sum of all $[D]$ with D a diagonal matrix in $\tilde{\Xi}(\infty)$. One proves that the map $\zeta_r^\dagger$: $\widehat{\mathcal{K}}^\dagger(\infty)\to\mathrm{End}_{\mathcal{H}}(\mathbf{\Omega}_\infty^{\otimes r})$ sending $\sum_{A\in\widetilde{\Xi}(\infty)}\beta_A[A]$ to $\sum_{A\in\Xi(\infty,r)}\beta_A[A]$ is an epimorphism; see [7, 6.9].

10 Highest Weight Representations of $\mathbf{U}(\infty)$

In this section, we discuss the "standard" representation theory of $\mathbf{U}(\infty)$. This includes the category $\mathcal{C}$ of weight modules, the category $\mathcal{C}^{\text{hi}}$ of weight modules with highest weights, the category $\mathcal{C}^{\text{int}}$ of integrable modules and the category $\mathcal{O}$, all of which are full subcategories of the category $\mathbf{U}(\infty)$-**Mod** of $\mathbf{U}(\infty)$-modules.

Let $X(\infty) = \{\lambda = (\lambda_i)_{i\in\mathbb{Z}} \mid \lambda_i \in \mathbb{Z}\}$ be the weight lattice, and let

$$X^+(\infty) = \{\lambda \in X(\infty) \mid \lambda_i \geqslant \lambda_{i+1} \text{ for all } i \in \mathbb{Z}\}$$

be the set of dominant weights. For $i \in \mathbb{Z}$, let as before $\boldsymbol{e}_i = (\ldots, 0, \underset{i}{1}, 0, \ldots)$ and $\alpha_i = \boldsymbol{e}_i - \boldsymbol{e}_{i+1}$. Then $R(\infty) = \{\boldsymbol{e}_i - \boldsymbol{e}_j \mid i \neq j\}$ is the root system of $\mathfrak{gl}_\infty$, and $R^+(\infty) = \{\boldsymbol{e}_i - \boldsymbol{e}_j \mid i < j\}$ is the associated positive system. Let $\Pi(\infty) = \{\alpha_i \mid i \in \mathbb{Z}\}$ be the set of all simple roots, and let $\leqslant$ be the partial ordering on $X(\infty)$ defined by setting, for all $\lambda, \mu \in X(\infty)$, $\mu \leqslant \lambda$ if $\lambda - \mu \in \mathbb{N}\Pi(\infty)$.

For a $\mathbf{U}(\infty)$-module M and $\lambda \in X(\infty)$, let $M_\lambda = \{x \in M \mid K_i x = \upsilon^{\lambda_i} x \text{ for } i \in \mathbb{Z}\}$. If $M_\lambda \neq 0$, then λ is called a *weight* of M and M_λ is called a *weight space*; if $M = \bigoplus_{\lambda\in X(\infty)} M_\lambda$, then M is called a *weight module*.

Let $\text{wt}(M) = \{\lambda \in X(\infty) \mid M_\lambda \neq 0\}$ denote the set of the weights of M. It is clear that for a $\mathbf{U}(\infty)$-module M and $i \in \mathbb{Z}$, we have

$$E_i M_\lambda \subseteq M_{\lambda+\alpha_i} \quad \text{and} \quad F_i M_\lambda \subseteq M_{\lambda-\alpha_i}. \tag{10.0.2}$$

Let $\mathcal{C}$ denote the full subcategory of $\mathbf{U}(\infty)$-**Mod** consisting of all weight modules, and let $\mathcal{C}^{\text{hi}}$ be the full subcategory of $\mathcal{C}$ whose objects are the weight $\mathbf{U}(\infty)$-module M with the property that, for any $x \in M$, there exists $n_0 \geqslant 1$ such that $u.x = 0$ whenever u is a monomial in the E_i's having degree at least n_0.

An *integrable* $\mathbf{U}(\infty)$-module M is a weight module satisfying that, for any $x \in M$ and any $i \in \mathbb{Z}$, there exists $n_0 \geqslant 1$ such that $E_i^n x = F_i^n x = 0$ for all $n \geqslant n_0$. Let $\mathcal{C}^{\text{int}}$ be the full subcategory of $\mathcal{C}$ consisting of integrable $\mathbf{U}(\infty)$-modules.

The category $\mathcal{O}$ is the full subcategory of $\mathcal{C}$ each of whose objects M has finite dimensional weight spaces and has weight set $\text{wt}(M) \subseteq \bigcup_{i=1}^s (-\infty, \lambda^{(i)}]$ for some $\lambda^{(1)}, \ldots, \lambda^{(s)} \in X(\infty)$. Here, for $\lambda \in X(\infty)$, $(-\infty, \lambda] := \{\mu \in X(\infty) \mid \mu \leqslant \lambda\}$.

We first observe the following; see [7, Sect. 7].

Proposition 10.1 ([7, 7.2]). *The category $\mathcal{O}$ is a full subcategory of $\mathcal{C}^{\text{hi}}$. Thus, we have the following flowchart for various chains of full subcategories:*

$$\begin{array}{ccccc}
 & & \mathcal{C} & & \\
 & \diagup & & \diagdown & \\
\mathcal{C}^{\text{hi}} & & & & \mathcal{C}^{\text{int}} \\
| & \diagdown & & & | \\
\mathcal{O} & & & & \mathcal{C}^{\text{hi}} \cap \mathcal{C}^{\text{int}} \\
 & \diagdown & & \diagup & \\
 & & \mathcal{O} \cap \mathcal{C}^{\text{int}} & &
\end{array}$$

For a $\mathbf{U}(\infty)$-module M, if there exist a $\lambda \in X(\infty)$ and a nonzero vector $x_0 \in M_\lambda$ such that $E_i x_0 = 0$, $K_i x_0 = v^{\lambda_i} x_0$ for all $i \in \mathbb{Z}$ and $\mathbf{U}(\infty)x_0 = M$, then M is called a *highest weight module*. The vector x_0 is called a highest weight vector. Using a standard argument (for finite dimensional highest weight modules), one sees easily that a highest weight $\mathbf{U}(\infty)$-module with highest weight λ is a weight module and $M = \bigoplus_{\mu \leqslant \lambda} M_\mu$. Moreover, $\dim M_\lambda = 1$ and M contains a unique maximal submodule.

For $\lambda \in X(\infty)$, let

$$M(\lambda) = \mathbf{U}(\infty)/\left(\sum_{i\in\mathbb{Z}} \mathbf{U}(\infty)E_i + \sum_{i\in\mathbb{Z}} \mathbf{U}(\infty)(K_i - v^{\lambda_i})\right),$$

which is called a *Verma module*. This is a highest weight module with the highest weight vector 1_λ, the image of the 1. Thus, $M(\lambda)$ has a unique irreducible quotient module $L(\lambda)$. Clearly, the modules $M(\lambda)$ and $L(\lambda)$ are all in the category $\mathcal{C}^{\mathrm{hi}}$ and in the category $\mathcal{O}$.

We have the following classification of irreducible modules in the category $\mathcal{C}^{\mathrm{hi}}$.

Theorem 10.2 ([7, 7.4]). *There is a bijection $\lambda \mapsto L(\lambda)$ between $X(\infty)$ and the set of isomorphism classes of irreducible $\mathbf{U}(\infty)$-modules in the category $\mathcal{C}^{\mathrm{hi}}$ (resp. in the category $\mathcal{O}$).*

The classification of irreducible integrable modules seems very hard. However, it is possible to classify all irreducible integrable modules in $\mathcal{C}^{\mathrm{hi}}$, i.e. all irreducible modules in $\mathcal{C}^{\mathrm{hi}} \cap \mathcal{C}^{\mathrm{int}}$. We shall see in the next section that there exist irreducible integrable modules which are not in $\mathcal{C}^{\mathrm{hi}}$.

For $\lambda \in X^+(\infty)$, let $I(\lambda)$ be the submodule of $M(\lambda)$ generated by $F_i^{(\lambda_i-\lambda_{i+1}+1)}$ $(i \in \mathbb{Z})$. By the commutator formula [18, 4.1 (a)]:

$$E_i^{(k)} F_i^{(l)} = \sum_{t=0}^{\min(k,l)} F_i^{(l-t)} \begin{bmatrix} \tilde{K}_i; 2t-k-l \\ t \end{bmatrix} E_i^{(k-t)},$$

we deduce $I(\lambda) = \sum_{i\in\mathbb{Z}} \mathbf{U}^-(\infty) F_i^{(\lambda_i-\lambda_{i+1}+1)} 1_\lambda$. Hence $I(\lambda)$ is a proper submodule of $M(\lambda)$. Let $\widetilde{L(\lambda)} = M(\lambda)/I(\lambda)$. By [21, 3.5.3], we have the following.

Lemma 10.3. *For $\lambda \in X^+(\infty)$, the module $\widetilde{L(\lambda)}$ is an integrable $\mathbf{U}(\infty)$-module. Hence, as a homomorphic image of $\widetilde{L(\lambda)}$, the module $L(\lambda)$ is an integrable $\mathbf{U}(\infty)$-module.*

We now have the following classification theorem whose proof requires the fact that, if M is a object in $\mathcal{C}^{\mathrm{int}}$ and $0 \neq x_0 \in M_\lambda$ satisfying $E_i x_0 = 0$ for all $i \in \mathbb{Z}$, then $\lambda \in X^+(\infty)$.

Theorem 10.4 ([7, 7.6 and 7.7]). *The map $\lambda \mapsto L(\lambda)$ defines a bijection between $X^+(\infty)$ and the set of isomorphism classes of irreducible $\mathbf{U}(\infty)$-modules in the* *category $\mathcal{C}^{\text{int}} \cap \mathcal{C}^{\text{hi}}$. Hence, it also defines a bijection between $X^+(\infty)$ and the set of isomorphism classes of irreducible $\mathbf{U}(\infty)$-modules in the category $\mathcal{C}^{\text{int}} \cap \mathcal{O}$. Moreover, we have $L(\lambda) \cong \widetilde{L(\lambda)}$ for all $\lambda \in X^+(\infty)$.*

Remarks 10.5. It can be proved (see [7, 7.7]) that, for $\lambda \in X^+(\infty)$, $L(\lambda)$ is isomorphic to the direct limit of finite dimensional irreducible $\mathbf{U}([-n,n])$-module $L(\lambda_{[-n,n]})$, where $\lambda_{[-n,n]} = (\lambda_i)_{-n \leqslant i \leqslant n}$, if $\lambda = (\lambda_i)_{i \in \mathbb{Z}}$. However, for a non-dominant weight λ, we could not prove that a similar isomorphism holds.

The following result shows that there are not many finite dimensional weight $\mathbf{U}(\infty)$-modules. Let $\mathcal{C}^{\text{fd}}$ be the category of finite dimensional weight $\mathbf{U}(\infty)$-modules.

Theorem 10.6 ([7, 7.10]). *The category $\mathcal{C}^{\text{fd}}$ is a completely reducible category, and the modules $L(m\mathbf{1})$ $(m \in \mathbb{Z})$, where $\mathbf{1} = (\ldots, 1, 1, \ldots, 1, \ldots) \in X(\infty)$, are all non-isomorphic irreducible $\mathbf{U}(\infty)$-modules in $\mathcal{C}^{\text{fd}}$.*

11 Polynomial Type Representations of U(∞)

Polynomial representations of $\mathbf{U}(n)$ are obtained via q-Schur algebra representations and are all highest weight modules. In this section, we will see that the $\mathbf{U}(\infty)$-modules arising from weight modules over infinite q-Schur algebras are in general not highest weight modules. Thus, we obtain a "nonstandard" representation theory for $\mathbf{U}(\infty)$.

Let r be a positive integer, and let $\mathcal{C}_r$ be the full subcategory of $\mathcal{C}$ consisting of $\mathbf{U}(\infty, r)$-modules which are weight modules when regarded as $\mathbf{U}(\infty)$-modules via $\zeta_r : \mathbf{U}(\infty) \to \mathbf{U}(\infty, r)$. We also define $\mathcal{C}^{\text{pol}}$ to be the full subcategory of $\mathcal{C}$ consisting of weight $\mathbf{U}(\infty)$-modules M such that $\text{wt}(M) \subseteq \mathbb{N}^\infty$. (Recall that the elements of $\mathbb{N}^\infty$ have finite support.) We call the objects in $\mathcal{C}^{\text{pol}}$ *polynomial representations*.

There is a close connection between polynomial representations and weight $\boldsymbol{\mathcal{S}}(\infty, r)$-modules. By definition, an $\boldsymbol{\mathcal{S}}(\infty, r)$-module M is called a weight $\boldsymbol{\mathcal{S}}(\infty, r)$-module if $M = \bigoplus_{\lambda \in \Lambda(\infty, r)} \mathsf{k}_\lambda M$. (Recall from Lemma 9.3(1) that $\mathsf{k}_\lambda = \phi^1_{\lambda\lambda}$.) Let $\boldsymbol{\mathcal{S}}(\infty, r)$-**mod** be the category of weight $\boldsymbol{\mathcal{S}}(\infty, r)$-modules. Since the quantum group $\mathbf{U}(\infty)$ maps to $\boldsymbol{\mathcal{S}}(\infty, r)$, every $\boldsymbol{\mathcal{S}}(\infty, r)$-module is naturally a $\mathbf{U}(\infty)$-module. Moreover, we have the following (which justifies the terminology of weight $\boldsymbol{\mathcal{S}}(\infty, r)$-modules).

Lemma 11.1. *Let M be a weight $\boldsymbol{\mathcal{S}}(\infty, r)$-module regarded naturally as a $\mathbf{U}(\infty)$-module. Then, for any $\lambda \in \Lambda(n, r)$, $\mathsf{k}_\lambda M = M_\lambda$. Hence, M is a weight $\mathbf{U}(\infty)$-module. Moreover, M is also an integrable $\mathbf{U}(\infty)$-module.*

We may define the category $\boldsymbol{\mathcal{K}}(\infty, r)$-**mod** of weight $\boldsymbol{\mathcal{K}}(\infty, r)$-modules. By regarding $\mathbf{U}(\infty, r)$ as a subalgebra of $\boldsymbol{\mathcal{S}}(\infty, r)$, we may define $\mathbf{U}(\infty, r)$-**mod** similarly. Note that $\mathbf{U}(\infty, r)$-**mod** is a full subcategory of $\mathcal{C}_r$.

Since every weight $\mathcal{K}(\infty, r)$-module induces naturally a weight $\boldsymbol{\mathcal{S}}(\infty, r)$-module using the fact that for any $x \in \boldsymbol{\mathcal{S}}(\infty, r)$ and $\lambda \in \Lambda(\infty, r)$, $x\mathsf{k}_\lambda \in \mathcal{K}(\infty, r)$, we have immediately the following category isomorphisms.

Lemma 11.2 ([7, 8.2]). *The category* $\mathcal{K}(\infty, r)$-**mod**, *and hence the category* $\mathrm{U}(\infty, r)$-**mod**, *is isomorphic to the category* $\boldsymbol{\mathcal{S}}(\infty, r)$-**mod**.

Thus, $\boldsymbol{\mathcal{S}}(\infty, r)$-**mod** can be regarded as a subcategory of $\mathcal{C}$.

We want to classify the irreducible modules in $\boldsymbol{\mathcal{S}}(\infty, r)$-**mod**. Let $\boldsymbol{\Omega}(\infty, r) := \boldsymbol{\mathcal{S}}(\infty, r)\mathsf{k}_\varpi$. Since $\boldsymbol{\mathcal{S}}(\infty, r)\mathsf{k}_\lambda = \mathcal{K}(n, r)\mathsf{k}_\lambda$ for all $\lambda \in \Lambda(\infty, r)$ (see 9.3), it follows that $\boldsymbol{\Omega}(\infty, r)$ is a weight $\boldsymbol{\mathcal{S}}(\infty, r)$-module. Moreover, it is an $(\boldsymbol{\mathcal{S}}(\infty, r), \boldsymbol{\mathcal{H}})$-bimodule.

Proposition 11.3. *The* $(\boldsymbol{\mathcal{S}}(\infty, r), \boldsymbol{\mathcal{H}})$*-bimodule* $\boldsymbol{\Omega}(\infty, r)$ *is isomorphic to* $\boldsymbol{\Omega}_\infty^{\otimes r}$.

For $\lambda, \mu \in \Lambda(\infty, r)$, we say that λ and μ are associated if μ can be derived from λ by reordering the parts of λ. Let $\Lambda^+(r) = \{\lambda \in \Lambda(\infty, r) \mid \lambda_i = 0 \text{ for } i \leqslant 0 \text{ and } \lambda_i \geqslant \lambda_{i+1} \text{ for } i \geqslant 1\}$. This is the set of all partitions of r. We define a map from $\Lambda(\infty, r)$ to $\Lambda^+(r)$ by sending λ to λ^+ where λ^+ is the unique element in $\Lambda^+(r)$ which is associated with λ.

For $\lambda \in \Lambda(\infty, r)$, let λ' be the partition dual to λ^+. Let $y_{\lambda'} = \sum_{w \in \mathfrak{S}_{\lambda'}} (-q)^{-l(w)} T_w$ and $z_\lambda = \phi^1_{\lambda\varpi} T_{w_\lambda} y_{\lambda'}$ where $q = \upsilon^2 \in \mathbb{Q}(\upsilon)$ and $w_\lambda \in \mathfrak{D}_{\lambda,\lambda'}$ satisfying $\mathfrak{S}_\lambda \cap w_\lambda \mathfrak{S}_{\lambda'} w_\lambda^{-1} = \{1\}$. Let $S^\lambda = z_\lambda \boldsymbol{\mathcal{H}}$ be the Specht module of $\boldsymbol{\mathcal{H}}$ associated with λ, and let $W(\infty, \lambda) = \boldsymbol{\mathcal{S}}(\infty, r) z_\lambda$. We call $W(\infty, \lambda)$ the *Weyl module* of $\boldsymbol{\mathcal{S}}(\infty, r)$.

The following results classify irreducible representations of three relevant categories.

Theorem 11.4. *Let* $\lambda \in \Lambda(\infty, r)$.

(1) [7, 8.4] $W(\infty, \lambda) \cong W(\infty, \lambda^+)$ *as* $\boldsymbol{\mathcal{S}}(\infty, r)$*-modules;*
(2) [7, 8.8 (1)] *the set* $\{W(\infty, \lambda) \mid \lambda \in \Lambda^+(r)\}$ *forms a complete set of irreducible modules in* $\boldsymbol{\mathcal{S}}(\infty, r)$-**mod***;*
(3) [7, 9.3 (1)] *the set* $\{W(\infty, \lambda) \mid \lambda \in \bigcup_{i=0}^r \Lambda^+(i)\}$ *forms a complete set of irreducible modules in* $\mathcal{C}_r$*;*
(4) [7, 9.3 (2)] *the set* $\{W(\infty, \lambda) \mid \lambda \in \bigcup_{i=0}^\infty \Lambda^+(i)\}$ *forms a complete set of irreducible modules in* $\mathcal{C}^{\mathrm{pol}}$.

The proof for Part (3) requires the following facts: (1) there is a surjective homomorphism, the transfer map, from $\mathbf{U}(\infty, r+1)$ to $\mathbf{U}(\infty, r)$ (see 8.3); (2) every irreducible polynomial representation of $\mathbf{U}(\infty)$ is an irreducible weight $\boldsymbol{\mathcal{S}}(\infty, r)$-module for some r.

Finally, we mention the following result.

Theorem 11.5 ([7, 8.8 (2)]). *Every weight* $\boldsymbol{\mathcal{S}}(\infty, r)$*-module is completely reducible.*

It would be interesting to point out that the category of finite dimensional weight $\mathbf{U}(n)$-modules possesses a quite rich structure. It covers all finite dimensional $\boldsymbol{\mathcal{S}}(n,r)$-modules which form a major constituent. As a contrast, the category $\mathcal{C}^{\mathrm{fd}}$ of finite dimensional $\mathbf{U}(\infty)$-representations is more or less trivial. However, the infinite dimensional $\mathbf{U}(\infty)$-module categories $\mathcal{C}^{\mathrm{hi}}$, $\mathcal{C}^{\mathrm{int}}$, $\mathcal{O}$ and $\boldsymbol{\mathcal{S}}(\infty,r)$-**mod** have inherited some of the features from their (finite dimensional) $\mathbf{U}(n)$ counterparts. For example, the complete reducibility continue to hold in $\boldsymbol{\mathcal{S}}(\infty,r)$-**mod**, and the irreducible objects in $\mathcal{C}^{\mathrm{hi}} \cap \mathcal{C}^{\mathrm{int}}$ are indexed by "dominant weights."

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Cherednik Algebras for Algebraic Curves

Michael Finkelberg and Victor Ginzburg

Abstract For any algebraic curve C and $n \geq 1$, Etingof introduced a "global" Cherednik algebra as a natural deformation of the cross product $\mathscr{D}(C^n) \rtimes \mathbb{S}_n$, of the algebra of differential operators on C^n and the symmetric group. We provide a construction of the global Cherednik algebra in terms of quantum Hamiltonian reduction. We study a category of *character $\mathscr{D}$-modules* on a representation scheme associated with C and define a Hamiltonian reduction functor from that category to category $\mathcal{O}$ for the global Cherednik algebra.

In the special case of the curve $C = \mathbb{C}^\times$, the global Cherednik algebra reduces to the trigonometric Cherednik algebra of type $\mathbf{A}_{n-1}$, and our character $\mathscr{D}$-modules become holonomic $\mathscr{D}$-modules on $GL_n(\mathbb{C}) \times \mathbb{C}^n$. The corresponding perverse sheaves are reminiscent of (and include as special cases) Lusztig's *character sheaves*.

Keywords $\mathscr{D}$-modules · Character sheaves · Cherednik algebras

Mathematics Subject Classifications (2000): 20Gxx (20C08)

1 Introduction

1.1 Global Cherednik Algebras

Associated with an integer $n \geq 1$ and an algebraic curve C, there is an interesting family, $\mathsf{H}_{\kappa,\psi}$, of sheaves of associative algebras on $C^{(n)} = C^n/\mathbb{S}_n$, the nth symmetric power of C. The algebras in question, referred to as *global Cherednik*

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algebras, see Sect. 3.1, are natural deformations of the cross-product $\mathscr{D}_\psi(C^n) \rtimes \mathbb{S}_n$, of the sheaf of (twisted) differential operators[1] on C^n and the symmetric group $\mathbb{S}_n$ that acts on $\mathscr{D}_\psi(C^n)$. The algebras $\mathsf{H}_{\kappa,\psi}$ were introduced by Etingof, [E], as "global counterparts" of rational Cherednik algebras studied in [EG].

The global Cherednik algebra $\mathsf{H}_{\kappa,\psi}$ contains an important *spherical subalgebra* $\mathsf{eH}_{\kappa,\psi}\mathsf{e}$, where e denotes the symmetriser idempotent in the group algebra of the group $\mathbb{S}_n$. We generalize [GG] and prove that the algebra $\mathsf{eH}_{\kappa,\psi}\mathsf{e}$ may be obtained as a quantum Hamiltonian reduction of $\mathscr{D}_{n\kappa,\psi}(\mathrm{rep}^n_C \times \mathbb{P}^{n-1})$, a sheaf of twisted differential operators on $\mathrm{rep}^n_C \times \mathbb{P}^{n-1}$, cf. Theorem 3.3.3.

Our result provides a strong link between categories of $\mathscr{D}_{n\kappa,\psi}(\mathrm{rep}^n_C \times \mathbb{P}^{n-1})$-modules and $\mathsf{H}_{\kappa,\psi}$-modules. Specifically, following the strategy of [GG], Sect. 7, we construct an exact functor

$$\mathbb{H} : \mathscr{D}_{n\kappa,\psi}(\mathrm{rep}^n_C \times \mathbb{P}^{n-1})\text{-mod} \longrightarrow \mathsf{H}_{\kappa,\psi}\text{-mod}, \tag{1.1.1}$$

called the functor of Hamiltonian reduction.

1.2 Character Sheaves

In mid 1980s, Lusztig introduced an important notion of *character sheaf* on a reductive algebraic group G. In more detail, write $\mathfrak{g}$ for the Lie algebra of G and use the Killing form to identify $\mathfrak{g}^* \cong \mathfrak{g}$. Let $\mathcal{N} \subset \mathfrak{g}^*$ be the image of the set of nilpotent elements in $\mathfrak{g}$, and let $G \times \mathcal{N} \subset G \times \mathfrak{g}^* = T^*G$ be the *nil-cone* in the total space of the cotangent bundle on G.

Recall further that, associated with any perverse sheaf M on G, one has its characteristic variety $SS(M) \subset T^*G$. A character sheaf is, by definition, an $\mathrm{Ad}G$-equivariant perverse sheaf M, on G, such that the corresponding characteristic variety is nilpotent, i.e. such that we have $SS(M) \subset G \times \mathcal{N}$.

We will be interested in the special case $G = GL_n$. Motivated by the geometric Langlands conjecture, Laumon, [La1], generalized the notion of character sheaf on GL_n to the "global setting" involving an arbitrary smooth algebraic curve. Given such a curve C, Laumon replaces the adjoint quotient stack $G/\mathrm{Ad}G$ by Coh^n_C, a certain stack of length n coherent sheaves on C. He then defines a *global nilpotent subvariety* of the cotangent bundle $T^*Coh^n_C$, cf. [La2], and considers the class of perverse sheaves M on Coh^n_C such that $SS(M)$ is contained in the global nilpotent subvariety.

In this chapter, we introduce character sheaves on $\mathrm{rep}^n_C \times \mathbb{P}^{n-1}$. Here, the scheme rep^n_C is an appropriate Quot scheme of length n sheaves on C, a close cousin of Coh^n_C, and $\mathbb{P}^{n-1}$ is an $n-1$-dimensional projective space. In Sect. 4.4, we define a version of "global nilpotent variety" $\mathbb{M}_{\mathsf{nil}} \subset T^*(\mathrm{rep}^n_C \times \mathbb{P}^{n-1})$, and introduce a class

[1] We refer the reader to [BB] and [K] for the basics of the theory of twisted differential operators.

of $\mathscr{D}$-modules on $\mathrm{rep}_C^n \times \mathbb{P}^{n-1}$, called *character $\mathscr{D}$-modules*, which have a nilpotent characteristic variety, i.e. are such that $SS(M) \subset \mathbb{M}_{\mathsf{nil}}$, see Definition 4.5.2.

The group $G = GL_n$ acts on both rep_C^n and $\mathbb{P}^{n-1}$ in a natural way. In analogy with the theory studied by Lusztig and Laumon, perverse sheaves associated with character $\mathscr{D}$-modules via the Riemann–Hilbert correspondence are locally constant along G-diagonal orbits in $\mathrm{rep}_C^n \times \mathbb{P}^{n-1}$.

A very special feature of the G-variety $\mathrm{rep}_C^n \times \mathbb{P}^{n-1}$ is that the corresponding nilpotent variety, $\mathbb{M}_{\mathsf{nil}}$, turns out to be a *Lagrangian* subvariety in $T^*(\mathrm{rep}_C^n \times \mathbb{P}^{n-1})$. This follows from a geometric result saying that the group GL_n acts diagonally on $\mathscr{N} \times \mathbb{C}^n$ with *finitely many* orbits, [GG], Corollary 2.2. These orbits may be parametrised by the pairs (λ, μ), of arbitrary partitions $\lambda = \lambda_1 + \cdots + \lambda_p$ and $\mu = \mu_1 + \cdots + \mu_q$, with total sum $\lambda + \mu = n$, see [AH, T].

1.3 The Trigonometric Case

Character $\mathscr{D}$-modules play an important role in representation theory of the global Cherednik algebra $\mathsf{H}_{\kappa,\psi}$. In more detail, there is a natural analogue, $\mathcal{O}(\mathsf{H}_{\kappa,\psi})$, of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ for the global Cherednik algebra, see Definition 4.6.1. We show (Proposition 4.6.2) that the Hamiltonian reduction functor (1.1.1) sends character $\mathscr{D}$-modules to objects of the category $\mathcal{O}(\mathsf{H}_{\kappa,\psi})$; moreover, the latter category gets identified, via the functor $\mathbb{H}$, with a quotient of the former by the Serre subcategory Ker $\mathbb{H}$.

In the special case of the curve $C = \mathbb{C}^\times$, the global Cherednik algebra reduces to H_κ, the *trigonometric* Cherednik algebra of type $\mathbf{A}_{n-1}$, see Sect. 5.1 for definitions, and the sheaves considered by Laumon become Lusztig's character sheaves on the group GL_n. Similarly, our character $\mathscr{D}$-modules become (twisted) $\mathscr{D}$-modules on $GL_n \times \mathbb{P}^{n-1}$.

Given a character sheaf on GL_n in the sense of Lusztig, one may pull-back the corresponding $\mathscr{D}$-module via the first projection $GL_n \times \mathbb{P}^{n-1} \to GL_n$. The resulting $\mathscr{D}$-module on $GL_n \times \mathbb{P}^{n-1}$ is a character $\mathscr{D}$-module in our sense. However, there are many other quite interesting character $\mathscr{D}$-modules on $GL_n \times \mathbb{P}^{n-1}$ which do not come from Lusztig's character sheaves on GL_n.

Sometimes, it is more convenient to replace GL_n by its subgroup SL_n. In that case, we prove that *cuspidal* character $\mathscr{D}$-modules correspond, via the Hamiltonian functor, to finite dimensional representations of the corresponding Cherednik algebra, cf. Corollary 4.6.6 and Theorem 5.5.1.

1.4 Convention

The trigonometric Cherednik algebra depends on one complex parameter, to be denoted $\kappa \in \mathbb{C}$. Such an algebra corresponds, via the quantum Hamiltonian reduction construction, to a sheaf of twisted differential operators on $SL_n \times \mathbb{P}^{n-1}$, which is also labelled by one complex parameter, to be denoted $c \in \mathbb{C}$. Throughout the chapter, we will use the normalization of the above parameters κ and c, such that the sheaf of TDO with parameter c gives rise to the trigonometric Cherednik algebra with parameter

$$\kappa = c/n. \tag{1.4.1}$$

2 A Representation Scheme

2.1 Basic Definitions

In this chapter, we work over $\mathbb{C}$ and write $\mathrm{Hom} = \mathrm{Hom}_{\mathbb{C}}$, $\mathrm{End} = \mathrm{End}_{\mathbb{C}}$, $\otimes = \otimes_{\mathbb{C}}$.
Let Y be a scheme with structure sheaf $\mathcal{O}_Y$ and coordinate ring $\mathcal{O}(Y)$.

Definition 2.1.1. Let V be a finite dimensional vector space, $\mathcal{F}$ a finite length (torsion) $\mathcal{O}_Y$-sheaf, and $\vartheta : V \xrightarrow{\sim} \Gamma(Y, \mathcal{F})$, a vector space isomorphism. Such a data $(V, \mathcal{F}, \vartheta)$ are called *representation* of $\mathcal{O}_Y$ in V.

Proposition 2.1.2. (i) *For each $n \geq 1$, there is a $GL_n(\mathbb{C})$-scheme rep^n_Y, of finite type, that parametrizes representations of $\mathcal{O}_Y$ in $\mathbb{C}^n$.*
(ii) *If Y is affine, then we have $\mathrm{rep}^n_Y = \mathrm{rep}^n \mathcal{O}(Y)$, the affine scheme that parametrizes algebra homomorphisms $\mathcal{O}(Y) \to \mathrm{End}\mathbb{C}^n$.*

Proof. The scheme rep^n_Y may be identified with an open subscheme of Grothendieck's Quot-scheme that parametrizes surjective morphisms $\mathbb{C}^n \otimes \mathcal{O}_Y \twoheadrightarrow \mathcal{F}$ such that the composition $\mathbb{C}^n \hookrightarrow \Gamma(Y, \mathbb{C}^n \otimes \mathcal{O}_Y) \twoheadrightarrow \Gamma(Y, \mathcal{F})$ is an isomorphism. The group GL_n acts on rep^n_Y by base change transformations, that is, by changing the isomorphism $\mathbb{C}^n \cong \Gamma(Y, \mathcal{F})$. □

Below, we fix $n \geq 1$, let $V = \mathbb{C}^n$, and $GL(V) = GL_n(\mathbb{C})$. Given a representation $(\mathcal{F}, \vartheta)$ in V, the isomorphism $\vartheta : V \xrightarrow{\sim} \Gamma(Y, \mathcal{F})$ makes the vector space V an $\mathcal{O}(Y)$-module. We often drop ϑ from the notation and write $\mathcal{F} \in \mathrm{rep}^n_Y$.

Example 2.1.3. For $Y := \mathbb{A}^1$, we have $\mathcal{O}(Y) = \mathbb{C}[t]$; hence, $\mathrm{rep}^n_Y = \mathrm{rep}^n \mathbb{C}[t] = \mathrm{End}V$.
For $Y := \mathbb{C}^\times$, we have $\mathcal{O}(Y) = \mathbb{C}[t, t^{-1}]$; hence, $\mathrm{rep}^n_Y = \mathrm{rep}^n \mathbb{C}[t, t^{-1}] = GL(V)$. In these examples, the action of the group $GL(V)$ is the conjugation-action.

2.2 Factorization Property

From now on, we will be concerned exclusively with the case where $Y = C$ is a smooth algebraic curve (either affine or complete).

Let $\mathbb{S}_n$ denote the symmetric group. Write C^n and $C^{(n)} = C^n/\mathbb{S}_n$ for the nth cartesian and symmetric power of C, respectively. Taking support cycle of a length n coherent sheaf on C gives a natural map

$$\varpi : \operatorname{rep}_C^n \longrightarrow C^{(n)}, \quad \mathcal{F} \mapsto \mathtt{Supp}\mathcal{F}. \tag{2.2.1}$$

It is straightforward to see that the map ϖ is an affine and surjective morphism of schemes, and that the group $GL(V)$ acts along the fibres of ϖ.

The collection of morphisms (2.2.1) for various values of the integer n enjoys an important *factorization property*. Specifically, let k, m, be a pair of positive integers and $D_1 \in C^{(k)}$, $D_2 \in C^{(m)}$, $|D_1| \cap |D_2| = \emptyset$, a pair of divisors with disjoint supports, that is, a pair of unordered collections of k and m points of C, respectively, which have no points in common. The factorization property says that there is a natural isomorphism

$$\varpi^{-1}(D_1 + D_2) \cong GL_{k+m} \overset{GL_k \times GL_m}{\times} \big(\varpi^{-1}(D_1) \times \varpi^{-1}(D_2)\big).$$

The factorization property also holds in families. To explain this, let ${}^{\heartsuit}C^{(k,m)} \subset C^{(k)} \times C^{(m)}$ be the open subset formed by all pairs of divisors with disjoint support. There is a natural composite projection ${}^{\heartsuit}C^{(k,m)} \hookrightarrow C^{(k)} \times C^{(m)} \twoheadrightarrow C^{(k+m)}$.

The family version of the factorization property reads

$$\operatorname{rep}_C^{k+m} \underset{C^{(k+m)}}{\times} {}^{\heartsuit}C^{(k,m)} \simeq \left(GL_{k+m} \overset{GL_k \times GL_m}{\times} (\operatorname{rep}_C^k \times \operatorname{rep}_C^m)\right) \underset{C^{(k)} \times C^{(m)}}{\times} {}^{\heartsuit}C^{(k,m)}. \tag{2.2.2}$$

2.3 Tangent and Cotangent Bundles

Let Ω_C be the sheaf of Kähler differentials on C. Given a finite length sheaf $\mathcal{F}$ we let $\mathcal{F}^\vee := \mathcal{E}xt^1(\mathcal{F}, \Omega_C)$, denote the Grothendieck–Serre dual of $\mathcal{F}$. Let $\jmath : C \hookrightarrow \bar{C}$ be an open imbedding of C into a complete curve. Then we have natural isomorphisms

$$\Gamma(C, \mathcal{F}^\vee) = \operatorname{Ext}^1(\bar{\mathcal{F}}, \Omega_{\bar{C}}) \cong \operatorname{Hom}(\mathcal{O}_{\bar{C}}, \bar{\mathcal{F}})^* = \Gamma(C, \mathcal{F})^*, \tag{2.3.1}$$

where we write $\bar{\mathcal{F}} = \jmath_*\mathcal{F}$ and where the isomorphism in the middle is provided by the Serre duality. In particular, for any $\mathcal{F} \in \operatorname{rep}_C^n$, one has canonical isomorphisms

$$\Gamma(C^2, \mathcal{F} \boxtimes \mathcal{F}^\vee) = \Gamma(C, \mathcal{F}) \otimes \Gamma(C, \mathcal{F}^\vee) \overset{\vartheta}{\tilde{\to}} V \otimes V^* = \operatorname{End}V. \tag{2.3.2}$$

Let $\Delta \subset C^2 := C \times C$ denote the diagonal divisor. Thus, on C^2, we have a triple of sheaves $\mathcal{O}_{C^2}(-\Delta) \hookrightarrow \mathcal{O}_{C^2} \hookrightarrow \mathcal{O}_{C^2}(\Delta)$. For any $\mathcal{F} \in \operatorname{rep}_C^n$, let $\mathcal{K}er_{\mathcal{F}}$ denote the kernel of the composite morphism $V \otimes \mathcal{O}_C \tilde{\to} \Gamma(C, \mathcal{F}) \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{F}$.

Lemma 2.3.3. *The scheme* rep^n_C *is smooth. For the tangent, resp. cotangent, space at a point* $\mathcal{F} \in \mathrm{rep}^n_C$, *there are canonical isomorphisms:*

$$\begin{aligned} T_{\mathcal{F}}(\mathrm{rep}^n_C) &\cong \mathrm{Hom}(\mathcal{K}er_{\mathcal{F}}, \mathcal{F}) \cong \Gamma\big(C^2, (\mathcal{F} \boxtimes \mathcal{F}^\vee)(\Delta)\big), \quad \text{resp.} \\ T^*_{\mathcal{F}}(\mathrm{rep}^n_C) &\cong \mathrm{Ext}^1(\mathcal{F}, \mathcal{K}er_{\mathcal{F}} \otimes \Omega_C) \cong \Gamma\big(C^2, (\mathcal{F} \boxtimes \mathcal{F}^\vee)(-\Delta)\big). \end{aligned}$$

Proof. It is well known that the Zariski tangent space to the scheme parametrizing surjective morphisms $V \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{F}$ is equal to the vector space $\mathrm{Hom}(\mathcal{K}er_{\mathcal{F}}, \mathcal{F})$. This proves the first isomorphism. From this, by Serre duality one obtains $T^*_{\mathcal{F}}(\mathrm{rep}^n_C) \cong \mathrm{Ext}^1(\mathcal{F}, \mathcal{K}er_{\mathcal{F}} \otimes \Omega_C)$.

To complete the proof, it suffices to prove the second formula for the tangent space $T_{\mathcal{F}}(\mathrm{rep}^n_C)$; the corresponding formula for $T^*_{\mathcal{F}}(\mathrm{rep}^n_C)$ would then follow by duality.

To prove the formula for $T_{\mathcal{F}}(\mathrm{rep}^n_C)$, we may assume that C is affine and put $A = \mathcal{O}(C)$. Also, let $V = \Gamma(C, \mathcal{F})$ and $K = \Gamma(C, \mathcal{K}er_{\mathcal{F}})$. Thus, we have $\mathrm{Hom}(\mathcal{K}er_{\mathcal{F}}, \mathcal{F}) = \mathrm{Hom}_A(K, V)$.

The sheaf extension $\mathcal{K}er_{\mathcal{F}} \hookrightarrow V \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{F}$ yields a short exact sequence of A-modules

$$0 \to K \longrightarrow A \otimes V \xrightarrow{\mathtt{act}} V \to 0,$$

where $\mathtt{act}$ is the action-map.

The action of A on V makes the vector space $\mathrm{End}V$ an A-bimodule. We write $\mathrm{Der}_A(A, \mathrm{End}V)$ for the space of derivations of the algebra A with coefficients in the A-bimodule $\mathrm{End}V$. Further, given an A-module map $f : K \to V$, for any element $a \in A$ we define a linear map

$$\delta_f(a) : V \to V, \quad v \mapsto f(a \otimes v - 1 \otimes av).$$

It is straightforward to see that the assignment $a \mapsto \delta_f(a)$ gives a derivation $\delta_f \in \mathrm{Der}_A(A, \mathrm{End}V)$. Moreover, this way, one obtains a canonical isomorphism

$$\mathrm{Hom}_A(K, V) \xrightarrow{\sim} \mathrm{Der}_A(A, \mathrm{End}V), \quad f \mapsto \delta_f.$$

The above isomorphism provides a well-known alternative interpretation of the tangent space to a representation scheme in the form $T_{\mathcal{F}}(\mathrm{rep}^n_C) \cong \mathrm{Der}_A(A, \mathrm{End}V)$.

We observe next that for any $a \in A$, one has $a \otimes 1 - 1 \otimes a \in \Gamma(C^2, \mathcal{O}_{C^2}(-\Delta)) \subset A \otimes A$. Further, for any section $s \in \Gamma\big(C^2, (\mathcal{F} \boxtimes \mathcal{F}^\vee)(\Delta)\big)$, the map

$$A \longrightarrow \Gamma\big(C^2, \mathcal{F} \boxtimes \mathcal{F}^\vee\big) = V^* \otimes V = \mathrm{End}V, \quad a \mapsto \xi^s(a) := s \cdot (a \otimes 1 - 1 \otimes a)$$

is easily seen to be a derivation. Moreover, one shows that any derivation has the form ξ^s for a unique section s.

Thus, combining all the above, we deduce the desired canonical isomorphisms

$$T_{\mathcal{F}}(\mathrm{rep}^n_C) \cong \mathrm{Hom}_A(K, V) \cong \mathrm{Der}_A(A, \mathrm{End}V) \cong \Gamma\big(C^2, (\mathcal{F} \boxtimes \mathcal{F}^\vee)(\Delta)\big).$$

In particular, for any $\mathcal{F} \in \operatorname{rep}_C^n$, we find that

$$\dim T_{\mathcal{F}}(\operatorname{rep}_C^n) = \dim \Gamma\big(C^2, (\mathcal{F} \boxtimes \mathcal{F}^\vee)(\Delta)\big) = \dim(V \otimes V^*) = n^2$$

is independent of $\mathcal{F}$. This implies that the scheme rep_C^n is smooth. $\square$

2.4 Example: Formal Disc

Given a formal power series $f = f(t) \in \mathbb{C}[[t]]$, we define the *difference-derivative* of f as a power series $Df(t', t'') := \dfrac{f(t') - f(t'')}{t' - t''} \in \mathbb{C}[[t', t'']]$.

For any $X \in \operatorname{End} V$, write $L_X, R_X : \operatorname{End} V \to \operatorname{End} V$, for the pair of linear maps of left, resp. right, multiplication by X in the algebra $\operatorname{End} V$. These maps commute, hence, for any polynomial $f \in \mathbb{C}[t]$, there is a well-defined linear operator $Df(L_X, R_X) : \operatorname{End} V \to \operatorname{End} V$. For instance, in the special case where $f(t) = t^m$, we find $Df(L_X, R_X)(Y) = X^{m-1}Y + X^{m-2}YX + \cdots + XYX^{m-2} + YX^{m-1}$.

Let C_x be the completion of a smooth curve C at a point $x \in C$, a formal scheme. A choice of local parameter on C_x amounts to a choice of algebra isomorphism $\mathcal{O}(C_x) = \mathbb{C}[[t]]$. Thus, for the corresponding representation schemes, we obtain $\operatorname{rep}^n \mathcal{O}(C_x) \cong \operatorname{rep}^n \mathbb{C}[[t]] = \widehat{\mathcal{N}}$, the completion of $\operatorname{End} V$ along the closed subscheme $\mathcal{N} \subset \operatorname{End} V$, of nilpotent endomorphisms.

The tangent space at a point of $\operatorname{rep}^n \mathcal{O}(C_x) = \widehat{\mathcal{N}}$ may be therefore identified with the vector space $\operatorname{End} V$. More precisely, the tangent bundle $T(\operatorname{rep}^n \mathcal{O}(C_x))$ is the completion of $\operatorname{End} V \times \operatorname{End} V$ along $\mathcal{N} \times \operatorname{End} V$. Abusing the notation slightly, we may write a point in the tangent bundle as a pair $(X, Y) \in \operatorname{End} V \times \operatorname{End} V$, where $X \in \operatorname{rep}^n \mathcal{O}(C_x) = \widehat{\mathcal{N}}$, and Y is a tangent vector at X. Thus, associated with such a point (X, Y) and any formal power series $f \in \mathbb{C}[[t]]$, there is a well-defined linear operator $Df(L_X, R_X)$, acting on an appropriate completion of $\operatorname{End} V$.

Now, let t and u be two different local parameters on the formal scheme C_x. Thus, one can write $u = f(t)$, for some $f \in \mathbb{C}[[t]]$. We have

Lemma 2.4.1. *The differential of f acts on the tangent bundle $T(\operatorname{rep}^n \mathcal{O}(C_x))$ by the formula*

$$df : (X, Y) \mapsto \big(f(X),\ Df(L_X, R_X)(Y)\big). \qquad \square$$

Remark 2.4.2. In the special case where $f(t) = e^t$, writing $\Phi(t) := (e^t - 1)/t$, we compute

$$\frac{f(t') - f(t'')}{t' - t''} = \frac{e^{t'} - e^{t''}}{t' - t''} = \frac{e^{t'-t''} - 1}{t' - t''} \cdot e^{t''} = \Phi(t' - t'') \cdot e^{t''}.$$

Therefore, for the differential of the exponential map $X \mapsto \exp X$, using the lemma and the notation $\operatorname{ad} X(Y) := XY - YX = (L_X - R_X)(Y)$, one recovers the standard formula

$$d\exp : (X, Y) \mapsto \big(f(X),\ Df(L_X, R_X)(Y)\big) = \Big(e^X,\ \Phi(\operatorname{ad} X)(Y) \cdot e^X\Big). \quad \diamond$$

2.5 Étale Structure

The assignment $C \mapsto \mathrm{rep}_C^n$ is a functor. Specifically, for any morphism of curves $p: C_1 \to C_2$, the push-forward of coherent sheaves induces a GL_n-equivariant morphism $\mathrm{rep}\,p: \mathrm{rep}_{C_1}^n \to \mathrm{rep}_{C_2}^n$, $\mathcal{F} \mapsto p_*\mathcal{F}$.

The following result says that for any smooth curve C, the representation scheme rep_C^n is locally isomorphic, in étale topology, to $\mathrm{rep}_{\mathbb{A}^1}^n = \mathfrak{gl}_n$, the corresponding scheme for the curve $C = \mathbb{A}^1$.

Proposition 2.5.1. *For any smooth curve C and $n \geq 1$, one can find a finite collection of open sub-curves $C_s \subset C$, $s = 1, \ldots, r$, such that the following holds.*

For each $s = 1, \ldots, r$, there is a Zariski open subset $U_s \subset \mathrm{rep}_{C_s}^n$, and a diagram of morphisms of curves

$$C \xleftarrow{\;q_s\;} C_s \xrightarrow{\;p_s\;} \mathbb{A}^1, \quad s = 1, \ldots, r,$$

which gives rise to a diagram of representation schemes

$$\mathrm{rep}_C^n \xleftarrow{\;\mathrm{rep}q_s\;} \mathrm{rep}_{C_s}^n \supset U_s \xrightarrow{\;\mathrm{rep}p_s\;} \mathrm{rep}_{\mathbb{A}^1}^n = \mathfrak{gl}_n.$$

The above data satisfy, for each $s = 1, \ldots, r$, the following properties:

- *The map q_s is an open imbedding, and $\mathrm{rep}_C^n = (\mathrm{rep}q_1)(U_1) \cup \cdots \cup (\mathrm{rep}q_s)(U_s)$ is an open cover;*
- *The map p_s is étale and the restriction of the morphism $\mathrm{rep}\,p_s$ to U_s is étale as well.*

This proposition is an immediate consequence of two lemmas below. Fix a smooth curve C.

Lemma 2.5.2. *Given a collection of pairwise distinct points $c_1, \ldots, c_N \in C$ there is an open subset $C' \subset C$ and an étale morphism $p: C' \to \mathbb{A}^1$ such that $c_1, \ldots, c_N \in C'$ and all the values $p(c_1), \ldots, p(c_N)$ are pairwise distinct.*

Proof. Pick a point $x \in C \smallsetminus \{c_1, \ldots, c_N\}$. Then $C \smallsetminus \{x\}$ is affine. Choose an embedding $\gamma: C \smallsetminus \{x\} \hookrightarrow \mathbb{A}^m$. Then the composition of γ with a general linear projection $\mathbb{A}^m \to \mathbb{A}^1$ is the desired p; while C' is the complement of the set of ramification points of p in $C \smallsetminus \{x\}$. $\square$

Given an étale morphism of curves $p: C \to \mathbb{A}^1$ define

$${}^0C^{(n)} := \{(c_1, \ldots, c_n) \in C^{(n)} \;\big|\; p(c_i) = p(c_j) \;\text{ iff }\; c_i = c_j,\ i, j = 1, \ldots, n\}.$$

This is clearly a Zariski open subset in $C^{(n)}$. Let $\varpi^{-1}({}^0C^{(n)})$ be its preimage in rep_C^n, cf. (2.2.1).

Lemma 2.5.3. *The morphism below induced by the map $p : C \to \mathbb{A}^1$ is étale,*

$$\operatorname{rep} p : \ \varpi^{-1}({}^0C^{(n)}) \to \operatorname{rep}^n_{\mathbb{A}^1} = \mathfrak{gl}_n.$$

Proof. To simplify notation, write $U := \varpi^{-1}({}^0C^{(n)})$ and $p_* := \operatorname{rep} p$.

We must check that the differential of p_* on the tangent spaces is invertible. Let $\underline{c} = (k_1c_1 + \cdots + k_lc_l) \in {}^0C^{(n)}$ (so that $c_1, \ldots, c_l$ are all distinct). Then $p(c_1), \ldots, p(c_l)$ are all distinct. If $\mathcal{F} \in U$, and $\varpi\mathcal{F} = \underline{c}$, then $p_*\mathcal{F} \in \operatorname{rep}^n_{\mathbb{A}^1}$, and $\varpi(p_*\mathcal{F}) = (k_1p(c_1) + \cdots + k_lp(c_l))$. As we know, $T_{\mathcal{F}}\operatorname{rep}^n_C = \operatorname{Hom}(\mathcal{K}er_{\mathcal{F}}, \mathcal{F})$.

Let C_{c_i} (resp. $\mathbb{A}^1_{p(c_i)}$) stand for the completion of C at c_i (resp. $\mathbb{A}^1$ at $p(c_i)$). Let $(\mathcal{K}er_{\mathcal{F}})_{c_i}$ (resp. $(\mathcal{F})_{c_i}$) denote the restriction of $\mathcal{K}er_{\mathcal{F}}$ (resp. $\mathcal{F}$) to C_{c_i}. Then the restriction to these completions induces an isomorphism

$$\operatorname{Hom}(\mathcal{K}er_{\mathcal{F}}, \mathcal{F}) = \bigoplus_{1 \le i \le l} \operatorname{Hom}((\mathcal{K}er_{\mathcal{F}})_{c_i}, (\mathcal{F})_{c_i}).$$

Similarly, we have

$$\operatorname{Hom}(\mathcal{K}er_{p_*\mathcal{F}}, p_*\mathcal{F}) = \bigoplus_{1 \le i \le l} \operatorname{Hom}((\mathcal{K}er_{p_*\mathcal{F}})_{p(c_i)}, (p_*\mathcal{F})_{p(c_i)}).$$

The map p being étale at c_i, it identifies C_{c_i} with $\mathbb{A}^1_{p(c_i)}$. Under this identification $(\mathcal{F})_{c_i}$ gets identified with $(p_*\mathcal{F})_{p(c_i)}$, and $(\mathcal{K}er_{\mathcal{F}})_{c_i}$ gets identified with $(\mathcal{K}er_{p_*\mathcal{F}})_{p(c_i)}$. Note that the latter identification is provided by the differential of the morphism $\operatorname{rep} p$ at the point $\mathcal{F} \in \operatorname{rep}^n_C$. The lemma follows. □

3 Cherednik Algebras Associated with Algebraic Curves

3.1 Global Cherednik Algebras

For any smooth algebraic curve C, Etingof defined in [E], 2.19, a sheaf of Cherednik algebras on the symmetric power $C^{(n)}$, $n \ge 1$.

To recall Etingof's definition, for any smooth variety Y, introduce a length two complex of sheaves $\Omega^{1,2}_Y := [\Omega^1_Y \xrightarrow{d} (\Omega^2_Y)_{\text{closed}}]$, concentrated in degrees 1 and 2. The sheaves of algebraic twisted differential operators (TDO) on Y are known to be parametrized (up to isomorphism) by elements of $H^2(Y, \Omega^{1,2}_Y)$, the second hypercohomology group, cf. e.g., [K].

Remark 3.1.1. For an affine curve C, we have $H^2(C, \Omega^{1,2}_C) = 0$; for a projective curve C, we have $H^2(C, \Omega^{1,2}_C) \cong \mathbb{C}$, where a generator is provided by the first Chern class of a degree 1 line bundle on C. ◊

Now, let C be a smooth algebraic curve. Given a class $\psi \in H^2(C, \Omega_C^{1,2})$, write $\psi^{\boxtimes n} \in H^2(C^n, \Omega_{C^n}^{1,2})^{\mathbb{S}_n}$ for the external product of n copies of ψ. Pull-back via the projection $C^n \twoheadrightarrow C^{(n)}$ induces an isomorphism $H^2(C^{(n)}, \Omega_{C^{(n)}}^{1,2}) \xrightarrow{\sim} H^2(C^n, \Omega_{C^n}^{1,2})^{\mathbb{S}_n} \subset H^2(C^n, \Omega_{C^n}^{1,2})$. Let $\psi_n \in H^2(C^{(n)}, \Omega_{C^{(n)}}^{1,2})$ denote the preimage of the class $\psi^{\boxtimes n}$ via the above isomorphism.

For any $n \geq 1$, $\kappa \in \mathbb{C}$, and $\psi \in H^2(C, \Omega_C^{1,2})$, Etingof defines the sheaf of global Cherednik algebras, as follows, cf. [E], 2.9. Let η be a 1-form on C^n such that for the cohomology class $\eta \in H^2(C^n, \Omega_{C^n}^{1,2})$ one has $d\eta = \psi^{\boxtimes n}$. Next, for any $i, j \in [1, n]$, let $\Delta_{ij} \subset C^n$ be the corresponding (ij)-diagonal, with equal ith and jth coordinates. Thus, $\Delta_n = \bigcup_{i \neq j} \Delta_{ij}$, is the big diagonal, and the image of Δ_n under the projection $C^n \twoheadrightarrow C^{(n)}$ is the discriminant divisor, $\mathbf{D} \subset C^{(n)}$.

Given a vector field v on C^n, for each pair (i, j), choose a rational function f_{ij}^v on C^n whose polar part at Δ_{ij} corresponds to v, as explained in *loc. cit.* 2.4. Associated with such a data, Etingof defines in [E], 2.9, the following Dunkl operator

$$D_v := \mathrm{Lie}_v + \langle v, \eta \rangle + \kappa \cdot n \cdot \sum_{i \neq j} (s_{ij} - 1) \otimes f_{ij}^v \in \mathscr{D}_{\psi_n}(C^n \smallsetminus \Delta_n) \rtimes \mathbb{S}_n.$$

Here Lie_v stands for the Lie derivative with respect to the vector field v, and the cross-product algebra $\mathscr{D}_{\psi_n}(C^n \smallsetminus \Delta_n) \rtimes \mathbb{S}_n$ on the right is viewed as a sheaf of associative algebras on $C^{(n)} \smallsetminus \mathbf{D}$.

The sheaf of Cherednik algebras is defined as a subsheaf of the sheaf $\jmath_\bullet \mathscr{D}_{\psi_n}(C^n \smallsetminus \Delta_n) \rtimes \mathbb{S}_n$, where $\jmath : C^{(n)} \smallsetminus \mathbf{D} \hookrightarrow C^{(n)}$ stands for the open imbedding. Specifically, following Etingof, we have

Definition 3.1.2. Let $\mathsf{H}_{\kappa,\psi_n}$, the sheaf of Cherednik algebras, be the sheaf, on $C^{(n)}$, of associative subalgebras, generated by all regular functions on C^n and by the Dunkl operators D_v, for all vector fields v and f_{ij}^v as above.

Next, let $\mathsf{e} \in \mathbb{C}[\mathbb{S}_n]$ be the idempotent projector to the trivial representation. The subalgebra $\mathsf{e}\mathsf{H}_{\kappa,\psi_n}\mathsf{e} \subset \mathsf{H}_{\kappa,\psi_n}$ is called *spherical subalgebra*. This is a sheaf of associative algebras on $C^{(n)}$ that may be identified naturally with a subsheaf of $\mathscr{D}_{\psi_n}(C^{(n)})$.

The following is a global analogue of a result due to Gordon and Stafford [GS], and Bezrukavnikov and Etingof [BE] in the case where $C = \mathbb{A}^1$.

Proposition 3.1.3. *For any $\kappa \in \mathbb{C} \smallsetminus [-1, 0)$, the functor below is a Morita equivalence*

$$\mathsf{H}_{\kappa,\psi_n}\text{-mod} \longrightarrow \mathsf{e}\mathsf{H}_{\kappa,\psi_n}\mathsf{e}\text{-mod}, \quad M \mapsto \mathsf{e}M.$$

Proof. It is a well-known fact that the Morita equivalence statement is equivalent to an equality $\mathsf{H}_{\kappa,\psi_n} = \mathsf{H}_{\kappa,\psi_n} \cdot \mathsf{e} \cdot \mathsf{H}_{\kappa,\psi_n}$. In any case, on $C^{(n)}$, one has an exact sequence of sheaves

$$0 \to \mathsf{H}_{\kappa,\psi_n} \cdot \mathsf{e} \cdot \mathsf{H}_{\kappa,\psi_n} \to \mathsf{H}_{\kappa,\psi_n} \to \mathsf{H}_{\kappa,\psi_n} / \mathsf{H}_{\kappa,\psi_n} \cdot \mathsf{e} \cdot \mathsf{H}_{\kappa,\psi_n} \to 0. \tag{3.1.4}$$

Proving that the sheaf on the right vanishes is a "local" problem. Thus, one can restrict (3.1.3) to an open subset in $C^{(n)}$. Then, we are in a position to use the above-cited result of Gordon and Stafford saying that Proposition 3.1.3 holds for the curve $C = \mathbb{A}^1$. In effect, by Proposition 2.5.1, each point in $C^{(n)}$ is contained in an open subset U with the following property. There exists an étale morphism $f : U \to (\mathbb{A}^1)^{(n)}$ such that the pull-back via f of the sheaf on the right of (3.1.3) for the curve $\mathbb{A}^1$ is equal to the corresponding sheaf for the curve C. $\square$

The sheaf $\mathsf{H}_{\kappa,\psi_n}$ comes equipped with an increasing filtration arising from the standard filtration by the order of differential operator, cf. [EG, E]. The filtration on $\mathsf{H}_{\kappa,\psi_n}$ induces, by restriction, an increasing filtration, $F_\bullet(\mathsf{e}\mathsf{H}_{\kappa,\psi_n}\mathsf{e})$, on the spherical subalgebra. Etingof proved a graded algebra isomorphism, cf. [E],

$$\mathtt{gr}^F(\mathsf{e}\mathsf{H}_{\kappa,\psi_n}\mathsf{e}) \cong p_*\mathcal{O}_Y, \quad Y := (T^*C)^{(n)} = (T^*(C^n))/\mathbb{S}_n, \tag{3.1.5}$$

where $p : (T^*C)^{(n)} \to C^{(n)}$ denotes the natural projection.

3.2 The Determinant Line Bundle

We fix a curve C, an integer $n \geq 1$, and let $V = \mathbb{C}^n$. Set $X_n := \mathrm{rep}^n_C \times V$, a smooth variety. Let $X_n^{\mathrm{cyc}} \subset X_n = \mathrm{rep}^n_C \times V$ be a subset formed by the triples $(\mathcal{F}, \vartheta, v)$ such that v is a cyclic vector, i.e. such that the morphism $\mathcal{O}_C \to \mathcal{F},\ f \mapsto fv$ is surjective. It is clear that the set X_n^{cyc} is a $GL(V)$-stable Zariski open subset of X_n.

Choose a basis of V, and identify $V = \mathbb{C}^n$, and $GL(V) = GL_n$, etc. Assume further that our curve C admits a *global* coordinate $t : C \hookrightarrow \mathbb{A}^1$. Then, associated with each pair $(\mathcal{F}, \vartheta, v) \in X_n^{\mathrm{cyc}}$, there is a *matrix* $\mathsf{g}(\mathcal{F}, \vartheta, v) \in GL_n$, whose kth row is given by the n-tuple of coordinates of the vector $t^{k-1}(v) \in \mathbb{C}^n = V = \Gamma(C, \mathcal{F})$, $k = 1, \ldots, n$.

Lemma 3.2.1. (i) *The map* $\varpi : X_n^{\mathrm{cyc}} \to C^{(n)}$, *cf.* (2.2.1), *makes the scheme* X_n^{cyc} *a* $GL(V)$*-torsor over* $C^{(n)}$.

(ii) *Given a basis in* V *and a* global *coordinate* $t : C \hookrightarrow \mathbb{A}^1$, *the map*

$$\mathsf{g} \times \varpi : X_n^{\mathrm{cyc}} \xrightarrow{\sim} GL_n \times C^{(n)}, \quad (\mathcal{F}, \vartheta, v) \mapsto \big(\mathsf{g}(\mathcal{F}, \vartheta, v),\ \mathtt{Supp}\mathcal{F}\big), \tag{3.2.2}$$

provides a $GL(V)$*-equivariant trivialization of the* $GL(V)$*-torsor from* (i).

Proof. The action of $GL(V)$ on $\mathrm{rep}^n_C \times V$ is given by $g(\mathcal{F}, \vartheta, v) = (\mathcal{F}, \vartheta \circ g^{-1}, gv)$.

To prove (i), we observe that the quotient $GL(V)\backslash X_n^{\mathrm{cyc}}$ is the moduli stack of quotient sheaves of length n of the structure sheaf $\mathcal{O}_C$. However, this stack is just the Grothendieck Quot scheme $\mathrm{Quot}^n_{\mathcal{O}_C}$ isomorphic to $C^{(n)}$. Part (i) follows. Part (ii) is immediate. $\square$

On $X_n = \mathrm{rep}^n_C \times V$, we have a trivial line bundle $\mathtt{det}$ with fibre $\wedge^n V$. This line bundle comes equipped with the natural $GL(V)$-equivariant structure given, for any

$a \in \wedge^n V$, by $g(\mathcal{F}, \vartheta, v; a) = (\mathcal{F}, \vartheta \circ g^{-1}, gv; (\mathtt{det}\, g) \cdot a)$. We define a *determinant bundle* to be the unique line bundle $\boldsymbol{\mathcal{L}}$ on $C^{(n)}$ such that $\varpi^* \boldsymbol{\mathcal{L}}$, the pull-back of $\boldsymbol{\mathcal{L}}$ via the projection $X_n^{\mathrm{cyc}} \twoheadrightarrow C^{(n)}$, is isomorphic to $\mathtt{det}\,|_{X_n^{\mathrm{cyc}}}$.

The square of determinant bundle $\boldsymbol{\mathcal{L}}^2$ has a canonical *rational* section $\boldsymbol{\delta}$, defined as follows. Let $(c_1, \ldots, c_n)$ be pairwise distinct points of C, and $\mathcal{F} = \mathcal{O}_C / \mathcal{O}_C(-c_1 - \cdots - c_n) = \mathcal{O}_{c_1} \oplus \cdots \oplus \mathcal{O}_{c_n}$, a point in X_n^{cyc}. The fibre of $\boldsymbol{\mathcal{L}}^2$ at the point $\varpi(\mathcal{F}) \in C^{(n)}$ is identified with the vector space $\boldsymbol{\mathcal{L}}^2_{\mathcal{F}} = (\mathcal{O}_{c_1} \otimes \cdots \otimes \mathcal{O}_{c_n})^{\otimes 2}$. We define $\boldsymbol{\delta}(c_1, \ldots, c_n) := (1_{c_1} \wedge \cdots \wedge 1_{c_n})^2$, where $1_{c_r} \in \mathcal{O}_{c_r}$ stands for the unit element.

We have the finite projection map $p : C^n \twoheadrightarrow C^{(n)}$, the (big) diagonal divisor, $\Delta_n \subset C^n$, and the discriminant divisor, $\mathbf{D} \subset C^{(n)}$. Note that, set theoretically, we have $|\mathbf{D}| = p(|\Delta_n|)$; however, for divisors, one has an equation $p^* \mathbf{D} = 2\Delta_n$.

Next, we put $X_n^{\mathrm{reg}} := X_n^{\mathrm{cyc}} \cap \varpi^{-1}(C^{(n)} \smallsetminus \mathbf{D})$. Clearly, we have $X_n^{\mathrm{reg}} \subset X_n^{\mathrm{cyc}} \subset X_n$. Given a global coordinate $t : C \hookrightarrow \mathbb{A}^1$, we write

$$\pi(c) := \prod_{1 \le i < j \le n} \big(t(c_i) - t(c_j)\big), \qquad \forall c = (c_1, \ldots, c_n) \in C^n.$$

for the Vandermonde determinant. Thus, $\pi^2(c)$ is a regular function on $C^{(n)}$.

Proposition 3.2.3. (i) *For any smooth connected curve C, we have*

$$X_n \smallsetminus X_n^{\mathrm{reg}} = \varpi^{-1}(\mathbf{D}) \cup (X_n \smallsetminus X_n^{\mathrm{cyc}}),$$

a union of two irreducible divisors.

(ii) *There exists a regular function $\mathbf{f} \in \mathcal{O}(X_n^{\mathrm{reg}})$ such that*

- *It has a zero of order 2 at the divisor $X_n \smallsetminus X_n^{\mathrm{cyc}}$, and a pole of order 1 at the divisor $\varpi^{-1}(\mathbf{D})$;*
- *It is a GL_n-semi-invariant, specifically, we have*

$$\mathbf{f}(g \cdot x) = (\mathtt{det}\, g)^2 \cdot \mathbf{f}(x), \qquad \forall g \in GL_n,\ x \in X_n^{\mathrm{reg}}.$$

(iii) *The function $\mathbf{f}$ is defined uniquely up to multiplication by the pull back of an invertible function on $C^{(n)}$. Furthermore, in a trivialization as in Lemma* 3.2.1 (ii)*, one can put*

$$\mathbf{f}(x) = \big(\mathtt{det}\, \mathsf{g}(x) \cdot \pi(\varpi(x))\big)^2, \quad x \in X_n^{\mathrm{reg}}.$$

Proof. The uniqueness statement follows from the semi-invariant property of $\mathbf{f}$. To prove the existence, recall the definition of the line $\boldsymbol{\mathcal{L}}$. By that definition, a section of $\boldsymbol{\mathcal{L}}$ is the same thing as a semi-invariant section of the trivial bundle with fibre $\wedge^n V$, on X_n^{cyc}. We let $\mathbf{f}$ be the rational section of the latter bundle corresponding, this way, to the rational section $\boldsymbol{\delta}$ of $\boldsymbol{\mathcal{L}}$. A choice of base in V gives a basis in $\wedge^n V$, and hence allows to view $\mathbf{f}$ as a semi-invariant rational function.

The order of zero of $\mathbf{f}$ at $X_n \smallsetminus X_n^{\mathrm{cyc}}$ is a local question, so we may assume the existence of a local coordinate t on C, trivializing our torsor as in 3.2.1 (ii). Then the divisor $X_n \smallsetminus X_n^{\mathrm{cyc}}$ is given by an equation $\mathtt{det}\ \mathtt{g}(x) = 0$, so we see that the zero of $\mathbf{f}$ at $X_n \smallsetminus X_n^{\mathrm{cyc}}$ indeed has order 2. The statement that $\mathbf{f}$ has a pole of order 1 at $\varpi^{-1}(\mathbf{D})$ immediately follows from the next lemma. In the presence of a local coordinate t, the explicit formula for the function $\mathtt{g}$ then follows from the uniqueness statement. □

Lemma 3.2.4. *There is a canonical isomorphism $\boldsymbol{\mathcal{L}}^{\otimes 2} \simeq \mathcal{O}_{C^{(n)}}(\mathbf{D})$.*

Proof. We have to construct an $\mathbb{S}_n$-equivariant section of $p^*\boldsymbol{\mathcal{L}}^{\otimes 2}$ which is regular nonvanishing on $C^n - \Delta_n$, and has a second order pole at (each component of) Δ_n. Clearly, $\boldsymbol{\delta}$ is a $\mathbb{S}_n$-invariant regular nonvanishing section of $p^*\boldsymbol{\mathcal{L}}^{\otimes 2}|_{C^n \smallsetminus \Delta_n}$. To compute the order of the pole of $\boldsymbol{\delta}$ at the diagonal $c_i = c_j$, it suffices to consider the case where $C = \mathbb{A}^1,\ n = 2$. In the latter case, a straightforward calculation shows that $\boldsymbol{\delta}$ has a second-order pole. □

3.3 Quantum Hamiltonian Reduction

The goal of this section is to provide a construction of the sheaf of spherical Cherednik subalgebras in terms of quantum Hamiltonian reduction, in the spirit of [GG].

Let $c_1(\boldsymbol{\mathcal{L}}) \in H^2(C^n, \Omega^{1,2}_{C^{(n)}})^{\mathbb{S}_n}$ denote the image of the first Chern class of the determinant line bundle $\boldsymbol{\mathcal{L}}$. For a complex number κ, we will sometimes denote the class $\kappa \cdot c_1(\boldsymbol{\mathcal{L}}) \in H^2(C^n, \Omega^{1,2}_{C^n})^{\mathbb{S}_n}$ simply by κ, if there is no risk of confusion. Let $\psi \in H^2(C, \Omega^{1,2}_C)^{\mathbb{S}_n}$. Note that if C is affine, then $\psi = 0$, and if C is projective, then ψ is proportional to the first Chern class $c_1(\mathscr{L})$ of a degree one line bundle $\mathscr{L}$ on C, so that $\psi = k \cdot c_1(\mathscr{L})$. The line bundle $\mathscr{L}^{\boxtimes n}$ on C^n is $\mathbb{S}_n$-equivariant. Let $\mathscr{L}^{(n)}$ denote the subsheaf of $\mathbb{S}_n$-invariants in $p_*\mathscr{L}^{\boxtimes n}$, the direct image sheaf on $C^{(n)}$.

We set $\psi_n = k{\cdot}c_1(\mathscr{L}^{(n)}) \in H^2(C^{(n)}, \Omega^{1,2}_{C^{(n)}})$, and let $\varpi^*(\psi_n) \in H^2(X_n, \Omega^{1,2}_{X_n})$, be its pull-back via the support-morphism (2.2.1), cf. [E], 2.9. Let $\mathscr{D}_{\psi_n}(X_n)$ denote the sheaf on $X_n = \mathrm{rep}^n_C \times V$, of twisted differential operators associated with the class $\varpi^*(\psi_n)$.

We have a GL_n-action along the fibres of the support-morphism. Therefore, the TDO associated with a pull-back class comes equipped with a natural GL_n-equivariant structure. More precisely, [BB], Lemma 1.8.7, implies that the pair $(\mathscr{D}_{\psi_n}(X_n), GL_n)$ has the canonical structure of a Harish–Chandra algebra on X_n, in the sense of [BB], Sect. 1.8.3. It follows, in particular, that there is a natural Lie algebra morphism $\mathfrak{gl}_n \to \mathscr{D}_{\psi_n}(X_n),\ u \mapsto \vec{u}$, compatible with the GL_n-action on X_n.

For any $\kappa \in \mathbb{C}$, the assignment $u \mapsto \vec{u} - \kappa \cdot tr(u) \cdot 1$ gives another Lie algebra morphism $\mathfrak{gl}_n \to \mathscr{D}_{\psi_n}(X_n)$. Let $\mathfrak{g}_\kappa \subset \mathscr{D}_{\psi_n}(X_n)$ denote the image of the latter morphism. Thus, $\mathfrak{g}_\kappa$ is a Lie subalgebra of first-order twisted differential operators, and we write $\mathscr{D}_{\psi_n}(X_n)\mathfrak{g}_\kappa$ for the left ideal in $\mathscr{D}_{\psi_n}(X_n)$ generated by the vector space $\mathfrak{g}_\kappa$.

Let $\mathscr{D}_{\psi_n}(X_n^{\mathrm{cyc}})$ be the restriction of the TDO $\mathscr{D}_{\psi_n}(X_n)$ to the open subset $X_n^{\mathrm{cyc}} \subset X_n$. Let $\varpi_\bullet \mathscr{D}_{\psi_n}(X_n^{\mathrm{cyc}})$ denote the sheaf-theoretic direct image, a sheaf of filtered associative algebras on $C^{(n)}$ equipped with a GL_n-action. We have the left ideal $\mathscr{D}_{\psi_n}(X_n^{\mathrm{cyc}})\mathfrak{g}_\kappa$, in $\mathscr{D}_{\psi_n}(X_n^{\mathrm{cyc}})$, and the corresponding GL_n-stable left ideal $\varpi_\bullet\mathscr{D}_{\psi_n}(X_n^{\mathrm{cyc}})\mathfrak{g}_\kappa \subset \varpi_\bullet\mathscr{D}_{\psi_n}(X_n^{\mathrm{cyc}})$.

The TDO $\mathscr{D}_\psi(X_n)$ may be identified with a subsheaf of the direct image of $\mathscr{D}_\psi(X_n^{\mathrm{cyc}})$ under the open embedding $X_n^{\mathrm{cyc}} \hookrightarrow X_n$. This way, one obtains a restriction morphism

$$r:\ \varpi_\bullet\mathscr{D}_\psi(X_n)/\varpi_\bullet\mathscr{D}_\psi(X_n)\mathfrak{g}_\kappa \to \varpi_\bullet\mathscr{D}_\psi(X_n^{\mathrm{cyc}})/\varpi_\bullet\mathscr{D}_\psi(X_n^{\mathrm{cyc}})\mathfrak{g}_\kappa.$$

We are now going to generalize [GG] formula (6.15), and construct, for any $\kappa \in \mathbb{C}$, $\psi \in H^2(C^n, \Omega_C^{1,2})$, a canonical "radial part" isomorphism

$$\mathsf{rad}:\ \left(\varpi_\bullet\mathscr{D}_{\psi_n}(X_n^{\mathrm{reg}})/\varpi_\bullet\mathscr{D}_{\psi_n}(X_n^{\mathrm{reg}})\mathfrak{g}_\kappa\right)^{GL_n} \xrightarrow{\sim} \mathscr{D}_{\psi_n}(C^{(n)} \smallsetminus \mathbf{D}), \quad u \mapsto \mathring{u}_\kappa, \tag{3.3.1}$$

of sheaves of filtered associative algebras on $X_n^{\mathrm{reg}}/GL_n = C^{(n)} \smallsetminus \mathbf{D}$.

To this end, assume first that the class $\psi = c_1(\mathscr{L})$ is the first Chern class of a line bundle $\mathscr{L}$ on C. Then, we have $\psi_n = c_1(\mathscr{L}^{(n)})$. Furthermore, we have the pull-back $\varpi^*\mathscr{L}^{(n)}$, a line bundle on X_n^{cyc} equipped with a natural GL_n-equivariant structure. Write $\mathscr{D}(C^{(n)}, \mathscr{L}^{(n)})$, resp. $\mathscr{D}(X_n^{\mathrm{cyc}}, \varpi^*\mathscr{L}^{(n)})$ for the sheaf of TDO on $C^{(n)}$, acting in the line bundle $\mathscr{L}^{(n)}$, resp. TDO on X_n^{reg}, acting in the line bundle $\varpi^*\mathscr{L}^{(n)}$.

It is clear that, on X_n^{reg}, one has a sheaf isomorphism

$$\begin{aligned}&\mathscr{L}^{(n)} \xrightarrow{\sim} \widetilde{\mathscr{L}}^{(n)},\ \ s \mapsto \mathbf{f}\cdot\varpi^*(s),\\ &\qquad\text{where } \widetilde{\mathscr{L}}^{(n)} := \{\widetilde{s} \in \varpi_*\varpi^*\mathscr{L}^{(n)}\ \big|\ g(\widetilde{s}) = (\mathtt{det}\, g)\cdot\widetilde{s},\ \forall g \in GL_n\}.\end{aligned}$$

Note further that for any GL_n-invariant twisted differential operator $u \in \mathscr{D}(X_n^{\mathrm{reg}}, \varpi^*\mathscr{L}^{(n)})$, we have $u(\widetilde{\mathscr{L}}^{(n)}) \subset \widetilde{\mathscr{L}}^{(n)}$. We deduce that for any $\kappa \in \mathbb{C}$, and any GL_n-invariant twisted differential operator $u \in \mathscr{D}(X_n^{\mathrm{reg}}, \varpi^*\mathscr{L}^{(n)})$, the assignment

$$\mathring{u}_\kappa:\ \mathscr{L}^{(n)} \to \mathscr{L}^{(n)}, \quad s \mapsto \mathbf{f}^{-\kappa}\cdot u(\mathbf{f}^\kappa\cdot s),$$

is well-defined and, moreover, it is given by a uniquely determined twisted differential operator $\mathring{u}_\kappa \in \mathscr{D}_{\psi_n}(C^{(n)} \smallsetminus \mathbf{D})$.

The above construction of the map $u \mapsto \mathring{u}_\kappa$ may be adapted to cover the general case, where the class ψ is not necessarily of the form $c_1(\mathscr{L})$ for a line bundle $\mathscr{L}$. The map $u \mapsto \mathring{u}_\kappa$ is, by definition, the radial part homomorphism rad that appears in (3.3.1).

The group $\mathbb{S}_n$ acts freely on $C^n \smallsetminus \Delta_n$, and we have $(C^n \smallsetminus \Delta_n)/\mathbb{S}_n = C^{(n)} \smallsetminus \mathbf{D}$. Hence, $\mathscr{D}_{\psi_n}(C^{(n)} \smallsetminus \mathbf{D}) \simeq \mathscr{D}_{\psi^{\boxtimes n}}(C^n \smallsetminus \Delta_n)^{\mathbb{S}_n}$. The lift of the determinant line bundle $\boldsymbol{\mathcal{L}}$ to C^n has a canonical rational section $\boldsymbol{\delta}^{\frac{1}{2}}$. The restriction of this section to $C^n \smallsetminus \Delta_n$ is invertible, and we let $\mathsf{twist}: \mathscr{D}_{\psi_n}(C^{(n)} \smallsetminus \mathbf{D}) \xrightarrow{\sim} \mathscr{D}_{\psi_n+1}(C^{(n)} \smallsetminus \mathbf{D})$ be an isomorphism of TDO induced via conjugation by $\boldsymbol{\delta}^{\frac{1}{2}}$.

Combining all the above, we obtain the following morphism of sheaves of filtered algebras on $C^{(n)}$, where j stands for the open embedding $C^{(n)} \smallsetminus \mathbf{D} \hookrightarrow C^{(n)}$:

$$\mathsf{twist} \circ \mathsf{rad} \circ r \,:\, \left(\frac{\varpi_\bullet \mathscr{D}_\psi(X_n)}{\varpi_\bullet \mathscr{D}_\psi(X_n)\mathfrak{g}_\kappa}\right)^{GL_n} \longrightarrow \mathsf{j}_* \mathscr{D}_{\psi_n+1}(C^{(n)} \smallsetminus \mathbf{D}). \tag{3.3.2}$$

Here, the algebra on the left is the quantum Hamiltonian reduction of $\mathscr{D}_\psi(X_n^{\mathrm{reg}})$ at the point $\kappa \cdot tr \in \mathfrak{gl}_n^*$.

Our main result about quantum Hamiltonian reduction reads

Theorem 3.3.3. *The image of the composite morphism in (3.3.2) is equal to the spherical Cherednik subalgebra. Moreover, this composite yields a filtered algebra isomorphism*

$$\big(\varpi_\bullet \mathscr{D}_{\psi_n}(X_n)/\varpi_\bullet \mathscr{D}_{\psi_n}(X_n)\mathfrak{g}_\kappa\big)^{GL_n} \xrightarrow{\sim} \mathsf{eH}_{\kappa,\psi_n+1}\mathsf{e},$$

as well as the associated graded algebra isomorphism

$$\mathtt{gr}\big(\varpi_\bullet \mathscr{D}_{\psi_n}(X_n)/\varpi_\bullet \mathscr{D}_{\psi_n}(X_n)\mathfrak{g}_\kappa\big)^{GL_n} \xrightarrow{\sim} \mathtt{gr}(\mathsf{eH}_{\kappa,\psi_n+1}\mathsf{e}).$$

3.4 Proof of Theorem 3.3.3

It will be convenient to introduce the following simplified notation. Let $\mathcal{H}_{n,\kappa,\psi} \subset \mathsf{j}_* \mathscr{D}_{\psi_n+1}(C^{(n)} \smallsetminus \mathbf{D})$ be the image of the composite morphism in (3.3.2) (note the shift $\psi_n + 1$ on the right); also, for the spherical subalgebra, write

$$\mathsf{A}_{n,\kappa,\psi} := \mathsf{eH}_{\kappa,\psi_n+1}\mathsf{e}. \tag{3.4.1}$$

We will use the factorization isomorphism (2.2.2) for $k + m = n$. Write $j : {}^{\heartsuit}C^{(k,m)} \hookrightarrow C^{(k,m)} := C^{(k)} \times C^{(m)}$ for the open imbedding and $pr : {}^{\heartsuit}C^{(k,m)} \hookrightarrow C^{(k,m)} \twoheadrightarrow C^{(n)}$ for the composite projection.

A natural factorization property for the determinant bundle gives rise to a canonical isomorphism of sheaves of TDO,

$$pr^* \mathscr{D}_{\psi_n+1}(C^{(n)}) \simeq j^*(\mathscr{D}_{\psi_k+1}(C^{(k)}) \boxtimes \mathscr{D}_{\psi_m+1}(C^{(m)})).$$

Lemma 3.4.2. *The above isomorphism restricts to the isomorphism of subalgebras* $pr^*\mathcal{H}_{n,\kappa,\psi} \simeq j^*(\mathcal{H}_{k,\kappa,\psi} \boxtimes \mathcal{H}_{m,\kappa,\psi})$.

Proof. Let $(\mathcal{F}_1, v_1) \in X_k$ and $(\mathcal{F}_2, v_2) \in X_m$ be such that the sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ have disjoint supports. Then, the stabilizer in GL_n of the point $(\mathcal{F}_1 \oplus \mathcal{F}_2, v_1 \oplus v_2) \in X_n$, $n = k + m$, is equal to the product of the stabilizers of $(\mathcal{F}_1, v_1)$ and $(\mathcal{F}_2, v_2)$ in

GL_k and GL_m, respectively. This yields the following factorization isomorphism, cf. (2.2.2),

$$\left(GL_{k+m} \overset{GL_k\times GL_m}{\times} (X_k \times X_m)\right) \underset{C^{(k)}\times C^{(m)}}{\times} {}^{\heartsuit}C^{(k,m)} \simeq X_n \times_{C^{(n)}} {}^{\heartsuit}C^{(k,m)},$$
$$(g, \mathcal{F}_1, \vartheta_1, v_1, \mathcal{F}_2, \vartheta_2, v_2) \longmapsto (\mathcal{F}_1 \oplus \mathcal{F}_2, (\vartheta_1 \oplus \vartheta_2) \circ g^{-1}, g(v_1 \oplus v_2)).$$

Now, the desired isomorphism of subalgebras in the statement of the lemma is a particular case of the following situation. We have a subgroup $G' \subset G''$ (in our case $G' = GL_k \times GL_m$, $G'' = GL_n$), and a G'-variety X' with a TDO $\mathscr{D}'$ equipped with an action of G' (in our case $X' = (X_k \times X_m) \underset{C^{(k)}\times C^{(m)}}{\times} {}^{\heartsuit}C^{(k,m)}$, and $\mathscr{D}' = j^*(\mathscr{D}_{\psi_k} \boxtimes \mathscr{D}_{\psi_m})$). We have a G''-invariant linear functional χ'' on the Lie algebra $\mathfrak{g}''$ whose restriction to $\mathfrak{g}'$ is denoted by χ' (in our case $\chi'' = (\kappa - 1)\cdot tr$). We set $X'' = G'' \overset{G'}{\times} X'$; it is equipped with the TDO $\mathscr{D}''$ lifted from X', acted upon by G'' (in our case $X'' = X_n \times_{C^{(n)}} {}^{\heartsuit}C^{(k,m)}$, and $\mathscr{D}'' = pr^*\mathscr{D}_{\psi_n}$). Then it is easy to show that one has $(\mathscr{D}''/\mathscr{D}''\mathfrak{g}''_{\chi''})^{G''} \simeq (\mathscr{D}'/\mathscr{D}'\mathfrak{g}'_{\chi'})^{G'}$. ◻

Lemma 3.4.3. *Let $U \subset C^{(n)}$ be an open subset and let $D_1 \in \Gamma(U, \mathcal{H}_{n,\kappa,\psi}) \subset \mathscr{D}_{\psi_n+1}(U)$, resp. $D_2 \in \Gamma(U, \mathsf{A}_{n,\kappa,\psi})$, be a pair of second-order twisted differential operator on U with equal principal symbols.*

Then, the difference $D_1 - D_2$ is a zero-order differential operator; more precisely, it is the operator of multiplication by a regular function on U.

Proof. It is enough to prove the statement of the lemma in the formal neighbourhood of each point $\underline{c} = (k_1c_1 + \cdots + k_lc_l) \in U \subset C^{(n)}$, where $c_1, \ldots, c_l$ are pairwise distinct and $k_1 + \cdots + k_l = n$. By induction in n, we are reduced to the diagonal case $\underline{c} = (nc)$. Then it suffices to take the formal disk around c for C. In the latter case, our claim follows from the explicit calculation in [GGS], Sect. 5, and the proof of Proposition 2.18 in [E]. ◻

Proof of Theorem 3.3.3. We equip the sheaf $\mathcal{H}_{n,\kappa,\psi}$, defined above (3.1.5), with the standard increasing filtration induced by the standard increasing filtration on $\mathscr{D}_{\psi_n}(X_n)$, by the order of differential operator. Let $\mathrm{gr}\mathcal{H}_{n,\kappa,\psi}$ denote the associated graded sheaf. We also have the increasing filtration on the algebra $\mathsf{A}_{n,\kappa,\psi}$, cf. (3.1.5). Furthermore, Lemma 3.4.3 yields $F_2\mathsf{A}_{n,\kappa,\psi} = F_2\mathcal{H}_{n,\kappa,\psi}$.

Next, we observe that the spherical Cherednik subalgebra $\mathsf{A}_{n,\kappa,\psi}$ is generated by the subsheaf $F_2\mathsf{A}_{n,\kappa,\psi}$. Indeed, it is enough to prove the equality of $\mathsf{A}_{n,\kappa,\psi}$ with the subsheaf generated by $F_2\mathsf{A}_{n,\kappa,\psi}$ in the formal neighbourhood of any point $\underline{c} \in C^{(n)}$. Arguing by induction in n as in the proof of the lemma, it suffices to consider the case $\underline{c} = (nc)$. We then take the formal disk around c for C. In such a case, the desired statement is proved in Sect. 10 of [EG].

Thus, we have filtered algebra morphisms

$$\mathsf{A}_{n,\kappa,\psi} \hookrightarrow \mathcal{H}_{n,\kappa,\psi} \twoheadleftarrow (\varpi_\bullet\mathscr{D}_{\psi_n}(X_n)/\varpi_\bullet\mathscr{D}_{\psi_n}(X_n)\mathfrak{g}_\kappa)^{GL_n}.$$

To prove that these morphisms are isomorphisms, it suffices to show that the induced morphisms of associated graded algebras,

$$\mathtt{gr}\mathsf{A}_{n,\kappa,\psi} \to \mathtt{gr}\mathcal{H}_{n,\kappa,\psi} \leftarrow \mathtt{gr}(\varpi_{\bullet}\mathscr{D}_{\psi_n}(X_n)/\varpi_{\bullet}\mathscr{D}_{\psi_n}(X_n)\mathfrak{g}_{\kappa})^{GL_n},$$

are isomorphisms. Reasoning as above, we may further reduce the proof of this last statement to the case where C is the formal disk around c. The latter case follows from [GG], page 40. The theorem is proved. □

4 Character Sheaves

4.1 The Moment Map

Fix a smooth curve C and let $\Delta \hookrightarrow C^2 = C \times C$ be the diagonal.

Recall the smooth scheme $X_n = \mathrm{rep}^n_C \times V$, and let $T^*X_n = (T^*\mathrm{rep}^n_C) \times V \times V^*$ denote the total space of the cotangent bundle on X_n. We will write a point of T^*X_n as a quadruple

$$(\mathcal{F}, y, i, j) \quad \text{where } \mathcal{F} \in \mathrm{rep}^n_C,\ \ y \in \Gamma\big(C^2, (\mathcal{F} \boxtimes \mathcal{F}^\vee)(-\Delta)\big),\ i \in V,\ \ j \in V^*.$$

The group $GL(V)$ acts diagonally on $\mathrm{rep}^n_C \times V$. This gives a Hamiltonian $GL(V)$-action on T^*X_n, with moment map μ. Given $\mathcal{F} \in \mathrm{rep}^n_C$, we will also use the map

$$\nu_{\mathcal{F}} :\ \Gamma\big(C^2, (\mathcal{F} \boxtimes \mathcal{F}^\vee)(-\Delta)\big) \hookrightarrow \Gamma(C^2, \mathcal{F} \boxtimes \mathcal{F}^\vee) \xrightarrow{\sim} \mathrm{End}V, \tag{4.1.1}$$

induced by the sheaf imbedding $\mathcal{O}_{C^2}(-\Delta) \hookrightarrow \mathcal{O}_{C^2}$, cf. (2.3.2).

Lemma 4.1.2. *The moment map μ is given by the formula, cf.* (4.1.1)*,*

$$\mu :\ (T^*\mathrm{rep}^n_C) \times V \times V^* \to \mathrm{End}V = \mathrm{Lie}GL(V), \quad (\mathcal{F}, y, i, j) \mapsto \nu_{\mathcal{F}}(y) + i \otimes j.$$

We leave the proof to the reader.

Example 4.1.3. In the special case $C = \mathbb{A}^1$, we have $\mathrm{rep}^n_C = \mathrm{End}V \cong \mathfrak{gl}_n$. In this case, the moment map reads, see [GG],

$$\mu :\ \mathfrak{gl}_n \times \mathfrak{gl}_n \times V \times V^* \ \longrightarrow\ \mathfrak{gl}_n, \quad (x, y, i, j) \mapsto [x, y] + i \otimes j.$$

Similarly, in the case $C = \mathbb{C}^\times$, the moment map reads

$$\mu :\ GL_n \times \mathfrak{gl}_n \times V \times V^* \ \longrightarrow\ \mathfrak{gl}_n, \quad (x, y, i, j) \mapsto xyx^{-1} - y + i \otimes j.$$

4.2 A Categorical Quotient

The goal of this subsection is to construct an isomorphism $(T^*C)^{(n)} \simeq \mu^{-1}(0)//GL_n$ (the categorical quotient). To stress the dependence on n, we will sometimes write μ_n for the moment map μ.

Note that we have the direct sum morphism

$$X_k \times X_m \to X_{k+m}, \ (\mathcal{F}, \vartheta_1, v_1; \mathcal{G}, \vartheta_2, v_2) \mapsto (\mathcal{F} \oplus \mathcal{G}, \vartheta_1 \oplus \vartheta_2, v_1 \oplus v_2).$$

This morphism induces a similar direct sum morphism $T^*X_k \times T^*X_m \to T^*X_{k+m}$,

$$\Gamma(\mathcal{F} \boxtimes \mathcal{F}^\vee(-\Delta)) \times \Gamma(\mathcal{G} \boxtimes \mathcal{G}^\vee(-\Delta)) \ni (y_1, y_2) \mapsto y_1 \oplus y_2 \in \Gamma((\mathcal{F} \oplus \mathcal{G}) \boxtimes (\mathcal{F} \oplus \mathcal{G})^\vee(-\Delta)),$$

where we have used simplified notation $\Gamma(-) = \Gamma(C^2, -)$.

Clearly, we have $T^*X_1 = T^*C \times \mathbb{A}^1 \times (\mathbb{A}^1)^*$. Iterating the direct sum morphism n times, we obtain a morphism $(T^*C \times \mathbb{A}^1 \times (\mathbb{A}^1)^*)^n \to T^*X_n$. Restricting this map further to the product of n copies of the subset $T^*C \simeq T^*C \times \{1\} \times \{0\} \subset T^*C \times \mathbb{A}^1 \times (\mathbb{A}^1)^*$, we obtain a morphism $(T^*C)^n \to T^*X_n$. The image of the latter morphism is clearly contained in $\mu_n^{-1}(0)$. It is also clear that the composite morphism $(T^*C)^n \to T^*X_n \to T^*X_n//GL_n$ factors through the symmetrization projection $(T^*C)^n \to (T^*C)^{(n)}$.

This way, we have constructed a morphism $\varepsilon : \ (T^*C)^{(n)} \to \mu_n^{-1}(0)//GL_n$.

Lemma 4.2.1. *$\varepsilon : \ (T^*C)^{(n)} \to \mu_n^{-1}(0)//GL_n$ is an isomorphism.*

Proof. Use the factorization and the "local" result for $C = \mathbb{A}^1$ proved in [GG] 2.8. □

Notation 4.2.2. We let π_n denote the natural projection $\mu_n^{-1}(0) \to \mu_n^{-1}(0)//GL_n \simeq (T^*C)^{(n)}$. If the value of n is clear from the context, we will simply write $\pi : \ \mu^{-1}(0) \to (T^*C)^{(n)}$.

4.3 Flags

Fix a smooth curve C and a sheaf $\mathcal{F} \in \mathrm{rep}_C^n$. Let $\mathcal{F}_\bullet : \ 0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$ be a complete flag of subsheaves, $\mathrm{length}(\mathcal{F}_r) = r$. Thus, for each $r = 1, 2, \ldots, n$, we have a sheaf imbedding $i_r : \mathcal{F}_r \hookrightarrow \mathcal{F}$ and also the dual projection $i_r^\vee : \mathcal{F}^\vee \twoheadrightarrow \mathcal{F}_r^\vee$.

Thus, for the spaces of global sections of sheaves on C^2, we have a diagram

$$\Gamma\big((\mathcal{F}_{r-1} \boxtimes \mathcal{F}_r^\vee)(-\Delta)\big) \xhookrightarrow{i_{r-1}} \Gamma\big((\mathcal{F} \boxtimes \mathcal{F}_r^\vee)(-\Delta)\big) \xleftarrow{i_r^\vee} \Gamma\big((\mathcal{F} \boxtimes \mathcal{F}^\vee)(-\Delta)\big).$$

Definition 4.3.1. (a) Let $\widetilde{\mathrm{rep}}_C^n$ be the moduli scheme of pairs $(\mathcal{F}, \mathcal{F}_\bullet)$, where $\mathcal{F} \in \mathrm{rep}_C^n$, and $\mathcal{F}_\bullet$ is a flag of subsheaves.

(b) Fix $\mathcal{F} \in \mathrm{rep}_C^n$, and an element $y \in \Gamma\big(C^2, (\mathcal{F} \boxtimes \mathcal{F}^\vee)(-\Delta)\big)$. A flag $\mathcal{F}_\bullet$ is said to be a *nil-flag* for the pair $(\mathcal{F}, y)$ if we have

$$i_r^\vee(y) \in i_r\Big(\Gamma\big(C^2, (\mathcal{F}_{r-1} \boxtimes \mathcal{F}_r^\vee)(-\Delta)\big)\Big), \quad \forall r = 1, 2, \ldots, n.$$

We have a natural forgetful morphism $\widetilde{\mathrm{rep}_C^n} \to \mathrm{rep}_C^n$, a $GL(V)$-equivariant proper morphism which is an analogue of Grothendieck simultaneous resolution, cf. [La1] 3.2.

We further extend the above morphism to a map $\phi : \widetilde{\mathrm{rep}_C^n} \times V \to X_n = \mathrm{rep}_C^n \times V$, identical on the second factor V. For the corresponding cotangent bundles, one obtains a standard diagram

$$T^*X_n \xleftarrow{p_1} \phi^*(T^*X_n) \xhookrightarrow{p_2} T^*(\widetilde{\mathrm{rep}_C^n} \times V),$$

where the map p_2 is a natural closed imbedding, and p_1 is a proper morphism. Thus, $\phi^*(T^*X_n)$ is the smooth variety parametrizing quintuples $(\mathcal{F}, \mathcal{F}_\bullet, y, i, j)$.

Let $Z \subset T^*(\widetilde{\mathrm{rep}_C^n} \times V)$ be a closed subscheme formed by the quintuples $(\mathcal{F}, \mathcal{F}_\bullet, y, i, j)$ such that $\mathcal{F}_\bullet$ is a nil-flag for y. Now, the set $p_1^{-1}(\mu^{-1}(0))$ is clearly a closed algebraic subvariety in $\phi^*T^*(\mathrm{rep}_C^n \times V)$, and so is $p_1^{-1}(\mu^{-1}(0)) \cap p_2^{-1}(Z)$. The map $p_1 : \; p_1^{-1}(\mu^{-1}(0)) \cap p_2^{-1}(Z) \to T^*X_n$ is a proper projection.

Observe next that the zero section embedding $C \hookrightarrow T^*C$ induces an embedding $C^{(n)} \hookrightarrow (T^*C)^{(n)}$, and recall the map π from Notation 4.2.2.

Proposition 4.3.2. *For a quadruple $(\mathcal{F}, y, i, j) \in \mu^{-1}(0)$, the following are equivalent:*

- *The pair $(\mathcal{F}, y)$ has a nil-flag;*
- *$(\mathcal{F}, y, i, j)$ belongs to the image of the projection* $p_1 : p_1^{-1}(\mu^{-1}(0)) \cap p_2^{-1}(Z) \to T^*X_n$;
- $\pi(\mathcal{F}, y, i, j) \in C^{(n)} \subset (T^*C)^{(n)}$.

This proposition will be proved in the next section.

4.4 A Lagrangian Subvariety

We define a *nil-cone* $\mathbb{M}_{\mathsf{nil}}(C) \subset T^*X_n$ to be the set of points satisfying the equivalent conditions of Proposition 4.3.2, with the natural structure of a reduced closed subscheme of T^*X_n arising from the third condition of the proposition.

In the special case of the curve $C = \mathbb{A}^1$, we have $\mathrm{rep}_C^n = \mathrm{End}V \cong \mathfrak{gl}_n$, and the corresponding variety $\mathbb{M}_{\mathsf{nil}}(C)$ is nothing but the Lagrangian subvariety introduced in [GG], (1.3). In more detail, write $\mathcal{N} \subset \mathfrak{gl}_n$ for the nilpotent variety. Then, one has

Lemma 4.4.1. (i) *Let $C = \mathbb{A}^1$. Then, we have*

$$\mathbb{M}_{\mathsf{nil}}(\mathbb{A}^1)=\{(x,y,i,j)\in\mathfrak{gl}_n\times\mathfrak{gl}_n\times V\times V^* \mid [x,y]+i\otimes j=0 \ \& \ y\in\mathcal{N}\}. \tag{4.4.2}$$

(ii) *Similarly, in the case $C = \mathbb{C}^\times$, we have*

$$\mathbb{M}_{\mathsf{nil}}(\mathbb{C}^\times)\simeq\{(x,y,i,j)\in GL_n\times\mathfrak{gl}_n\times V\times V^* \mid xyx^{-1}-y+i\otimes j=0 \ \& \ y\in\mathcal{N}\}.$$

Proof. In the case $C = \mathbb{A}^1$, the morphism $\phi : \widetilde{\mathrm{rep}^n_{\mathbb{A}^1}} \to \mathrm{rep}^n_{\mathbb{A}^1} = \mathfrak{gl}_n$ becomes the Grothendieck simultaneous resolution, cf. [La1] (3.2). In this case Proposition 4.3.2 is known. Hence, we only have to check that for any quadruple (x,y,i,j) as in (4.4.2) there exists a complete flag $V_\bullet$ such that $xV_k \subset V_k$, and $yV_k \subset V_{k-1}$ for any $k = 1,\dots,n$. But this is also well known, see e.g., [EG], Lemma 12.7. Part (i) follows. Part (ii) is proved similarly. □

Proof of Proposition 4.3.2. The first two conditions are clearly equivalent. The first and third conditions are compatible with factorization, and so are reduced to the "local" case $C = \mathbb{A}^1$, see Example 4.1.3. Then they are both equivalent to the nilpotency of y, cf. Lemma 4.4.1. □

Proposition 4.4.3. $\mathbb{M}_{\mathsf{nil}}(C)$ *is the reduced scheme of a Lagrangian locally complete intersection subscheme in* $T^*(\mathrm{rep}^n_C \times V)$.

We recall that it has been proved in [GG], Theorem 1.2, that the right-hand side in (4.4.2) is the reduced scheme of a lagrangian complete intersection in $\mathfrak{gl}_n \times \mathfrak{gl}_n \times V \times V^*$.

Proof of Proposition 4.4.3. We are going to reduce the statement of the proposition to the above-mentioned special case of $C = \mathbb{A}^1$, see (4.4.2).

To this end, fix a curve C'. Proposition 2.5.1 implies that there exists a finite collection of Zariski open subsets $C \subset C'$ and, for each subset C, an étale morphism $p : C \to \mathbb{A}^1$ and a Zariski open subset $\mathcal{U} \subset \mathrm{rep}^n_C \times V$, such that the following holds:

- The scheme $\mathrm{rep}^n_{C'} \times V$ is covered by the images in $\mathrm{rep}^n_{C'}$ of the subsets $\mathcal{U}$, corresponding to the curves C from our collection;
- For each C, there is an étale morphism of curves $p : C \to \mathbb{A}^1$ such that the induced morphism below is also étale,

$$\mathrm{rep}\,p\times\mathrm{Id}_V : \mathrm{rep}^n_C \times V \longrightarrow X := \mathrm{rep}^n_{\mathbb{A}^1}\times V = \mathfrak{gl}(V)\times V.$$

Then, we have $T^*\mathcal{U} = \mathcal{U}\times_X T^*X$; hence, the map $\mathrm{rep}\,p\times\mathrm{Id}_V$ induces a $GL(V)$-equivariant étale morphism $p_* : T^*\mathcal{U} \to T^*X$. The moment map $T^*\mathcal{U} \to \mathrm{End}V$ factors through $T^*\mathcal{U} \to T^*X \to \mathrm{End}V$, and $\mathbb{M}_{\mathsf{nil}}(C) \cap T^*\mathcal{U}$ is the preimage of $\mathbb{M}_{\mathsf{nil}}(\mathbb{A}^1)$ in $T^*\mathcal{U}$. Thus, $\mathbb{M}_{\mathsf{nil}}(C) \cap T^*\mathcal{U}$ is the reduced scheme of a lagrangian locally complete intersection in $T^*\mathcal{U}$. The proposition follows. □

4.5 Projectivization

One can perform the quantum hamiltonian reduction of Theorem 3.3.3 in two steps: first with respect to the central subgroup $\mathbb{C}^\times \subset GL_n$, of scalar matrices, and then with respect to the subgroup $SL_n \subset GL_n$. The subgroup $\mathbb{C}^\times \subset GL_n$ acts trivially on rep^n_C; it also acts naturally on $V = \mathbb{C}^n$, by dilations. We put

$$V^\circ := V \smallsetminus \{0\}; \quad \mathbb{P} = \mathbb{P}(V) = V^\circ/\mathbb{C}^\times; \quad \mathfrak{X}_n = \operatorname{rep}^n_C \times \mathbb{P}.$$

We have a natural Hamiltonian $\mathbb{C}^\times$-action on T^*V, a symplectic manifold, and Hamiltonian reduction procedure replaces T^*V by $T^*\mathbb{P}$.

Similarly, the diagonal $\mathbb{C}^\times$-action on $\operatorname{rep}^n_C \times V^\circ$ gives rise to a Hamiltonian $\mathbb{C}^\times$-action on $T^*(\operatorname{rep}^n_C \times V^\circ)$, with moment map μ. It is clear that the space

$$T^*\mathfrak{X}_n = \mu^{-1}(0)/\mathbb{C}^\times = \{(x, y, i, j) \in SL_n \times \mathfrak{sl}_n \times V^\circ \times V^* \mid \langle j, i\rangle = 0\}/\mathbb{C}^\times$$

is a Hamiltonian reduction of the symplectic manifold $T^*(\operatorname{rep}^n_C \times V^\circ)$.

One may also consider the Hamiltonian reduction of the Lagrangian subscheme $\mathbb{M}_{\mathsf{nil}} \subset T^*X_n$. This way one gets a closed Lagrangian subscheme $\mathfrak{M}_{\mathsf{nil}}(C) := \big(\mathbb{M}_{\mathsf{nil}}(C) \cap \mu^{-1}(0)\big)/\mathbb{C}^\times \subset T^*\mathfrak{X}_n$.

Recall that, by Hodge theory, there are natural isomorphisms $H^2(\mathbb{P}, \Omega^{1,2}_{\mathbb{P}}) \cong H^{1,1}(\mathbb{P}, \mathbb{C}) = \mathbb{C}$. Therefore, for any $\psi \in H^2(C, \Omega^{1,2}_C)$ and $c \in \mathbb{C} = H^2(\mathbb{P}, \Omega^{1,2}_{\mathbb{P}})$, there is a well-defined class $(\psi_n, c) \in H^2(\mathfrak{X}_n, \Omega^{1,2}_{\mathfrak{X}_n})$, and the corresponding TDO $\mathscr{D}_{\psi,c}(\mathfrak{X}_n)$, on $\mathfrak{X}_n$.

We have a natural algebra isomorphism

$$\mathscr{D}_{\psi,c}(\mathfrak{X}_n) \cong \big(\mathscr{D}_{\psi_n}(X_n)/\mathscr{D}_{\psi_n}(X_n)\cdot(\mathsf{eu} - c)\big)^{\mathbb{C}^\times}, \tag{4.5.1}$$

where eu is the Euler vector field on V that corresponds to the action of the identity matrix $\mathrm{Id} \in \mathfrak{gl}(V)$. The algebra on the right-hand side of (4.5.1) is a quantum hamiltonian reduction with respect to the group $\mathbb{C}^\times$, at the point $\kappa = c/n$.

Definition 4.5.2. An $SL(V)$-equivariant twisted $\mathscr{D}_{\psi,c}(\mathfrak{X}_n)$-module is called a *character $\mathscr{D}$-module* if its characteristic variety is contained in $\mathfrak{M}_{\mathsf{nil}}(C) = \big(\mathbb{M}_{\mathsf{nil}}(C) \cap \mu^{-1}(0)\big)/\mathbb{C}^\times$.

Let $\mathscr{C}_{\psi,c}$ denote the (abelian) category of character $\mathscr{D}_{\psi,c}(\mathfrak{X}_n)$-modules.

It is clear, by Proposition 4.3.2, that any object of the category $\mathscr{C}_{\psi,c}$ is a holonomic $\mathscr{D}$-module; in particular, such an object has finite length. See [FG] for more details.

Remark 4.5.3. For a general curve C and a general pair $(\psi, c) \in H^2(C, \Omega^{1,2}_C) \times \mathbb{C}$, the category $\mathscr{C}_{\psi,c}$ may have no nonzero objects at all. It is an interesting open problem to analyze which nonzero objects of the category $\mathscr{C}_{0,0}$ admit (at least formal) deformation in the direction of some $(\psi, c) \neq (0, 0)$.

In the special case of the curve $C = \mathbb{C}^\times$, however, we have $H^2(C, \Omega_C^{1,2}) = 0$. Thus, there is only one nontrivial parameter, $c \in \mathbb{C}$. It turns out that, for any $c \in \mathbb{C}$, there is a lot of nonzero character $\mathscr{D}$-modules on $GL_n \times \mathbb{P}$; moreover, all these $\mathscr{D}$-modules have regular singularities. $\diamondsuit$

4.6 Hamiltonian Reduction Functor

We now introduce a version of category $\mathcal{O}$ for $\mathsf{A}_{\kappa,\psi} = \mathsf{e}\mathsf{H}_{\kappa,\psi_n+1}\mathsf{e}$, the spherical Cherednik algebra associated with a smooth curve C, see (3.4.1).

Definition 4.6.1. Let $\mathcal{O}(\mathsf{A}_{\kappa,\psi})$ be the full subcategory of the abelian category of left $\mathsf{A}_{\kappa,\psi}$-modules whose objects are coherent as $\mathcal{O}_{C^{(n)}}$-modules.

For any $SL(V)$-equivariant $\mathscr{D}_{\psi,c}(\mathfrak{X}_n)$-module $\mathcal{F}$, one has the sheaf-theoretic push-forward $\varpi_\bullet\mathcal{F}$, a sheaf on $C^{(n)}$, cf. (2.2.1). The latter sheaf comes equipped with a natural locally finite (rational) $SL(V)$-action, and we write $(\varpi_\bullet\mathcal{F})^{SL(V)} \subset \varpi_\bullet\mathcal{F}$, for the $\mathcal{O}_{C^{(n)}}$-subsheaf of $SL(V)$-fixed sections.

Thanks to Theorem 3.3.3, one can apply the general formalism of Hamiltonian reduction, as outlined in [GG], Sect. 7, to the spherical Cherednik algebra. Specifically, we have the following result

Proposition 4.6.2. (i) *One has the following* exact *functor of* Hamiltonian reduction*:*

$$\mathbb{H} : \mathscr{C}_{\psi,c} \longrightarrow \mathcal{O}(\mathsf{A}_{\kappa,\psi}), \quad \mathcal{F} \mapsto \mathbb{H}(\mathcal{F}) = \mathcal{F}(\varpi_\bullet\mathcal{F})^{SL(V)}, \qquad \kappa = c/n.$$

Moreover, this functor induces an equivalence $\mathscr{C}_{\psi,c}/\ker\mathbb{H} \xrightarrow{\sim} \mathcal{O}(\mathsf{A}_{\kappa,\psi})$.

(ii) *The functor* $\mathbb{H}$ *has a left adjoint functor*

$${}^\top\mathbb{H} : \mathcal{O}(\mathsf{A}_{\kappa,\psi}) \to \mathscr{C}_{\psi,c}, \quad M \mapsto (\mathscr{D}_{\psi,c}(\mathfrak{X}_n)/\mathscr{D}_{\psi,c}(\mathfrak{X}_n)\mathfrak{g}_\kappa) \otimes_{\mathsf{A}_{\kappa,\psi}} M.$$

Moreover, for any $M \in \mathcal{O}(\mathsf{A}_{\kappa,\psi})$*, the canonical adjunction* $\mathbb{H} \circ {}^\top\mathbb{H}(M) \to M$ *is an isomorphism.*

To prove Proposition 4.6.2, first recall the projection $T^*C^n \twoheadrightarrow (T^*C)^{(n)}$. Abusing the language, we will refer to the image of the zero section of T^*C^n under the projection as the "zero section" of $(T^*C)^{(n)}$.

Recall that the spherical algebra $\mathsf{A}_{\kappa,\psi}$ has an increasing filtration. Therefore, given a coherent $\mathsf{A}_{\kappa,\psi}$-module M, one has a well-defined notion of *characteristic variety*, $SS(M) \subset \mathrm{Spec}(\mathtt{gr}^F \mathsf{A}_{\kappa,\psi})$. We may (and will) use isomorphism (3.1.5) and view $SS(M)$ as a closed algebraic subset of $(T^*C)^{(n)}$. Then, the following is clear

Lemma 4.6.3. *A coherent* $\mathsf{A}_{\kappa,\psi}$*-module* M *is an object of* $\mathcal{O}(\mathsf{A}_{\kappa,\psi})$ *if and only if* $SS(M)$ *is contained in the zero section of* $(T^*C)^{(n)}$. $\square$

Proof of Proposition 4.6.2. The assignment $\mathbb{H} : \mathcal{F} \mapsto (\varpi_\bullet \mathcal{F})^{SL(V)}$ clearly gives a functor from the category of $SL(V)$-equivariant coherent $\mathscr{D}_{\psi,c}(\mathfrak{X}_n)$-modules to the category of quasi-coherent $\mathcal{O}_{C^{(n)}}$-modules. This functor is *exact* since the support-morphism ϖ is affine and $SL(V)$ is a reductive group acting rationally on $\varpi_\bullet \mathcal{F}$. Thus, our Theorem 3.3.3 combined with [GG], Proposition 7.1, implies that $(\varpi_\bullet \mathcal{F})^{SL(V)}$ has a natural structure of coherent $\mathsf{A}_{\kappa,\psi}$-module.

We may use Lemma 4.2.1 and the map π introduced after the lemma to obtain a diagram

$$\mathrm{Spec}\big(\mathtt{gr}\mathscr{D}_{\psi,c}(\mathfrak{X}_n)\big) = T^*\mathfrak{X}_n \supset \mu^{-1}(0) \xrightarrow{\pi} (T^*C)^{(n)} = \mathrm{Spec}\big(\mathtt{gr}\mathsf{A}_{\kappa,\psi}\big). \tag{4.6.4}$$

We remark that the statement of Lemma 4.2.1 involves the space X_n rather than $\mathfrak{X}_n$; the above diagram is obtained from a similar diagram for the subset $\mathrm{rep}^n_C \times V^\circ \subset X_n$, by Hamiltonian reduction with respect to the $\mathbb{C}^\times$-action on $T^*(\mathrm{rep}^n_C \times V^\circ)$ induced by the natural action of the group $\mathbb{C}^\times \subset GL(V)$ on $\mathrm{rep}^n_C \times V^\circ$. Furthermore, from Lemma 4.2.1, we deduce that the map π in (4.6.4) induces an isomorphism $\mu^{-1}(0)/\!/SL(V) \cong (T^*C)^{(n)}$.

Now let $\mathcal{F}$ be an $SL(V)$-equivariant coherent $\mathscr{D}_{\psi,c}(\mathfrak{X}_n)$-module. Choose a good increasing filtration on $\mathcal{F}$ by $SL(V)$-equivariant $\mathcal{O}_{\mathfrak{X}_n}$-coherent subsheaves and view $\mathtt{gr}\mathcal{F}$, the associated graded object, as a coherent sheaf on $T^*\mathfrak{X}_n$. It follows that $SS(\mathcal{F}) = \mathtt{Supp}(\mathtt{gr}\mathcal{F}) \subset \mu^{-1}(0)$.

Each of the functors $\varpi_\bullet$ and $(-)^{SL(V)}$ is exact. We deduce $\mathtt{gr}((\varpi_\bullet \mathcal{F})^{SL(V)}) \cong (\pi_\bullet \mathtt{gr}\mathcal{F})^{SL(V)}$. Moreover, the isomorphism $(T^*C)^{(n)} \cong \mu^{-1}(0)/\!/SL(V)$ insures that $(\pi_\bullet \mathtt{gr}\mathcal{F})^{SL(V)}$ is a coherent sheaf on $(T^*C)^{(n)}$. Hence, the filtration on $\mathbb{H}(\mathcal{F}) = (\varpi_\bullet \mathcal{F})^{SL(V)}$, induced by the one on $\mathcal{F}$, is a good filtration; that is, $\mathtt{gr}\mathbb{H}(\mathcal{F})$ is a coherent $\mathtt{gr}\mathsf{A}_{\kappa,\psi}$-module, cf. (4.6.4). We conclude that $SS(\mathbb{H}(\mathcal{F})) \subset \pi(SS(\mathcal{F}))$.

The above implies that, for any character $\mathscr{D}$-module $\mathcal{F}$, one has

$$SS(\mathbb{H}(\mathcal{F})) \subset \pi(SS(\mathcal{F})) \subset \pi(\mathfrak{M}_{\mathsf{nil}}(C)) = \text{zero section of } (T^*C)^{(n)}.$$

The first claim of part (i) of the proposition follows from these inclusions and Lemma 4.6.3. Part (ii) is proved similarly; using that, for any coherent $\mathsf{A}_{\kappa,\psi}$-module M, one has

$$SS({}^\top\mathbb{H}(M)) \subset \pi^{-1}(SS(M)). \tag{4.6.5}$$

At this point, the second claim of part (i) is a general consequence of the existence of a left adjoint functor, cf. [GG], Proposition 7.6, and we are done. □

Corollary 4.6.6. *Let $\mathcal{F}$ be a simple character $\mathscr{D}$-module such that $\mathbb{H}(\mathcal{F}) \neq 0$. Then, we have*

$$\Gamma(C^{(n)}, \mathbb{H}(\mathcal{F})) < \infty \quad \iff \quad \varpi(\mathtt{Supp}\mathcal{F}) \text{ is a finite subset of } C^{(n)}.$$

Proof. The space of global sections of a coherent $\mathcal{O}_{C^{(n)}}$-module is finite dimensional if and only if the module has finite support. Furthermore, we know that $\mathbb{H}(\mathcal{F})$ is a coherent $\mathcal{O}_{C^{(n)}}$-module and, moreover, it is clear that $\mathtt{Supp}\mathbb{H}(\mathcal{F}) \subset \varpi(\mathtt{Supp}\mathcal{F})$. This gives the implication "$\Leftarrow$."

To prove the opposite implication, let $M := \mathbb{H}(\mathcal{F})$. We have a *nonzero* morphism ${}^{\top}\mathbb{H}(M) = {}^{\top}\mathbb{H} \circ \mathbb{H}(\mathcal{F}) \to \mathcal{F}$ that corresponds to $\mathrm{Id}_{\mathbb{H}(\mathcal{F})}$ via the adjunction isomorphism

$$\mathrm{Hom}_{\mathscr{C}_{\psi,c}}({}^{\top}\mathbb{H} \circ \mathbb{H}(\mathcal{F}), \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}(\mathsf{A}_{\kappa,\psi})}(\mathbb{H}(\mathcal{F}), \mathbb{H}(\mathcal{F})).$$

This yields the following inclusions

$$\varpi(\mathtt{Supp}\mathcal{F}) \subset \varpi(\mathtt{Supp}\,{}^{\top}\mathbb{H}(M)) \subset \varpi\big(\varpi^{-1}(\mathtt{Supp}M)\big) = \mathtt{Supp}M = \text{finite set}.$$

Here, the leftmost inclusion follows since $\mathcal{F}$ is simple; hence the map ${}^{\top}\mathbb{H}(M) \to \mathcal{F}$ is surjective, and the rightmost equality is due to our assumption that the sheaf $\mathbb{H}(\mathcal{F})$ has finite support. The implication "$\Rightarrow$" follows. □

5 The Trigonometric Case

5.1 Trigonometric Cherednik Algebra

From now on, we consider a special case of the curve $C = \mathbb{C}^{\times}$. Thus, we have $\mathrm{rep}^n_{\mathbb{C}^\times} \simeq GL_n$.

Let $H \subset GL_n$ be the maximal torus formed by diagonal matrices, and let $\mathfrak{h}$ be the Lie algebra of H. Let $\{\mathsf{y}_1, \dots, \mathsf{y}_n\}$ and $\{x_1, \dots, x_n\}$ be dual bases of $\mathfrak{h}$ and $\mathfrak{h}^*$, respectively. The coordinate ring $\mathbb{C}[H]$, of the torus H, may be identified with $\mathbb{C}[\mathbf{x}_1^{\pm 1}, \dots, \mathbf{x}_n^{\pm 1}]$, the Laurent polynomial ring in the variables $\mathbf{x}_i = \exp(x_i)$. Given an n-tuple $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n$, we write $\mathbf{x}^{\nu} := \mathbf{x}_1^{\nu_1} \cdots \mathbf{x}_n^{\nu_n} \in \mathbb{C}[\mathbf{x}_1^{\pm 1}, \dots, \mathbf{x}_n^{\pm 1}]$, for the corresponding monomial. Let $\mathbb{T} = SL_n \cap H$ be a maximal torus in SL_n, and let $\mathfrak{t} := \mathrm{Lie}\mathbb{T}$. Let $P = \mathrm{Hom}(H, \mathbb{C}^{\times})$, resp. P_0, and P^0, be the weight lattice of the group GL_n, resp. of the group SL_n, and PGL_n. Thus, $P^0 \subset P_0$, and we put $\Omega := P_0/P^0 \cong \mathbb{Z}/n\mathbb{Z}$. We form semi-direct products $W^e = P \rtimes W$ (an extended affine Weyl group), resp. $W^a := P^0 \rtimes W$, and $W_0^e := P_0 \rtimes W$. One has an isomorphism $W_0^e = W^a \rtimes \Omega$.

We fix $\kappa \in \mathbb{C}$. The *trigonometric Cherednik algebra* $\mathsf{H}_{\kappa}^{\mathrm{trig}}(GL_n)$ *of type* GL_n is generated by the subalgebras $\mathbb{C}[W^e] := \mathbb{C}[H] \rtimes \mathbb{C}[\mathbb{S}_n]$ and $\mathrm{Sym}(\mathfrak{h}) = \mathbb{C}[\mathsf{y}_1, \dots, \mathsf{y}_n]$ with relations, see e.g., [AST], 1.3.7, or [Su], Sect. 2:

$$\begin{aligned}
&s_i \cdot \mathsf{y} - s_i(\mathsf{y}) \cdot s_i = -\kappa\langle x_i - x_{i+1}, \mathsf{y}\rangle, && \forall \mathsf{y} \in \mathfrak{h},\ 1 \le i < n;\\
&[\mathsf{y}_i, \mathbf{x}_j] = \kappa \mathbf{x}_j \cdot s_{ij}, && 1 \le i \ne j \le n;\\
&[\mathsf{y}_k, \mathbf{x}_k] = \mathbf{x}_k - \kappa \mathbf{x}_k \cdot \sum_{i \in [1,n] \smallsetminus \{k\}} s_{ik}, && 1 \le k \le n.
\end{aligned}$$

Recall that $\mathbb{P} := \mathbb{P}(V)$, where we put $V = \mathbb{C}^n$. According to [GG], (6.13), one has a canonical isomorphism

$$(\mathscr{D}(GL_n \times V)/\mathscr{D}(GL_n \times V)\mathfrak{g}_\kappa)^{GL_n} \simeq (\mathscr{D}_{n\kappa}(GL_n \times \mathbb{P})/\mathscr{D}_{n\kappa}(GL_n \times \mathbb{P})\mathfrak{g}_\kappa)^{SL_n},$$

where $\mathscr{D}_{n\kappa}(GL_n \times \mathbb{P}) = \mathscr{D}(GL_n) \otimes \mathscr{D}_{n\kappa}(\mathbb{P})$ stands for the sheaf of twisted differential operators on $GL_n \times \mathbb{P}$, with a twist $n \cdot \kappa$ along $\mathbb{P}$.

Corollary 5.1.1. *The spherical trigonometric Cherednik subalgebra* $\mathsf{eH}_\kappa^{\mathrm{trig}}(GL_n)\mathsf{e}$ *is isomorphic to the quantum Hamiltonian reduction* $(\mathscr{D}_{n\kappa}(GL_n \times \mathbb{P})/\mathscr{D}_{n\kappa}(GL_n \times \mathbb{P})\mathfrak{g}_\kappa)^{SL_n}$.

Proof. Recall that $\rho := \frac{1}{2}\sum_{\alpha\in R^+} \alpha \in \mathfrak{h}^*$. For $\mathsf{y} \in \mathfrak{h}$, let ∂_y denote the corresponding translation invariant vector field on H.

Introduce the *Dunkl–Cherednik operator* T_y^κ as an endomorphism of $\mathbb{C}[H]$ defined as follows:

$$T_\mathsf{y}^\kappa := \partial_\mathsf{y} - \kappa\langle\rho,\mathsf{y}\rangle + \kappa \sum_{1\le i<j\le n} \frac{\langle x_i - x_j, \mathsf{y}\rangle}{1 - \mathbf{x}_i^{-1}\mathbf{x}_j}(1 - s_{ij}) \tag{5.1.2}$$

Let $A(GL_n)$ be a subalgebra of $\mathrm{End}(\mathbb{C}[H])$ generated by the operators corresponding to the action of elements $w \in \mathbb{S}_n$, by the commutative algebra $\mathbb{C}[H]$, of multiplication operators, and by all the operators $T_\mathsf{y}^\kappa,\ \mathsf{y} \in \mathfrak{h}$.

The following key result is due to Cherednik; for a nice exposition, see e.g., [Op], 3.7:

The assignment

$$\mathbf{x}^\nu w \mapsto \exp(\nu)w, \quad \mathfrak{h} \ni \mathsf{y} \mapsto T_\mathsf{y}^\kappa, \qquad \nu \in \mathbb{Z}^n,\ w \in \mathbb{S}_n,\ \mathsf{y} \in \mathfrak{h}$$

extends to an algebra isomorphism $\mathsf{H}_\kappa^{\mathrm{trig}}(GL_n) \xrightarrow{\sim} A(GL_n)$.

To complete the proof, note that the curve $(\mathbb{C}^\times)^{(n)}$ being affine, one can replace the sheaf of Cherednik algebras by the corresponding algebra of global sections. Moreover, since $H^2(\mathbb{C}^\times, \Omega^{1,2}_{\mathbb{C}^\times}) = 0$, the parameter ψ in $\mathsf{H}_{\kappa,\psi}$ vanishes. Thus, Theorem 3.3.3 says that the Hamiltonian reduction algebra $(\mathscr{D}_{n\kappa}(GL_n \times \mathbb{P})/\mathscr{D}_{n\kappa}(GL_n \times \mathbb{P})\mathfrak{g}_\kappa)^{SL_n}$ is isomorphic to the algebra $A(GL_n)$. □

5.2 SL_n and PGL_n Cases

The *trigonometric Cherednik algebra* $\mathsf{H}_\kappa^{\mathrm{trig}}(PGL_n)$ *of* PGL_n*-type* is defined as a subalgebra in $\mathsf{H}_\kappa^{\mathrm{trig}}(GL_n)$ generated by $\mathbb{C}[W^a]$ and $\mathrm{Sym}(\mathfrak{t})$. Equivalently, $\mathsf{H}_\kappa^{\mathrm{trig}}(PGL_n)$ is generated by the subalgebras $\mathbb{C}[W^a]$ and $\mathrm{Sym}(\mathfrak{t})$ with relations

$$s_i \cdot \mathsf{y} - s_i(\mathsf{y}) \cdot s_i = -\kappa\langle x_i - x_{i+1}, \mathsf{y}\rangle, \quad \forall \mathsf{y} \in \mathfrak{t},\ 1 \le i < n \tag{5.2.1}$$

$$[\mathsf{y},\mathbf{x}] = \langle\eta,\mathsf{y}\rangle\mathbf{x} - \kappa \sum_{1\le i<j\le n} \langle\alpha_{ij},\mathsf{y}\rangle \frac{\mathbf{x}-s_{ij}(\mathbf{x})}{1-\mathbf{x}_i^{-1}\mathbf{x}_j} s_{ij},$$
$$\forall \mathsf{y}\in\mathfrak{t},\ \mathbf{x}=\exp(\eta),\ \eta\in P^0\subset\mathbb{C}[W^a] \tag{5.2.2}$$

The *trigonometric Cherednik algebra* $\mathsf{H}_\kappa^{\mathrm{trig}}(SL_n)$ *of type* SL_n is generated by the subalgebras $\mathbb{C}[W_0^e]$ and $\mathrm{Sym}(\mathfrak{t})$ with relations

$$s_i\cdot\mathsf{y} - s_i(\mathsf{y})\cdot s_i = -\kappa\langle x_i - x_{i+1},\mathsf{y}\rangle,\quad \forall\mathsf{y}\in\mathfrak{t},\ 1\le i<n \tag{5.2.3}$$

$$\omega\cdot\mathsf{y} = \omega(\mathsf{y})\cdot\omega,\quad \forall\mathsf{y}\in\mathfrak{t},\ \omega\in\Omega \tag{5.2.4}$$

$$[\mathsf{y},\mathbf{x}] = \langle\eta,\mathsf{y}\rangle\mathbf{x} - \kappa \sum_{1\le i<j\le n} \langle\alpha_{ij},\mathsf{y}\rangle \frac{\mathbf{x}-s_{ij}(\mathbf{x})}{1-\mathbf{x}_i^{-1}\mathbf{x}_j} s_{ij},$$
$$\forall \mathsf{y}\in\mathfrak{t},\ \mathbf{x}=\exp(\eta),\ \eta\in P_0\subset\mathbb{C}[W_0^e] \tag{5.2.5}$$

For an equivalent definition, see e.g., [Op], 3.6.

The algebras $\mathsf{H}_\kappa^{\mathrm{trig}}(SL_n)$, $\mathsf{H}_\kappa^{\mathrm{trig}}(PGL_n)$ and $\mathsf{H}_\kappa^{\mathrm{trig}}(GL_n)$ are closely related to each other. To formulate the relations more precisely, let $\mathscr{D}(\mathbb{C}^\times)$ be the algebra of differential operators on $\mathbb{C}^\times$. Let $\mathfrak{x}$ be a coordinate on $\mathbb{C}^\times$, and $\mathfrak{y}=\mathfrak{x}\partial_{\mathfrak{x}}$. Then $\mathscr{D}(\mathbb{C}^\times)$ is generated by $\mathfrak{x}^{\pm1}$, $\mathfrak{y}$ with the relation $[\mathfrak{y},\mathfrak{x}]=\mathfrak{x}$.

We have an embedding $\mathbb{C}^\times\hookrightarrow H$ as the scalar (central) matrices in GL_n. Taking product with the embedding $\mathbb{T}\hookrightarrow H$, we obtain a morphism ($n$-fold covering) $\mathbb{T}\times\mathbb{C}^\times\twoheadrightarrow H$. At the level of functions, we have an embedding $\mathbb{C}[H]\hookrightarrow\mathbb{C}[\mathbb{T}]\otimes\mathbb{C}[\mathbb{C}^\times]$. In coordinates, we have $\mathbf{x}_i\mapsto\mathbf{x}_i\otimes\mathbf{x}_i$, $1\le i\le n$. The isogeny $\mathbb{T}\times\mathbb{C}^\times\twoheadrightarrow H$ induces an isomorphism of the Lie algebras of our tori. The inverse isomorphism $\mathfrak{h}\to\mathfrak{t}\oplus\mathbb{C}$ sends $\mathsf{y}_i\in\mathfrak{h}$ to $(\mathsf{y}_i-\frac{1}{n}\sum_{k=1}^n\mathsf{y}_k,\ \frac{1}{n}\mathfrak{y})$ (recall from the previous paragraph that $\mathbb{C}=\mathrm{Lie}\,\mathbb{C}^\times$ is equipped with the base $\{\mathfrak{y}\}$).

We consider the tensor product algebra $\mathsf{H}_\kappa^{\mathrm{trig}}(SL_n)\otimes\mathscr{D}(\mathbb{C}^\times)$ together with the following elements:

$$\Xi(\mathbf{x}_i) := \mathbf{x}_i\otimes\mathfrak{x};\quad \Xi(\mathsf{y}_i) := \left(\mathsf{y}_i - \frac{1}{n}\sum_{k=1}^n\mathsf{y}_k\right)\otimes 1 + 1\otimes\frac{1}{n}\mathfrak{y};\quad 1\le i\le n.$$

The following is a straightforward corollary of definitions.

Corollary 5.2.6. (i) *There is a natural isomorphism* $\mathsf{H}_\kappa^{\mathrm{trig}}(SL_n)\simeq\mathsf{H}_\kappa^{\mathrm{trig}}(PGL_n)\rtimes\Omega$.
(ii) *The map* $\mathbf{x}_i\mapsto\Xi(\mathbf{x}_i)$, $\mathsf{y}_i\mapsto\Xi(\mathsf{y}_i)$, $s_{ij}\mapsto s_{ij}$; $1\le i\ne j\le n$ *defines an injective homomorphism* $\Xi:\ \mathsf{H}_\kappa^{\mathrm{trig}}(GL_n)\hookrightarrow\mathsf{H}_\kappa^{\mathrm{trig}}(SL_n)\otimes\mathscr{D}(\mathbb{C}^\times)$. $\square$

Corollary 5.2.7. *For any* $c\in\mathbb{C}\smallsetminus[-1,0)$, *the functor below is a Morita equivalence*

$$\mathsf{H}_\kappa^{\mathrm{trig}}(SL_n)\text{-mod}\longrightarrow \mathsf{e}\mathsf{H}_\kappa^{\mathrm{trig}}(SL_n)\mathsf{e}\text{-mod},\quad M\mapsto\mathsf{e}M.$$

Proof. We need to prove that $\mathsf{H}^{\mathrm{trig}}_\kappa(SL_n) = \mathsf{H}^{\mathrm{trig}}_\kappa(SL_n)\cdot\mathsf{e}\cdot\mathsf{H}^{\mathrm{trig}}_\kappa(SL_n)$. The similar statement for the trigonometric Cherednik algebra $\mathsf{H}^{\mathrm{trig}}(GL_n)$ follows from Proposition 3.1.3 (for $C = \mathbb{C}^\times$). Therefore, we have an equality

$$1 = \sum_{k=1}^{r} u_k\cdot\mathsf{e}\cdot v_k, \quad u_1, \dots, u_r, v_1, \dots, v_r \in \mathsf{H}^{\mathrm{trig}}_\kappa(GL_n). \tag{5.2.8}$$

This implies, via the imbedding $\mathsf{H}^{\mathrm{trig}}_\kappa(GL_n) \hookrightarrow \mathsf{H}^{\mathrm{trig}}_\kappa(SL_n) \otimes \mathscr{D}(\mathbb{C}^\times)$ of Corollary 5.2.6 (ii), that a similar equality holds in the algebra $\mathsf{H}^{\mathrm{trig}}_\kappa(SL_n) \otimes \mathscr{D}(\mathbb{C}^\times)$. Thus, in $\mathsf{H}^{\mathrm{trig}}_\kappa(SL_n) \otimes \mathscr{D}(\mathbb{C}^\times)$ we have

$$1 \otimes 1 = \sum_{k=1}^{r} (u_k\cdot\mathsf{e}\cdot v_k) \otimes w_k, \quad u_k, v_k \in \mathsf{H}^{\mathrm{trig}}_\kappa(SL_n),\ w_k \in \mathscr{D}(\mathbb{C}^\times). \tag{5.2.9}$$

Choose a $\mathbb{C}$-linear countable basis $\{q_\nu\}_{\nu\in\mathbb{N}}$ of the vector space $\mathscr{D}(\mathbb{C}^\times)$ such that $q_1 = 1$. Expanding each of the elements $w_k \in \mathscr{D}(\mathbb{C}^\times)$ in this basis and equating the corresponding terms in (5.2.9), one deduces from (5.2.9) an equation of the form $1 = \sum_\ell a_\ell \cdot \mathsf{e} \cdot b_\ell$, where $a_\ell, b_\ell \in \mathsf{H}^{\mathrm{trig}}_\kappa(SL_n)$. Thus, we have shown that

$$\mathsf{H}^{\mathrm{trig}}_\kappa(SL_n) = \mathsf{H}^{\mathrm{trig}}_\kappa(SL_n)\cdot\mathsf{e}\cdot\mathsf{H}^{\mathrm{trig}}_\kappa(SL_n),$$

and the Morita equivalence for the algebra $\mathsf{H}^{\mathrm{trig}}_\kappa(SL_n)$ follows. □

5.3 *The SL_n-case*

We have the $\mathbb{S}_n$-equivariant product morphism $H = (\mathbb{C}^\times)^n \to \mathbb{C}^\times$ with the kernel $\mathbb{T} \subset H$. We can consider the restriction of the sheaf of Cherednik algebras H_κ to the closed subvariety $\mathbb{T}/\mathbb{S}_n \subset H/\mathbb{S}_n$. Abusing the notation, we also write H_κ for the corresponding algebra of global sections.

The proof of the following result copies the proof of Proposition 5.1.1.

Corollary 5.3.1. *The spherical trigonometric Cherednik subalgebra* $\mathsf{e}\mathsf{H}^{\mathrm{trig}}_\kappa(SL_n)\mathsf{e}$ *is isomorphic to the quantum Hamiltonian reduction* $(\mathscr{D}_{n\kappa}(SL_n \times \mathbb{P})/\mathscr{D}_{n\kappa}(SL_n \times \mathbb{P})\mathfrak{g}_\kappa)^{SL_n}$.

An important difference between the algebras $\mathsf{e}\mathsf{H}^{\mathrm{trig}}_\kappa(GL_n)\mathsf{e}$ and $\mathsf{e}\mathsf{H}^{\mathrm{trig}}_\kappa(SL_n)\mathsf{e}$ is that the latter may have finite dimensional representations, while the former cannot have such representations. In view of this, it is desirable to have a version of Hamiltonian reduction functor that would relate $\mathscr{D}$-modules on $SL_n \times \mathbb{P}$ (rather than on $GL_n \times \mathbb{P}$) with $\mathsf{e}\mathsf{H}^{\mathrm{trig}}_\kappa(SL_n)\mathsf{e}$-modules. To this end, one needs first to introduce a Lagrangian nil-cone in $T^*(SL_n \times \mathbb{P})$, and the corresponding notion of character $\mathscr{D}$-module on $SL_n \times \mathbb{P}$.

To define the nil-cone, we intersect $\mathbb{M}_{\mathsf{nil}}(\mathbb{C}^\times) \subset GL_n \times \mathfrak{gl}_n \times V^\circ \times V^* = T^*(\mathrm{rep}^n_{\mathbb{C}^\times} \times V^\circ)$, the nil-cone for the group GL_n, with $SL_n \times \mathfrak{sl}_n \times V^\circ \times V^*$. Let $\mathbb{M}_{\mathsf{nil}} \subset T^*(SL_n \times \mathbb{P})$ be the Hamiltonian reduction of the resulting variety with respect to the Hamiltonian $\mathbb{C}^\times$-action on the factor $T^*V^\circ = V^\circ \times V^*$.

We write $\mathscr{D}_c := \mathscr{D}(SL_n) \otimes \mathscr{D}_c(\mathbb{P})$. An SL_n-equivariant $\mathscr{D}_c$-module is called character $\mathscr{D}$-module provided its characteristic variety is contained in $\mathbb{M}_{\mathsf{nil}}$.

5.4 A Springer Type Construction

We are going to introduce certain analogues of Springer resolution in our present setting. The constructions discussed below work more generally in the framework of an arbitrary smooth curve C. However, to simplify the exposition, we restrict ourselves to the case $C = \mathbb{C}^\times$.

Write $\mathcal{B}$ for the flag variety, that is, the variety of complete flags $F = (0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = V)$, where $\dim F_k = k, \forall k = 0, \ldots, n$. In the trigonometric case, the variety $\widetilde{\mathrm{rep}^n_C}$, introduced in Sect. 4.3, reduces to $\widetilde{SL}_n = \{(g, F) \in SL_n \times \mathcal{B} \mid g(F_k) \subset F_k,\ \forall k = 0, \ldots, n\}$. The first projection yields a proper morphism $\widetilde{SL}_n \twoheadrightarrow SL_n$ known as the *Grothendieck–Springer resolution*.

Now, fix an integer $1 \leq m \leq n$. We define a closed subvariety $\mathfrak{X}_{n,m} \subset \mathfrak{X}_n$ as follows:

$$\mathfrak{X}_{n,m} := \{(g, \ell) \in SL_n \times \mathbb{P} \mid \dim(\mathbb{C}[g]\ell) \leq m\},$$

where $\mathbb{C}[g]\ell \subset V$ denotes the minimal g-stable subspace in V that contains the line $\ell \subset V$.

Next, we put $\widetilde{\mathfrak{X}}_n = \widetilde{SL}_n \times \mathbb{P}$ and define a closed subvariety $\widetilde{\mathfrak{X}}_{n,m} \subset \widetilde{\mathfrak{X}}_n$ as follows:

$$\widetilde{\mathfrak{X}}_{n,m} := \{(g, F, \ell) \in SL_n \times \mathcal{B} \times \mathbb{P} \mid \ell \in F_m,\ \&\ g(F_k) \subset F_k,\ \forall k = 0, \ldots, n\}.$$

We have a diagram

$$\begin{tikzcd}
 & \widetilde{\mathfrak{X}}_{n,m} \arrow[dl, "(g,F,\ell)\mapsto(g,\ell)"'] \arrow[dr, "(g,F,\ell)\mapsto(F,\ell)"] & \\
SL_n \times \mathbb{P} & & \mathcal{B} \times \mathbb{P}.
\end{tikzcd}$$

We observe that:

- The projection $\widetilde{\mathfrak{X}}_{n,m} \to \mathcal{B} \times \mathbb{P}$, $(g, F, \ell) \mapsto (F, \ell)$ makes $\widetilde{\mathfrak{X}}_{n,m}$ a locally trivial fibration over the base $\{(F, \ell) \in \mathcal{B} \times \mathbb{P} \mid \ell \subset F_m\}$. Both the fibre and the base are smooth.

  Thus, $\widetilde{\mathfrak{X}}_{n,m}$ is smooth.

- The image of the projection $\widetilde{\mathfrak{X}}_{n,m} \to SL_n \times \mathbb{P}$ is contained in $\mathfrak{X}_{n,m}$; hence the projection gives a well-defined morphism:

$$\pi_{n,m} : \widetilde{\mathfrak{X}}_{n,m} \longrightarrow \mathfrak{X}_{n,m}, \quad (g, F, v) \mapsto (g, v).$$

Let $\mathfrak{X}^{\text{reg}}_{n,m}$ be an open subset of $\mathfrak{X}_{n,m}$ formed by the pairs (g, ℓ) such that g is a matrix with n pairwise distinct eigenvalues. Let $\widetilde{\mathfrak{X}}^{\text{reg}}_{n,m} = \pi^{-1}_{n,m}(\mathfrak{X}^{\text{reg}}_{n,m})$, an open subset in $\widetilde{\mathfrak{X}}_{n,m}$.

Proposition 5.4.1. (i) *The map* $\pi_{n,m} : \widetilde{\mathfrak{X}}_{n,m} \to \mathfrak{X}_{n,m}$ *is a dominant proper morphism, which is* small *in the sense of Goresky–MacPherson.*
(ii) *The restriction* $\pi_{n,m} : \widetilde{\mathfrak{X}}^{\text{reg}}_{n,m} \to \mathfrak{X}^{\text{reg}}_{n,m}$ *is a Galois covering with the Galois group* $\mathbb{S}_m \times \mathbb{S}_{n-m}$.

Proof. Clearly, $\pi_{n,m}$ is proper and has a dense image. Hence, it is dominant.

Let $Z := \widetilde{\mathfrak{X}}_{n,m} \times_{\mathfrak{X}_{n,m}} \widetilde{\mathfrak{X}}_{n,m}$; so we have a projection $Z \to \mathfrak{X}_{n,m}$. To prove that $\pi_{n,m}$ is *semi*small, one must check that $\dim Z \le n^2+m$. To prove that $\pi_{n,m}$ is small, one must show in addition that each irreducible component of Z, of dimension $n^2 + m$, dominates $\mathfrak{X}_{n,m}$. The argument is very similar to the standard proof of smallness of the Grothendieck–Springer resolution.

In more detail, for $w \in \mathbb{S}_n$ we denote by Z_w the locally closed subvariety of Z formed by the quadruples (g, F, F', ℓ) such that the flags F and F' are in relative position w (and such that $\ell \subset F_m \cap F'_m$, and $g(F) = F$, $g(F') = F'$). Then we have $Z = \bigsqcup_{w\in\mathbb{S}_n} Z_w$.

We may view Z_w as a fibration over an SL_n-orbit in $\mathcal{B} \times \mathcal{B}$, the cartesian square of flag variety. We see immediately that $\dim Z_w \le n^2 + m$ with an exact equality if and only if $F_m = F'_m$. The latter equality holds if and only if $w \in \mathbb{S}_m \times \mathbb{S}_{n-m} \subset \mathbb{S}_n$. For such a w, it is clear that Z_w dominates $\mathfrak{X}_{n,m}$. This completes the proof of (i).

Now part (ii) follows easily from the above description of irreducible components. $\square$

Let Σ_N denote the set of partitions of an integer N. For any partition $\lambda \in \Sigma_m$, resp. $\mu \in \Sigma_{n-m}$, write L_λ, resp. L_μ, for an irreducible representation of the Symmetric group $\mathbb{S}_m$, resp. $\mathbb{S}_{n-m}$, associated with that partition in a standard way. Thus, $L_\lambda \boxtimes L_\mu$ is an irreducible representation of the group $\mathbb{S}_m \times \mathbb{S}_{n-m}$.

Let $\mathbb{C}_{\widetilde{\mathfrak{X}}^{\text{reg}}_{n,m}}$ be the constant sheaf on $\widetilde{\mathfrak{X}}^{\text{reg}}_{n,m}$. According to Proposition 5.4.1 (ii), we have:

$$(\pi_{n,m})_* \mathbb{C}_{\widetilde{\mathfrak{X}}^{\text{reg}}_{n,m}} = \bigoplus_{(\lambda,\mu)\in\Sigma_m\times\Sigma_{n-m}} (L_\lambda \boxtimes L_\mu) \otimes \mathscr{L}_{\lambda,\mu},$$

where $\mathscr{L}_{\lambda,\mu}$ is an irreducible local system on $\mathfrak{X}^{\text{reg}}_{n,m}$ with monodromy $L_\lambda \boxtimes L_\mu$.

Now, let $\mathbb{C}_{\widetilde{\mathfrak{X}}_{n,m}}[\dim \widetilde{\mathfrak{X}}_{n,m}]$ be the constant sheaf on $\widetilde{\mathfrak{X}}_{n,m}$, with shift normalization as a perverse sheaf. Then, from part (i) of Proposition 5.4.1, using the definition of an intersection cohomology complex, we deduce

Corollary 5.4.2. *There is a direct sum decomposition*

$$(\pi_{n,m})_* \mathbb{C}_{\widetilde{\mathfrak{X}}_{n,m}}[\dim \widetilde{\mathfrak{X}}_{n,m}] = \bigoplus_{(\lambda,\mu)\in\Sigma_m\times\Sigma_{n-m}} (L_\lambda \boxtimes L_\mu) \otimes IC(\mathscr{L}_{\lambda,\mu}). \qquad \square$$

Here, $IC(-)$ denotes the intersection cohomology extension of a local system.

One can also translate the statement of the corollary into a $\mathscr{D}$-module language. To this end, write $i_{n,m} : \widetilde{\mathfrak{X}}_{n,m} \hookrightarrow \widetilde{\mathfrak{X}}_n = \widetilde{SL}_n \times \mathbb{P}$ for an obvious closed embedding. For any *integer* $c \in \mathbb{Z}$, one has a $\mathscr{D}_{0,c}$-module $\mathcal{O}(c)_{n,m} := i^*_{n,m}\big(\mathcal{O}_{\widetilde{SL}_n} \boxtimes \mathcal{O}_{\mathbb{P}}(c)\big)[m-n]$, on $\widetilde{\mathfrak{X}}_{n,m}$.

Corollary 5.4.2 yields the following result.

Corollary 5.4.3. *The direct image $(\pi_{n,m})_*\mathcal{O}(c)_{n,m}$ is a semisimple $\mathscr{D}_{0,c}$-module on $\mathfrak{X}_{n,m}$ and one has a direct sum decomposition*

$$(\pi_{n,m})_*\mathcal{O}(c)_{n,m} = \bigoplus_{(\lambda,\mu)\in\Sigma_m\times\Sigma_{n-m}} (L_\lambda \boxtimes L_\mu) \otimes \mathcal{F}_{\lambda,\mu},$$

where $\mathcal{F}_{\lambda,\mu}$ is an irreducible character $\mathscr{D}_{0,c}$-module on $\mathfrak{X}_n$. $\square$

5.5 Cuspidal $\mathscr{D}$-Modules

The goal of this subsection is to describe character $\mathscr{D}_c$-modules which have finite dimensional Hamiltonian reduction. These $\mathscr{D}_c$-modules turn out to be closely related to *cuspidal* character sheaves on SL_n.

In more detail, write $Z(SL_n)$ for the centre of the group SL_n. Thus, $Z(SL_n)$ is a cyclic group, the group of scalar matrices of the form $z \cdot \mathrm{Id}$, where $z \in \mathbb{C}$ is an nth root of unity.

Let $\mathbf{U} \subset SL_n$ be the unipotent cone, and let $\jmath : \mathbf{U}^{\mathrm{reg}} \hookrightarrow \mathbf{U}$ be an open imbedding of the conjugacy class formed by the regular unipotent elements. The fundamental group of $\mathbf{U}^{\mathrm{reg}}$ may be identified canonically with $Z(SL_n)$. For each integer $p = 0, 1, \ldots, n-1$, there is a group homomorphism $Z(SL_n) \to \mathbb{C}^\times$, $z \cdot \mathrm{Id} \mapsto z^p$. Let L_p be the corresponding rank one SL_n-equivariant local system, on $\mathbf{U}^{\mathrm{reg}}$, with monodromy $\theta = \exp(\frac{2\pi\sqrt{-1}p}{n})$.

From now on, we assume that $(p,n) = 1$, i.e. that θ is a primitive nth root of unity. Then, the local system L_p is known to be *clean*; that is, for $\mathscr{D}$-modules on SL_n, one has $\jmath_!\mathsf{L}_p \cong \jmath_{!*}\mathsf{L}_p \cong \jmath_*\mathsf{L}_p$, cf. [L] or [Os]. Given a central element $z \in Z(SL_n)$, we have the conjugacy class $z\mathbf{U}^{\mathrm{reg}} \subset SL_n$, the z-translate of $\mathbf{U}^{\mathrm{reg}}$, and let $z\jmath_!\mathsf{L}_p$ denote the corresponding translated $\mathscr{D}$-module supported on the closure of $z\mathbf{U}^{\mathrm{reg}}$. According to Lusztig [L], $z\jmath_!\mathsf{L}_p$ is a cuspidal character $\mathscr{D}$-module on SL_n.

Furthermore, for any integer $c \in \mathbb{Z}$, we may form $z\mathcal{L}_{p,c} := (z\jmath_!\mathsf{L}_p) \boxtimes \mathcal{O}(c)$, a twisted $\mathscr{D}$-module on $SL_n \times \mathbb{P}$. Thus, $z\mathcal{L}_{p,c}$ is a simple character $\mathscr{D}_c$-module.

Theorem 5.5.1. (i) *Let c be a nonnegative real number and let $\mathcal{F}$ be a nonzero simple character $\mathscr{D}_c$-module. Then, the following properties are equivalent:*

a. *The support of $\mathcal{F}$ is contained in $(Z(SL_n) \cdot \mathbf{U}) \times \mathbb{P}$;*
b. *We have $\mathcal{F} \cong z\mathcal{L}_{p,c}$, for some $z \in Z(SL_n)$ and some integers p, c such that p is prime to n, and $0 < p < n$.*

(ii) *For a simple character $\mathscr{D}_c$-module $\mathcal{F}$, we have*

$$\mathbb{H}(\mathcal{F}) \neq 0 \text{ and } \dim \mathbb{H}(\mathcal{F}) < \infty \Longleftrightarrow (1) \text{ and } (2) \text{ hold and, we have } n|(c-p). \tag{5.5.2}$$

(iii) *Let $\kappa = c/n$, where c and n are mutually prime integers. Then, the functor $\mathbb{H}$ yields a one-to-one correspondence between simple character $\mathscr{D}_c(SL_n \times \mathbb{P})$-modules of the form $z\mathcal{L}_{p,c}$, with $n|(c-p)$, and finite dimensional irreducible $\mathsf{eH}_\kappa^{\mathrm{trig}}(SL_n)\mathsf{e}$-modules, respectively.*

Proof. The implication (2) $\Rightarrow$ (1) of part (i) is clear. We now prove that (1) $\Rightarrow$ (2).

The group GL_n acts on $\mathbf{U} \times V^\circ$ with finitely many orbits, see [GG], Corollary 2.2. The open orbit $\mathbb{O}_0$ is formed by the pairs (X, v) where $X \in \mathbf{U}^{\mathrm{reg}}$, and $v \in V \setminus \mathrm{Im}(X-1)$. The action of GL_n on $\mathbb{O}_0$ is free (and transitive). Let $\mathbb{O}$ be a unique GL_n-orbit open in the support of $\mathcal{F}$. The singular support of $\mathcal{F}$ contains the conormal bundle $T^*_{\mathbb{O}}(SL_n \times V^\circ)$. According to [GG], Theorem 4.3, if $\mathbb{O}$ lies in $\mathbf{U} \times V^\circ$, and $T^*_{\mathbb{O}}(SL_n \times V^\circ)$ lies in $\mathscr{M}_{\mathsf{nil}} := \mathbb{M}_{\mathsf{nil}}(\mathbb{C}^\times) \cap T^*(SL_n \times V^\circ)$, then $\mathbb{O} = \mathbb{O}_0$.

We see that $\mathcal{F}$ must be the minimal extension of an irreducible local system on $\mathbb{O}_0$. Since $\mathbb{O}_0$ is a GL_n-torsor, its fundamental group is $\mathbb{Z}$, and an irreducible local system is one-dimensional with monodromy ϑ; it is GL_n-monodromic with monodromy ϑ. We will denote such local system by L_ϑ. It remains to prove that $\vartheta = \theta$. Note that the boundary of $\mathbb{O}_0$ contains a codimension 1 orbit $\mathbb{O}_1$ formed by the pairs (X, v) such that $X \in \mathbf{U}^{\mathrm{reg}}$, $v \in \mathrm{Im}(X-1) \setminus \mathrm{Im}(X-1)^2$. The monodromy of L_ϑ around $\mathbb{O}_1$ equals ϑ^n. Thus, if $\vartheta^n \neq 1$, the singular support of the minimal extension of L_ϑ necessarily contains $T^*_{\mathbb{O}_1}(SL_n \times V^\circ)$. As we have just seen, the latter is not contained in $\mathbb{M}_{\mathsf{nil}}$; a contradiction. Hence $\vartheta^n = 1$.

Now the monodromy of L_ϑ along lines in V° is also equal to $\vartheta^n = 1$. Hence, L_ϑ is a pullback to $\mathbf{U}^{\mathrm{reg}} \times V^\circ$ of a local system on $\mathbf{U}^{\mathrm{reg}}$ with monodromy ϑ. We conclude that the minimal extension of this latter local system must be a classical character sheaf on SL_n; so ϑ must be a primitive root of unity θ. This completes the proof of part (i) of the theorem.

We prove part (ii). It follows from [CEE], Theorem 9.19, that $\mathbb{H}(\mathcal{L}_{p,c}) \neq 0$ if the integer c is prime to n, and p is the residue of c modulo n. Now the first statement is immediate from Corollary 4.6.6. Let M be a finite dimensional $\mathsf{eH}_\kappa^{\mathrm{trig}}(SL_n)\mathsf{e}$-module and let $\mathcal{F} := {}^\top\mathbb{H}(M)$. Then, $\mathcal{F}$ is a character $\mathscr{D}$-module by Proposition 4.6.2. Thus, $\mathcal{F}$ has finite length. Let $\mathcal{F}_1, \ldots, \mathcal{F}_r$ be the collection of simple subquotients of $\mathcal{F}$, counted with multiplicities. The inclusion in (4.6.5) combined with Corollary 4.6.6 implies that each of the $\mathscr{D}$-modules $\mathcal{F}_i$ satisfies the equivalent conditions (1)–(3) of part (i) of the theorem, and hence has the form $\mathcal{F}_i = z_i \mathcal{L}_{p_i, c_i}$, for some integers c_i prime to n with residues c_i modulo n, and some $z_i \in Z(SL_n)$. Therefore, as we have mentioned earlier, for any $i = 1, \ldots, r$, one has $\mathbb{H}(\mathcal{F}_i) = \mathbb{H}(z_i \mathcal{L}_{p_i, c_i}) \neq 0$.

On the other hand, the functor of Hamiltonian reduction is exact and we know that $\mathbb{H}({}^\top\mathbb{H}(M)) = M$, by Proposition 4.6.2. We deduce that $\mathcal{F}_i = 0$ for all i except one. Thus, $\mathcal{F}$ is a simple character $\mathscr{D}$-module and part (ii) follows. The statements of part (iii) are now clear. □

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A Temperley–Lieb Analogue for the BMW Algebra

G.I. Lehrer and R.B. Zhang

To Toshiaki Shoji on his 60th birthday

Abstract The Temperley–Lieb algebra may be thought of as a quotient of the Hecke algebra of type A, acting on tensor space as the commutant of the usual action of quantum $\mathfrak{sl}_2$ on $(\mathbb{C}(q)^2)^{\otimes n}$. We define and study a quotient of the Birman–Wenzl–Murakami algebra, which plays an analogous role for the three-dimensional representation of quantum $\mathfrak{sl}_2$. In the course of the discussion, we prove some general results about the radical of a cellular algebra, which may be of independent interest.

Keywords Quantum group · Cellular algebra · Tensor · Temperley–Lieb

Mathematics Subject Classifications (2000): 17B37, 20G42

1 Introduction

Let $\mathfrak{g}$ be a finite dimensional simple complex Lie algebra, $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra, and $\mathcal{U}_q = \mathcal{U}_q(\mathfrak{g})$ its Drinfeld–Jimbo quantisation [D], the latter being an algebra over the function field $\mathcal{K} := \mathbb{C}(q^{\frac{1}{2}})$, q an indeterminate. As explained in [LZ, Sect. 6], the finite dimensional $\mathfrak{g}$-modules correspond bijectively to the "type $(1, 1, \ldots, 1)$ modules" of $\mathcal{U}_q$, with corresponding modules having the same character (which is an element of the weight lattice of $\mathfrak{g}$).

Let V be an irreducible finite dimensional $\mathfrak{g}$-module, and V_q its q-analogue (the corresponding $\mathcal{U}_q$-module). It is known that there is an action of the r-string braid group B_r on the tensor space $V_q^{\otimes r}$ which commutes with the action of $\mathcal{U}_q$, and

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in [LZ] a sufficient condition was given in order that B_r span $\mathrm{End}_{\mathcal{U}_q} V_q^{\otimes r}$. This condition, that V be "strongly multiplicity free" (see [LZ, Sect. 3]), was shown to be satisfied when V is any irreducible module for $\mathfrak{sl}_2$.

When $V = V(1)$, the (natural) two-dimensional $\mathfrak{sl}_2$-module, it is known that the two braid generators satisfy a quadratic relation, and together with the quantum analogue of the relation which expresses the vanishing of alternating tensors of rank ≥ 3 these relations give a presentation of the Temperley–Lieb algebra (see Sect. 2 below and [GL03]). In this work, we study the algebra that occurs when we start with the three-dimensional $\mathfrak{sl}_2$-module $V(2)$. It follows from our earlier work that this algebra is a quotient of the Birman–Wenzl–Murakami (BMW) algebra [BW], and we give a presentation for a quotient of the latter, whose semisimple quotient specialises generically to the endomorphism algebra of tensor space. One of the major differences between our case and the classical Temperley–Lieb case is that neither the BMW algebra we start with nor its Brauer specialisation at $q = 1$ is semisimple.

This work makes extensive use of the cellular structure of the BMW algebra and its "classical" specialisation, the Brauer algebra [Br]. We use specialisation arguments to relate the quantum and classical ($q = 1$) situations. Because of this, and also because we have in mind applications of this work to cases where the modules concerned may not be semisimple, we shall work in integral lattices for the modules we encounter, and with integral forms of the endomorphism algebras. Such constructions are closely related to the "Lusztig form" of the irreducible $\mathcal{U}_q$-modules [L2].

In Sect. 5 we prove some general results concerning the radical of a cellular algebra. These characterise it quite explicitly and give a general criterion for an ideal to contain the radical. These results may have some interest, independently of the rest of this work.

2 Dimensions

Let $\mathfrak{g} = \mathfrak{sl}_2$ and write $V(d)$ for the $(d + 1)$-dimensional irreducible representation of $\mathfrak{g}$ on homogeneous polynomials of degree d in two variables, say x and y. The standard generators e, f, h of $\mathfrak{sl}_2$ act as $x\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$, respectively.

2.1 Dimension of the Endomorphism Algebra

We start by giving a (well-known) recursive formula for $\dim_{\mathbb{C}} \mathrm{End}_{\mathfrak{g}} V(d)^{\otimes r} = \dim_{\mathbb{C}(z)} \mathrm{End}_{\mathcal{U}_q} V(d)_q^{\otimes r}$. From the classical Clebsch–Gordan formula, we have for $n \geq m$,

$$V(n) \otimes V(m) \cong V(n+m) \oplus V(n+m-2) \oplus \cdots \oplus V(n-m). \qquad (2.1)$$

Define the coefficients $m_n^r(i)$ by

$$V(n)^{\otimes r} \cong \bigoplus_{i \equiv rn (\bmod 2)} m_n^r(i) V(i), \tag{2.2}$$

and the dimension

$$d(n,r) := \dim \mathrm{End}_{\mathfrak{g}}(V(n)^{\otimes r}). \tag{2.3}$$

Then $d(n,r) = \sum_{i \equiv rn (\bmod 2)} m_n^r(i)^2$. Since all the modules $V(n)$ are self dual, it also follows from Schur's lemma that $(V(n)^{\otimes r}, V(n)^{\otimes r})_{\mathfrak{g}} = (V(n)^{\otimes 2r}, V(0))_{\mathfrak{g}}$, where $(\ , \ )_{\mathfrak{g}}$ denotes multiplicity. Thus,

$$d(n,r) = \sum_{i \equiv rn (\bmod 2)} m_n^r(i)^2 = m_n^{2r}(0).$$

Using (2.1), it is a straightforward combinatorial exercise to prove the following recursive formula. Let x be an indeterminate and write $[n]_x = \frac{x^n - x^{-n}}{x - x^{-1}} \in \mathbb{Z}[x^{\pm 1}]$ for the "x-analogue" of $n \in \mathbb{Z}$. Then define the integers $a_n^r(k)$ by $[n+1]_x^r = \sum_k a_n^r(k) x^k$, where the sum is over k such that $-nr \leq k \leq nr$ and $k \equiv nr$ (mod 2). Finally, define the integers $b_n^r(k)$ $(-nr \leq k \leq nr)$ by downward recursion on k as follows.

Set $b_n^r(nr) = a_n^r(nr) = 1$, and then $\sum_{i \geq k}^{nr} b_n^r(i) = a_n^r(k)$. Equivalently, $b_n^r(k) = a_n^r(k) - a_n^r(k+1)$.

Proposition 2.1. *We have, for all n, r and k,*

$$m_n^r(k) = b_n^r(k).$$

2.2 *The Case $n = 1$*

In this case, one verifies easily that

$$a_1^r(k) = \begin{cases} \binom{r}{\frac{k+r}{2}} & \text{if } k \equiv r \pmod 2 \\ 0 & \text{otherwise} \end{cases}.$$

It is then straightforward to compute that

$$m_1^r(k) = \binom{r}{\frac{r+k}{2}} \frac{2(k+1)}{r+k+2}.$$

In particular,

$$d(1,r) = \dim \mathrm{End}_{\mathfrak{sl}_2} V(1)^{\otimes r} = m_1^{2r}(0) = \frac{1}{r+1}\binom{2r}{r}. \tag{2.4}$$

2.3 The Case $n = 2$

With the above notation, we have

$$a_2^r(2\ell) = \sum_{k \geq \frac{\ell+r}{2}}^{r} \frac{r!}{(2k-(\ell+r))!(\ell+r-k)!(r-k)!}$$

In this case, we have from Proposition 2.1 that $m_2^r(2\ell) = a_2^r(2\ell) - a_2^r(2\ell+2)$. This relation easily yields the following formula for $d(2,r)$.

$$\begin{aligned} d(2,r) &= \dim \mathrm{End}_{\mathfrak{sl}_2} V(2)^{\otimes r} = m_2^{2r}(0) \\ &= \binom{2r}{r} + \sum_{p=0}^{r-1} \binom{2r}{2p}\binom{2p}{p} \frac{3p-2r+1}{p+1}. \end{aligned} \tag{2.5}$$

Thus for $r = 1, 2, 3, 4, 5$, the respective dimensions are $1, 3, 15, 91$ and 603.

3 Some Generators and Relations for $\mathrm{End}_{\mathcal{U}_q}(V(n)_q^{\otimes r})$

In this section, we review the results of [LZ] which pertain to the structure of the endomorphism algebras we wish to study.

3.1 The General Case

Recall that with $\mathfrak{g}$ and $\mathcal{U}_q$ as above, given any $\mathcal{U}_q$-module V_q, there is an operator $\check{R} \in \mathrm{End}_{\mathcal{U}_q}(V_q \otimes V_q)$, known as an "$R$-matrix" (see [LZ, Sect. 6.2]). Denote by R_i the element $\mathrm{id}_{V_q}^{\otimes i-1} \otimes \check{R} \otimes \mathrm{id}_{V_q}^{\otimes r-i-1}$ of $\mathrm{End}_{\mathcal{U}_q} V_q^{\otimes r}$ $(i = 1, \ldots, r-1)$. It is well known that the R_i satisfy the braid relations:

$$\begin{aligned} R_i R_j &= R_j R_i \text{ if } |i-j| \geq 2 \\ R_i R_{i+1} R_i &= R_{i+1} R_i R_{i+1} \text{ for } 1 \leq i \leq r-1. \end{aligned} \tag{3.1}$$

Moreover, if V_q is strongly multiplicity free (for the definition see [LZ, Sect. 7]), it follows from [LZ, Theorem 7.5] that the endomorphisms R_i generate $\mathrm{End}_{\mathcal{U}_q}(V_q^{\otimes r})$. Assume henceforth that V_q is strongly multiplicity free. The following facts may be found in [LZ, Sects. 3 and 7].

First, $V_q = L_{\lambda_0}$, the unique irreducible module for $\mathcal{U}_q$ with highest weight λ_0, and $V_q \otimes V_q$ is multiplicity free as $\mathcal{U}_q$-module. Write

$$V_q \otimes V_q \cong \bigoplus_{\mu \in \mathcal{P}(\lambda_0)} L_\mu,$$

where $\mathcal{P}_{\lambda_0}$ is the relevant set of dominant weights of $\mathfrak{g}$, and L_μ is the irreducible $\mathcal{U}_q$- module with highest weight μ. Let $P(\mu)$ be the projection : $V_q \otimes V_q \longrightarrow L_\mu$. These projections clearly span $\mathrm{End}_{\mathcal{U}_q}(V_q \otimes V_q)$, and we have (see [LZ, (6.10)])

$$\check{R} = \sum_{\mu \in \mathcal{P}_{\lambda_0}} \varepsilon(\mu) q^{\frac{1}{2}(\chi_\mu(C) - 2\chi_{\lambda_0}(C))} P(\mu), \tag{3.2}$$

where $C \in \mathcal{U}(\mathfrak{g})$ is the classical quadratic Casimir element, $\chi_\lambda(C)$ is the scalar through which C acts on the (classical) irreducible $\mathcal{U}(\mathfrak{g})$-module with highest weight λ and $\varepsilon(\mu)$ is the sign occurring in the action of the interchange s on the classical limit $V \otimes V$ of $V_q \otimes V_q$ as $q \to 1$.

3.2 *Relations for the Case* $\mathfrak{g} = \mathfrak{sl}_2$

It was proved in [LZ] that all irreducible modules for $\mathcal{U}_q(\mathfrak{sl}_2)$ are strongly multiplicity free. In this subsection, we make explicit the relations above when $\mathfrak{g} = \mathfrak{sl}_2$ and $V_q = V(n)_q$. These statements are all well known. As above, we think of $V(n)$ as the space $\mathbb{C}[x, y]_n$ of homogeneous polynomials of degree n. This has highest weight n, with $h = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ acting on x^n as highest weight vector. We have seen that

$$V(n) \otimes V(n) = \bigoplus_{\ell=0}^{n} V(2\ell). \tag{3.3}$$

It is easy to compute the highest weight vectors in the summands of (3.3), which leads to

3.1. *The endomorphism* $s : v \otimes w \mapsto w \otimes v$ *of* $V(n) \otimes V(n)$ *acts on* $V(2\ell)$ *as* $(-1)^{n+\ell}$. *That is, in the notation of Sect. 3.1,* $\varepsilon(2\ell) = (-1)^{d+\ell}$.

Next, if $\theta = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ is the Euler form, the Casimir C is given by $C = \theta + \frac{1}{2}\theta^2$. It follows that

3.2. *The Casimir* C *acts on* $V(n)$ *as multiplication by* $\chi_n(C) = \frac{1}{2}n(n+2)$.

Applying the statements in Sect. 3.1 here, we obtain

Proposition 3.3. *Let* V *be the irreducible* $\mathfrak{sl}_2$ *module* $V(n)$, *with* $V(n)_q$ *its quantum analogue. Then* $V(n)_q \otimes V(n)_q = \bigoplus_{\ell=0}^{n} V(2\ell)_q$, *and if* $\check{R}$ *is the* R*-matrix acting on* $V(n)_q \otimes V(n)_q$, *then*

$$\check{R} = \sum_{\ell=0}^{n} (-1)^{n+\ell} q^{\frac{1}{2}(\ell(2\ell+2) - n(n+2))} P(2\ell), \tag{3.4}$$

where $P(2\ell)$ *is the projection to the component* $V(2\ell)_q$.

Now let $E_q(n,r) := \mathrm{End}_{\mathcal{U}_q}(V(n)_q^{\otimes r})$, and let $R_i \in E_q(n,r)$ be the endomorphism defined above ($i = 1, \dots, r-1$). We have seen that the R_i generate $E_q(n,r)$, and that they satisfy the relations (3.1). From the relation (3.4), we deduce that for all i,

$$\prod_{\ell=0}^{n} \left(R_i - (-1)^{n+\ell} q^{\frac{1}{2}(\ell(2\ell+2)-n(n+2))} \right) = 0. \tag{3.5}$$

Writing $T_i := (-1)^n q^{\frac{1}{2}n(n+2)} R_i$, the above relation simplifies to

$$\prod_{\ell=0}^{n} \left(T_i - (-1)^{\ell} q^{\ell(\ell+1)} \right) = 0. \tag{3.6}$$

Now the relations (3.1) and (3.5) do not provide a presentation for $E_q(n,r)$, and it is one of our objectives to determine further relations among the R_i. These will suffice to present $E_q(n,r)$ as an associative algebra generated by the R_i only in some special cases.

3.3 Basic Facts About $\mathcal{U}_q(\mathfrak{sl}_2)$

It will be convenient to establish notation for the discussion below by recalling the following basic facts. The algebra $\mathcal{U}_q := \mathcal{U}_q(\mathfrak{sl}_2)$ has generators e, f and $k^{\pm 1}$, with relations $kek^{-1} = q^2 e$, $kfk^{-1} = q^{-2} f$, and $ef - fe = \frac{k-k^{-1}}{q-q^{-1}}$. The weight lattice P is identified with $\mathbb{Z}$, and for $\lambda, \mu \in P$, $(\lambda, \mu) = \frac{1}{2}\lambda\mu$. In general, if M is a $\mathcal{U}_q$-module with weights $\lambda_1, \dots, \lambda_d$ ($d = \dim M$), then the quantum dimension of (i.e. quantum trace of the identity on) M is $\dim_q M = \sum_{i=1}^{d} q^{-(2\rho,\lambda_i)}$, where 2ρ is the sum of the positive roots, in this case 2. Hence, $\dim_q V(n)_q = q^n + q^{n-2} + \cdots + q^{-n} = [n+1]_q$. The comultiplication is given by $\Delta(e) = e \otimes k + 1 \otimes e$, $\Delta(f) = f \otimes 1 + k^{-1} \otimes f$, $\Delta(k) = k \otimes k$.

3.4 The Case $n = 1$: The Temperley–Lieb Algebra

In this case, the relation (3.5) reads

$$(R_i + q^{\frac{-3}{2}})(R_i - q^{\frac{1}{2}}) = 0.$$

Renormalising by setting $T_i = q^{\frac{1}{2}} R_i$ ($i = 1, 2, \dots, r-1$), we obtain

$$(T_i + q^{-1})(T_i - q) = 0. \tag{3.7}$$

Now it is well known that the associative $\mathbb{C}(q)$-algebra with generators $T_1, \dots, T_{r-1}$ and relations (3.1) and (3.5) is the Hecke algebra $H_r(q)$ of type

A_{r-1} with parameter q. The algebra $H_r(q)$ has a $\mathbb{C}(q)$-basis $\{T_w \mid w \in \mathrm{Sym}_r\}$, and we may therefore speak of the action of $T_w(= q^{\frac{\ell(w)}{2}} R_w)$ (where $\ell(w)$ is the usual length function in Sym_r) on $V_q^{\otimes r}$.

Evidently we have a surjection $\phi : H_r(q) \longrightarrow E_q(1,r)$, and we shall determine $\mathrm{Ker}\,\phi$. The next statement is just the quantum analogue of the fact that there are no non-zero alternating tensors in $V^{\otimes 3}$ if V is two-dimensional.

Lemma 3.4. *Let $V = V(1)_q$ be the two-dimensional irreducible $\mathcal{U}_q(\mathfrak{sl}_2)$-module. In the notation above, let $E(\varepsilon) = \sum_{w \in \mathrm{Sym}_3} (-q)^{-\ell(w)} T_w \in H_3(q)$. Then $E(\varepsilon)$ acts as zero on $V^{\otimes 3}$.*

Proof. Let $v_1 \in V$ have weight 1, and take $v_2 = f v_1$ as the complementary basis element, which has weight -1. We have $V \otimes V \cong L_0 \oplus L_1$, where L_0 is the trivial $\mathcal{U}_q$ module, and L_1 is the irreducible $\mathcal{U}_q$-module of dimension 3. By computing the action of $\Delta(e)$ and $\Delta(f)$, one sees easily that L_0 is spanned by $qv_1 \otimes v_2 - v_2 \otimes v_1$, and L_1 has basis $v_1 \otimes v_1$, $v_2 \otimes v_2$ and $v_2 \otimes v_1 + q^{-1} v_1 \otimes v_2$.

Now $T = q^{\frac{1}{2}} \check{R}$ acts on L_0 as $-q^{-1}$ and on L_1 as q. If $P(0)$, $P(1)$ are the projections of $V \otimes V$ onto L_0, L_1, respectively, this implies that $P(0) = -\frac{1}{q+q^{-1}}(T - q)$, and $P(1) = \frac{1}{q+q^{-1}}(T + q^{-1})$. As above, write $P_i(j)$ for the projection of $V^{\otimes r}$ obtained by applying $P(j)$ to the $(i, i+1)$ factors of $V^{\otimes r}$ $(i = 1, \ldots, r-1;\ \ j = 0, 1)$. Then $P_i(1) = -\frac{1}{q+q^{-1}}(T_i + q^{-1})$, and so on.

Next, observe that since $(T_i + q^{-1})E(\varepsilon) = E(\varepsilon)(T_i + q^{-1}) = 0$ for $i = 1, 2$, we have $P_i(1)E(\varepsilon) = E(\varepsilon)P_i(1) = 0$ for $i = 1, 2$. Since $P_i(0) + P_i(1) = \mathrm{id}_{V^{\otimes 3}}$, it follows that

$$\begin{aligned} E(\varepsilon)V^{\otimes 3} &\subseteq P_1(0)V^{\otimes 3} \cap P_2(0)V^{\otimes 3} \\ &= L_0 \otimes V \cap V \otimes L_0. \end{aligned}$$

Now $L_0 \otimes V$ and $V \otimes L_0$ are two irreducible submodules of $V^{\otimes 3}$. Hence, they either coincide or have zero intersection. But $L_0 \otimes V$ has basis $\{(qv_1 \otimes v_2 - v_2 \otimes v_1) \otimes v_i \mid i = 1, 2\}$. Hence, $qv_1 \otimes v_1 \otimes v_2 - v_1 \otimes v_2 \otimes v_1$ is in $V \otimes L_0$, but not in $L_0 \otimes V$.

It follows that $E(\varepsilon)V^{\otimes 3} = 0$. □

This enables us to prove

Theorem 3.5. *Let V be the two-dimensional irreducible representation of $\mathcal{U}_q(\mathfrak{sl}_2)$. For each integer $r \geq 2$, let $E_q(1,r) = \mathrm{End}_{\mathcal{U}_q(\mathfrak{sl}_2)}(V^{\otimes r})$. Then $E_q(1,r)$ is isomorphic to the Temperley–Lieb algebra $TL_r(q)$ (cf.* [GL04, p. 144]).

Proof. We have seen above that $E_q(1,r)$ is generated as $\mathcal{K}$-algebra by the endomorphisms $T_1, \ldots, T_{r-1}$. By (3.7), they generate a quotient of the Hecke algebra $H_r(q)$ and by Lemma 3.4, that quotient is actually a quotient of $H_r(q)/I$, where I is the ideal generated by the element $E(\varepsilon)$ defined above. But up to the automorphism $T_i \mapsto T_i' = -T_i + q - q^{-1}$, this is precisely the idempotent E_1 of [GL04, (4.9)]. It follows (see, e.g., [GL04, (4.17)]) that the quotient $H_r(q)/I$ is isomorphic to $TL_r(q)$.

But this latter algebra is well known (cf. [GL96] or [GL03]) to have dimension $\frac{1}{r+1}\binom{2r}{r}$, which by (2.4) above is the dimension of $E_q(1,r)$. The theorem follows. □

It follows that in this case, the endomorphism algebra has a well understood cellular structure (see [GL96]).

4 The Case $n = 2$: Action of the BMW Algebra

In this section, we take V to be $V_q(2)$, the three-dimensional irreducible module for $\mathcal{U}_q(\mathfrak{sl}_2)$. In accord with the notation of the last section, we write $E_q(2,r) := \mathrm{End}_{\mathcal{U}_q(\mathfrak{sl}_2)}(V_q(2)^{\otimes r})$.

4.1 The Setup and Some Relations

In this situation, $V \otimes V \cong V_0 \oplus V_1 \oplus V_2$, where V_0 is the trivial module, and V_1, V_2 are, respectively, the three- and five-dimensional irreducible modules. As above, we therefore have operators R_i, $P_i(j)$ $(i = 1,\dots,r-1;\ j = 0,1,2)$ on $V^{\otimes r}$, where $P_i(j)$ is the projection $V \otimes V \to V_j$, applied to the $(i, i+1)$ factors of $V^{\otimes r}$, appropriately tensored with the identity endomorphism of V.

The R_i here satisfy the braid relations (as they always do), and the cubic relation

$$(R_i - q^{-4})(R_i + q^{-2})(R_i - q^2) = 0. \tag{4.1}$$

Now if L is any strongly multiplicity free $\mathcal{U}_q$-module such that the trivial module L_0 is a summand of $L \otimes L$, and $f \in \mathrm{End}_{\mathcal{U}_q}(L \otimes L)$, then writing $P_i(0)$ for the projection $L \otimes L \to L_0$, applied to the $(i, i+1)$ components of $L^{\otimes r}$, we have

$$P_i(0) f_{i\pm1} P_i(0) = \frac{1}{(\dim_q(L))^2} \tau_{q,L\otimes L}(f) P_i(0), \tag{4.2}$$

where $\tau_{q,M}$ denotes the quantum trace of an endomorphism of the $\mathcal{U}_q$-module M, and f_i is f applied to the $(i, i+1)$ components on $L^{\otimes r}$.

To apply (4.2) to the case when $f = \check{R}$, we shall use

$$\begin{aligned}\tau_{q,V(n)_q^{\otimes 2}}(\check{R}) &= q^{\frac{1}{2}n(n+2)}[n+1]_q \\ &= q^{\frac{1}{2}n(n+2)} \frac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}}.\end{aligned} \tag{4.3}$$

This may be proved in several different ways, including the use of the explicit expression given in Proposition 3.3 for $\check{R}$.

Applying (4.2) to the cases $f = \check{R}$ and $f = P(0)$ in turn, we obtain for our case $L = V_q(2)$,

$$P_i(0)R_{i\pm1}P_i(0) = q^4[3]_q{}^{-1}P_i(0) \text{ for } i = 1, \dots, r, \tag{4.4}$$

and

$$P_i(0)P_{i\pm1}(j)P_i(0) = [2j+1]_q[3]_q^{-2}P_i(0) \text{ for } i = 1, \dots, r \text{ and } j = 0, 1, 2. \tag{4.5}$$

In (4.4) and (4.5), the applicable range of values for i is understood to be such that $P_{k(i)}(j)$ and $R_{k(i)}$ make sense for the relevant functions $k(i)$ of i.

Since $\check{R}$ acts on V_0, V_1 and V_2, respectively, as $q^{-4}, -q^{-2}$ and q^2, we also have

$$P_i(0) = \frac{q^8(R_i + q^{-2})(R_i - q^2)}{(1+q^2)(1-q^6)}. \tag{4.6}$$

4.2 The BMW Algebra

We recall some basic facts concerning the BMW algebra, suitably adapted to our context. Let $\mathcal{K} = \mathbb{C}(q^{\frac{1}{2}})$ as above, and let $\mathcal{A}$ be the ring $\mathbb{C}[y^{\pm1}, z]$, where y, z are indeterminates.

The BMW algebra $BMW_r(y,z)$ over $\mathcal{A}$ is the associative $\mathcal{A}$-algebra with generators $g_1^{\pm1}, \dots, g_{r-1}^{\pm1}$ and $e_1, \dots, e_{r-1}$, subject to the following relations:

The braid relations for the g_i:

$$\begin{aligned} g_i g_j &= g_j g_i \text{ if } |i-j| \geq 2 \\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} \text{ for } 1 \leq i \leq r-1; \end{aligned} \tag{4.7}$$

The Kauffman skein relations:

$$g_i - g_i{}^{-1} = z(1 - e_i) \text{ for all } i; \tag{4.8}$$

The de-looping relations:

$$\begin{aligned} g_i e_i &= e_i g_i = y e_i; \\ e_i g_{i-1}^{\pm1} e_i &= y^{\mp1} e_i; \\ e_i g_{i+1}^{\pm1} e_i &= y^{\mp1} e_i. \end{aligned} \tag{4.9}$$

The next four relations are easy consequences of the previous three.

$$e_i e_{i\pm1} e_i = e_i; \tag{4.10}$$

$$(g_i - y)(g_i^2 - z g_i - 1) = 0; \tag{4.11}$$

$$z e_i^2 = (z + y^{-1} - y)e_i, \tag{4.12}$$

$$-yze_i = g_i^2 - z g_i - 1. \tag{4.13}$$

It is easy to show that $BMW_r(y,z)$ may be defined using the relations (4.7), (4.9), (4.11) and (4.13) instead of (4.7), (4.8) and (4.9), i.e. that (4.8) is a consequence of (4.11) and (4.13).

We shall require a particular specialisation of $BMW_r(y,z)$ to a subring $\mathcal{A}_q$ of $\mathcal{K}$, which is defined as follows. Let $\mathcal{S}$ be the multiplicative subset of $\mathbb{C}[q,q^{-1}]$ generated by $[2]_q$, $[3]_q$ and $[3]_q-1$. Let $\mathcal{A}_q := \mathbb{C}[q,q^{-1}]_{\mathcal{S}} := \mathbb{C}[q,q^{-1},[2]_q{}^{-1},[3]_q{}^{-1},(q^2+q^{-2})^{-1}]$ be the localisation of $\mathbb{C}[q,q^{-1}]$ at $\mathcal{S}$.

Now, let $\psi:\mathbb{C}[y^{\pm 1},z]\longrightarrow\mathcal{A}_q$ be the homomorphism defined by $y\mapsto q^{-4}$, $z\mapsto q^2-q^{-2}$. Then ψ makes $\mathcal{A}_q$ into an $\mathcal{A}$-module, and the specialisation $BMW_r(q):=\mathcal{A}_q\otimes_{\mathcal{A}}BMW_r(y,z)$ is the $\mathcal{A}_q$-algebra with generators which we denote, by abuse of notation, $g_i^{\pm 1}, e_i$ $(i=1,\dots,r-1)$ and relations (4.14) below, with the relations (4.15) being consequences of (4.14).

$$
\begin{aligned}
g_ig_j &= g_jg_i \text{ if } |i-j|\geq 2\\
g_ig_{i+1}g_i &= g_{i+1}g_ig_{i+1} \text{ for } 1\leq i\leq r-1\\
g_i-g_i{}^{-1} &= (q^2-q^{-2})(1-e_i) \text{ for all } i\\
g_ie_i &= e_ig_i = q^{-4}e_i\\
e_ig_{i-1}^{\pm 1}e_i &= q^{\pm 4}e_i\\
e_ig_{i+1}^{\pm 1}e_i &= q^{\pm 4}e_i.
\end{aligned}
\tag{4.14}
$$

$$
\begin{aligned}
e_ie_{i\pm 1}e_i &= e_i\\
(g_i-q^2)(g_i+q^{-2}) &= -q^{-4}(q^2-q^{-2})e_i\\
(g_i-q^{-4})(g_i-q^2)(g_i+q^{-2}) &= 0\\
e_i^2 &= (q^2+1+q^{-2})e_i.
\end{aligned}
\tag{4.15}
$$

We shall be concerned with the following two specialisations of $BMW_r(q)$.

Definition 4.1. Let $\phi_q:\mathcal{A}_q\longrightarrow\mathcal{K}=\mathbb{C}(q^{\frac{1}{2}})$ be the inclusion map, and let $\phi_1: BMW_r(q)\longrightarrow\mathbb{C}$ be the $\mathbb{C}$-algebra homomorphism defined by $q\mapsto 1$. Define the specialisations $BMW_r(\mathcal{K}):=\mathcal{K}\otimes_{\phi_q}BMW_r(q)$, and $BMW_r(1):=\mathbb{C}\otimes_{\phi_1}BMW_r(q)$.

The next statement is straightforward.

Lemma 4.2. (1) *The algebra $BMW_r(q)$ may be regarded as an $\mathcal{A}_q$-lattice in the $\mathcal{K}$-algebra $BMW_r(\mathcal{K})$.*

(2) *The specialisation $BMW_r(1)$ is isomorphic to the Brauer algebra $B_r(3)$ over $\mathbb{C}$.*

(3) *Let $\mathcal{I}$ be the two-sided ideal of $BMW_r(q)$ generated by $e_1,\dots,e_{r-1}$. There is a surjection $BMW_r(q)\to H_r(q^2)$ of $\mathcal{A}_q$-algebras with kernel $\mathcal{I}$, where $H_r(q^2)$ is the Hecke algebra discussed above (Sect. 3.4).*

Proof. Note for the first two statements that by [X, Theorem 3.11 and its proof], $BMW_r(y,z)$ has a basis of "r-tangles" $\{T_d\}$, where d runs over the Brauer

r-diagrams, which form a basis of $B_r(\delta)$ over any ring. The same thing applies to $BMW_r(y,z)$; thus, $BMW_r(q)$ may be thought of as the subring of $BMW_r(\mathcal{K})$ consisting of the $\mathcal{A}_q$-linear combinations of the T_d, while $B_r(3)$ is realised as the set of $\mathbb{C}$-linear combinations of the diagrams d. □

Note that in view of the third relation in (4.15), the element $\frac{e_i}{[3]_q}$ of $BMW_r(q)$ is an idempotent. Moreover, it follows from (4.13) or (4.15) that

$$\frac{e_i}{[3]_q} = \frac{(g_i - q^2)(g_i + q^{-2})}{(q^{-4} - q^2)(q^{-4} + q^{-2})}. \tag{4.16}$$

Taking into account the cubic relation (4.11), or its specialisation in (4.15), we also have the idempotents d_i and c_i in $BMW_r(\mathcal{K})$, where

$$\begin{aligned} d_i &= \frac{(g_i - q^2)(g_i - q^{-4})}{(q^{-2} + q^2)(q^{-2} + q^{-4})} \\ c_i &= \frac{(g_i + q^{-2})(g_i - q^{-4})}{(q^2 + q^{-2})(q^2 - q^{-4})}. \end{aligned} \tag{4.17}$$

Lemma 4.3. *If $BMW_r(q)$ is thought of as an $\mathcal{A}_q$-submodule of $BMW_r(\mathcal{K})$ as in Lemma 4.2 (1), then the idempotents $e_i[3]_q{}^{-1}$, d_i and c_i all lie in $BMW_r(q)$.*

Proof. It is evident that $e_i[3]_q{}^{-1} \in BMW_r(q)$. Since $(q^{-2}+q^2)(q^{-2}+q^{-4})$ is invertible in $\mathcal{A}_q$, clearly $d_i \in BMW_r(q)$. But it is easily verified that $e_i[3]_q{}^{-1} + d_i + c_i = 1$, whence the result. □

The relevance of the above for the study of endomorphisms is evident from the next result.

Theorem 4.4. *With the above notation, there is a surjection η_q from the algebra $BMW_r(\mathcal{K}) \to E_q(2,r)$ which takes e_i to $[3]_q P_i(0)$ and g_i to R_i.*

Proof. In view of the above discussion, it remains only to show that the endomorphisms $\eta_q(g_i) = R_i$ and $\eta_q(e_i) = [3]_q P_i(0)$ satisfy the relations (4.7), (4.9), (4.11) and (4.13) for the appropriate y and z. Now the braid relations (4.7) are always satisfied by the R_i; further, (4.11) with the R_i replacing the g_i is just (4.1). Now a simple calculation shows that in our specialisation, $z^{-1}(z + y^{-1} - y) = [3]_q := \delta$. It follows that (4.13) may be written

$$\begin{aligned} \delta^{-1} e_i &= \frac{1}{\delta y z}(g_i - q^2)(g_i + q^{-2}) \\ &= \frac{-1}{[3]_q q^{-4}(q^2 - q^{-2})}(g_i - q^2)(g_i + q^{-2}) \\ &= \frac{-(q - q^{-1})q^4}{(q^3 - q^{-3})(q^2 - q^{-2})}(g_i - q^2)(g_i + q^{-2}) \\ &= \frac{q^8}{(1 + q^2)(1 - q^6)}(g_i - q^2)(g_i + q^{-2}). \end{aligned}$$

Thus, (4.13) with $\delta P_i(0)$ replacing e_i is just (4.6). Finally, the first delooping relation follows immediately from (4.11) and (4.13), while the other two follow from (4.4). □

We wish to illuminate which relations are necessary in addition to those which define $BMW_r(\mathcal{K})$, to obtain $E_q(2,r)$, i.e. we wish to study $\mathrm{Ker}(\eta_q)$. Note that the specialisation of Lemma 4.2 (2) is the classical limit as $q \to 1$ of $BMW_r(q)$, and that this is just the Brauer algebra with parameter $\delta_{q\to 1} = 3$ in accord with [LZ, Sect. 3]. We shall study $\mathrm{Ker}(\eta_q)$ by first examining the classical case, and then use specialisation arguments. The cellular structure of the algebras involved will play an important role in what follows.

4.3 Tensor Notation and Quantum Action

In this subsection, we establish notation for basis elements of tensor powers, which is convenient for computation of the actions we consider. Since the $\mathfrak{sl}_2$-module $V(2)$ is the classical limit at $q \to 1$ of $V_q(2)$, we do this for the quantum case, and later obtain the classical one by putting $q = 1$.

Maintaining the notation of Sect. 3.3 and proceeding as in the proof of Lemma 3.4, let $v_{-1} \in V_q(2)$ be a basis element of the -2 weight space, and let e, f, k be the generators of $\mathcal{U}_q$ referred to in Sect. 3.3. Then $v_0 := ev_{-1}$ and $v_1 := ev_0$ have weights $0, 2$, respectively, and $\{v_0, v_{\pm 1}\}$ is a basis of $V_q(2)$. Moreover, it is easily verified that $fv_1 = (q+q^{-1})v_0$ and $fv_0 = (q+q^{-1})v_{-1}$. The tensor power $V_q(2)^{\otimes r}$ has a basis consisting of elements $v_{i_1,i_2,\dots,i_r} := v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}$. Note that $v_{i_1,i_2,\dots,i_r}$ is a weight element of weight $2(i_1 + \cdots + i_r)$ for the action of $\mathcal{U}_q(\mathfrak{sl}_2)$.

Now $V_q(2)^{\otimes 2}$ has a canonical decomposition $V_q(2)^{\otimes 2} = L(0)_q \oplus L(2)_q \oplus L(4)_q$, where $L(i)_q$ is isomorphic to $V(i)_q$ for all i. We shall give bases of the three components, which consist of weight vectors.

Lemma 4.5. *The three components of $V_q(2)^{\otimes 2}$ have bases as follows.*

(1) $L(0)_q$: $v_{-1,1} - q^2 v_{0,0} + q^2 v_{1,-1}$.
(2) $L(2)_q$: $v_{0,1} - q^2 v_{1,0}$; $v_{-1,1} - v_{1,-1} + (1-q^2)v_{0,0}$; $v_{-1,0} - q^2 v_{0,-1}$.
(3) $L(4)_q$: $v_{1,1}$; $v_{0,1} + q^{-2}v_{1,0}$; $v_{-1,1} + (1+q^{-2})v_{0,0} + q^{-4}v_{1,-1}$; $v_{-1,0} + q^{-2}v_{0,-1}$; $v_{-1,-1}$.

The corresponding statement for the classical case is obtained by putting $q = 1$ above.

The R-matrix $\check{R} = R_1$ acts on the three components above via the scalars $q^{-4}, -q^{-2}$ and q^2, respectively.

The proof is a routine calculation, which makes use of the fact that several of the basis elements above are characterised by the fact that they are annihilated by e and/or f.

It is useful to record the action of the endomorphism e_1 of $V_q(2)^{\otimes 2}$ (see Theorem 4.4) on the basis elements $v_{i,j}$.

Lemma 4.6. *The endomorphism e_1 of $V_q(2)^{\otimes 2}$ acts as follows. Let $w_0 = -q^2 v_{0,0} + q^2 v_{1,-1} + v_{-1,1} \in L(0)_q$. Then*

$$e_1 v_{i,j} = \begin{cases} w_0, & \text{if } (i,j) = (1,-1), \\ q^{-2} w_0, & \text{if } (i,j) = (-1,1), \\ -q^{-2} w_0, & \text{if } (i,j) = (0,0), \\ 0, & \text{if } i + j \neq 0. \end{cases}$$

Proof. Since e_1 acts as 0 on $L(2)_q$ and $L(4)_q$, and as $[3]_q$ on $L(0)_q$, this follows from an easy computation with the bases in Lemma 4.5. □

The next result will be used in the next section.

Lemma 4.7. *The $\mathcal{U}_q(\mathfrak{sl}_2)$-homomorphism $e_2 : L(2)_q \otimes L(2)_q \longrightarrow V_q(2) \otimes L(0)_q \otimes V_q(2)$ which is obtained by restricting e_2 to $L(2)_q \otimes L(2)_q \subset V_q(2)^{\otimes 4}$, is an isomorphism.*

Proof. Let u_i be the weight vector of weight $2i$ of $L(2)_q$ which is given in Lemma 4.5 ($i = 0, \pm 1$). Then $L(2)_q \otimes L(2)_q$ has basis $\{u_{i,j} := u_i \otimes u_j \mid i,j = 0, \pm 1\}$. Similarly, $V_q(2) \otimes L(0)_q \otimes V_q(2)$ has basis $\{x_{i,j} := v_i \otimes w_0 \otimes v_j \mid i,j = 0, \pm 1\}$.

Now from Lemma 4.6, $e_2 v_{i,j,a,b} = 0$ unless $j + a = 0$. This fact may be used to easily compute $e_2 u_{i,j}$ in terms of the $x_{a,b}$. The resulting 9×9 matrix of the linear map e_2 is then readily seen to have determinant $\pm q^{-4}(q^2 + q^{-2} - 1)(q^4 + 1 - q^{-2} + q^{-4})$, which is non-zero, whence the result. □

Corollary 4.8. *The statement in Lemma 4.7 remains true in the classical case ($q = 1$).*

This is clear since the determinant in the proof of Lemma 4.7 does not vanish at $q = 1$.

5 The Radical of a Cellular Algebra

In the next section, we shall discuss some cellular algebras which are not semisimple. This section is devoted to proving some general results about such algebras, which we use below. In this section only, we take $B = B(\Lambda, M, C, {}^*)$ to be any cellular algebra over a field $\mathbb{F}$, and prove some general results concerning its radical $\mathcal{R}$. These results may be of some interest independently of the rest of this work. We assume that the reader has some acquaintance with the general theory of cellular algebras (see [GL96, Sects. 1–3]); notation will be as is standard in cellular theory. In particular, for any element $\lambda \in \Lambda$, the corresponding cell module will be denoted by $W(\lambda)$ and its radical with respect to the canonical invariant bilinear form ϕ_λ by $R(\lambda)$. Since there is no essential loss of generality, we shall assume for ease of exposition, that B is quasi-hereditary.

Let $\lambda \in \Lambda$, and write $W(\lambda)^*$ for the dual of $W(\lambda)$; this is naturally a right B-module, and we have a vector space monomorphism (cf. [GL96, (2.2)(i)]) $C^\lambda : W(\lambda) \otimes_{\mathbb{F}} W(\lambda)^* \longrightarrow B$ defined by

$$C^\lambda(C_S \otimes C_T) = C^\lambda_{S,T} \text{ for } S, T \in M(\lambda). \tag{5.1}$$

Denote the image of C^λ by $B(\{\lambda\})$. This is a subspace of B, isomorphic as (B, B)-bimodule to $B(\leq \lambda)/B(< \lambda)$ (see [GL96, *loc. cit.*]), and we have a vector space isomorphism

$$B \xrightarrow{\sim} \bigoplus_{\lambda \in \Lambda} B(\{\lambda\}). \tag{5.2}$$

Note that $W(\lambda)$ and $W(\lambda)^*$ are equal as sets. We shall therefore differentiate between them only when actions are relevant.

Now let $\pi : B \longrightarrow \overline{B} := B/\mathcal{R}$ be the natural map from B to its largest semisimple quotient. Then $\overline{B} \cong \bigoplus_{\lambda \in \Lambda} \overline{B}(\lambda)$, where $\overline{B}(\lambda) \cong M_{l_\lambda}(\mathbb{F}) \cong \mathrm{End}_{\mathbb{F}}(L(\lambda))$. Thus π may be written $\pi = \oplus \pi_\lambda$, where $\pi_\lambda : B \longrightarrow \mathrm{End}_{\mathbb{F}}(L(\lambda))$ is the representation of B on $L(\lambda)$. We collect some elementary observations in the next lemma.

Lemma 5.1. (1) *The restriction* $\pi_\lambda : B(\{\lambda\}) \longrightarrow \mathrm{End}_{\mathbb{F}}(L(\lambda))$ *is a surjective linear map for each* $\lambda \in \Lambda$.

(2) *Let A be any semisimple $\mathbb{F}$-algebra and let $\sigma : B \longrightarrow A$ be a surjective homomorphism. Then σ factors through π as shown.*

$$\begin{tikzcd} B \arrow[rr, "\pi"] \arrow[dr, "\sigma"'] & & \overline{B} \cong \bigoplus_\lambda \overline{B}(\lambda) \arrow[dl, "\overline{\sigma}"] \\ & A & \end{tikzcd} \tag{5.3}$$

(3) *The restriction $\overline{\sigma}_\lambda$ of $\overline{\sigma}$ to $\overline{B}(\lambda)$ is either zero or an isomorphism.*

(4) *Denote by Λ^0 the set $\{\lambda \in \Lambda \mid \overline{\sigma}_\lambda$ is non-zero$\}$. Let $J = \mathrm{Ker}(\sigma)$. Then $\pi_\lambda(J) = 0$ if and only if $\lambda \in \Lambda^0$.*

(5) *The radical $\mathcal{R}$ is the set of elements of B which act as zero on each irreducible module $L(\lambda)$.*

Proof. The statement (1) follows from the cyclic nature of the cell modules ([GL96, (2.6)(i)]). The only other statement deserving of comment is (4), which follows immediately from the observation that $A \cong B/J \cong \overline{B}/\overline{J}$, where $\overline{J} = J/\mathcal{R}$ is a two-sided ideal of the semisimple algebra $\overline{B}$. The ideal $\overline{J}$ therefore acts trivially in precisely those irreducible representations of B which "survive" in the quotient, and non-trivially in the others. Note that (5) follows immediately from (4). □

Corollary 5.2. *Let $\eta : B \longrightarrow \mathrm{End}_{\mathbb{F}}(W)$ be a representation of B in the semisimple B-module W, and write $E = \mathrm{Im}(\eta)$. Let $N = \mathrm{Ker}(\eta : B \longrightarrow E)$. Then $E \cong$*

$B/N \cong \bigoplus_{\lambda\in\Lambda^0} \overline{B}(\lambda)$, where Λ^0 is the set of $\lambda \in \Lambda$ such that $L(\lambda)$ is a direct summand of W, regarded as an E-module. Moreover, Λ^0 is characterised as the set of $\lambda \in \Lambda$ such that N acts trivially on $L(\lambda)$.

To study the action of ideals on the $L(\lambda)$, we shall require the following results.

Lemma 5.3. *Assume that B is quasi-hereditary; i.e. that for all $\lambda \in \Lambda$, $\phi_\lambda \neq 0$.*

(1) *In the notation of (5.1), if x, y and z are elements of $W(\lambda)$, then $C^\lambda(x \otimes y)z = \phi_\lambda(y,z)x$.*
(2) *If x or y is in $R(\lambda)$, then $\pi_\lambda(C^\lambda(x \otimes y)) = 0$.*
(3) *The radical $\mathcal{R}$ of B has a filtration ($\mathcal{R}(\lambda) = \mathcal{R} \cap B(\leq \lambda)$) by two-sided ideals such that there is an isomorphism of (B, B)-bimodules*

$$\mathcal{R}(\lambda)/\mathcal{R}(<\lambda) \xrightarrow{\sim} W(\lambda) \otimes R(\lambda)^* + R(\lambda) \otimes W(\lambda)^* \subset W(\lambda) \otimes W(\lambda)^*.$$

Proof. (1) is just [GL96, (2.4)(iii)]. To see (2), note that if $y \in R(\lambda)$, then from (1), $C^\lambda(x \otimes y)z = \phi_\lambda(y,z)x = 0$ for all $z \in W(\lambda)$. If $x \in R(\lambda)$, then again by (1), $C^\lambda(x \otimes y)z \in R(\lambda)$, whence $C^\lambda(x \otimes y)$ acts as 0 on $L(\lambda) = W(\lambda)/R(\lambda)$.

Now suppose $b \in C^\lambda(W(\lambda) \otimes R(\lambda) + R(\lambda) \otimes W(\lambda))$. Then by (2), $\pi_\lambda(b) = 0$. We shall prove

$$\exists \text{ elements } b_{\lambda'} \in B(\{\lambda'\})\ (\lambda' < \lambda) \text{ such that } b + \sum_{\lambda'<\lambda} b_{\lambda'} \in \mathcal{R}. \tag{5.4}$$

We do this recursively as follows. Suppose we have a subset $\Gamma \subseteq \Lambda$ and an element $\sum_{\gamma\in\Gamma} b_\gamma \in \sum_{\gamma\in\Gamma} B(\{\gamma\})$ such that for any $\beta \in \Lambda$ which satisfies $\beta \geq \gamma$ for some $\gamma \in \Gamma$, we have $\pi_\beta(\sum_{\gamma\in\Gamma} b_\gamma) = 0$. We show that if there is $\mu \in \Lambda$ such that $\pi_\mu(\sum_{\gamma\in\Gamma} b_\gamma) \neq 0$, then we may increase Γ to obtain another set with the same properties. For this, take $\mu \in \Lambda$ such that $\pi_\mu(\sum_{\gamma\in\Gamma} b_\gamma) \neq 0$, and maximal with respect to this property. Note that since $\pi_\beta(b_\gamma) \neq 0$ implies that $\gamma \geq \beta$, we have $\mu \leq \gamma$ for some element $\gamma \in \Gamma$. By Lemma 5.1 (1), there is an element $b_\mu \in B(\{\mu\})$ such that $\pi_\mu(\sum_{\gamma\in\Gamma} b_\gamma) = \pi_\mu(-b_\mu)$. Let $\Gamma' = \Gamma \cup \{\mu\}$. If $\beta \geq \gamma'$ for some $\gamma' \in \Gamma$, we show that $\pi_\beta(\sum_{\gamma'\in\Gamma'} b_{\gamma'}) = 0$.

There are two cases. If $\gamma' \in \Gamma$, then $\pi_\beta(\sum_{\gamma\in\Gamma} b_\gamma) = 0$. If $\pi_\beta(b_\mu) \neq 0$, then $\beta \leq \mu$ and so $\gamma' \leq \beta \leq \mu$, whence by hypothesis $\pi_\mu(\sum_{\gamma\in\Gamma} b_\gamma) = 0$, a contradiction. Hence $\pi_\beta(b_\mu) = 0$, which proves the assertion in this case.

The remaining possibility is that $\gamma' = \mu$. In this case, since $\pi_\mu(\sum_{\gamma'\in\Gamma'} b_{\gamma'}) = 0$ by construction, we may suppose $\beta > \mu$. But then by the maximal nature of μ, $\pi_\beta(\sum_{\gamma\in\Gamma} b_\gamma) = 0$. Moreover, since $\beta > \mu$, $\pi_\beta(b_\mu) = 0$. Hence, Γ' and $\sum_{\gamma'\in\Gamma'} b_{\gamma'}$ have the same property as Γ and $\sum_{\gamma\in\Gamma} b_\gamma$. Note that Γ' is obtained from Γ by adding an element μ such that $\mu \leq \gamma$ for some $\gamma \in \Gamma$.

Now to prove the assertion (5.4), start with $\Gamma = \{\lambda\}$ and $b_\lambda = b$. The argument above shows that we may repeatedly add elements $\mu < \lambda$ to Γ, with corresponding $b_\mu \in B(\{\mu\})$, eventually coming to a set $\Gamma_{\max}$ such that $\sum_{\mu\in\Gamma_{\max}} b_\mu$ acts trivially on each $L(\beta)$ ($\beta \in \Lambda$).

This completes the proof of (5.4), and hence of (3). $\square$

The arguments used in the proof of the above lemma may be applied to yield the following result, in which we use the standard notation of [GL96] for cellular theory.

Theorem 5.4. *Let $B = (\Lambda, M, C, {}^*)$ be a cellular algebra over a field $\mathbb{F}$, and assume that B is quasi-hereditary, i.e. that the invariant form ϕ_λ on each cell module is non-zero. For $\lambda \in \Lambda$, denote by $W(\lambda)$ and $R(\lambda)$, respectively, the corresponding cell module and its radical.*

(1) *Let $\lambda \in \Lambda$ and take any elements $x \in W(\lambda)$, $z \in R(\lambda)$. Then there exist elements $r(x,z) \in C^\lambda(x \otimes z) + B(< \lambda)$ and $r(z,x) \in C^\lambda(z \otimes x) + B(< \lambda)$, both in $\mathcal{R}$, the radical of B.*
(2) *Let X be a subset of B such that for all $\lambda \in \Lambda$, $x \in W(\lambda)$ and $z \in R(\lambda)$, X contains elements $r(x,z)$ and $r(z,x)$ as in* (1). *Then the linear subspace of B spanned by X contains $\mathcal{R}$.*
(3) *Suppose J is a two-sided ideal of B such that $J^* = J$. Let $\Lambda^0 := \{\lambda \in \Lambda \mid JL(\lambda) = 0\}$. Then $J \supseteq \mathcal{R}$ if and only if, for all $\lambda \in \Lambda^0$, $R(\lambda) \subseteq JW(\lambda)$.*

Proof. The argument given in the proof of Lemma 5.3 (3) proves the statement (1).

For each $\lambda \in \Lambda$, let $w_\lambda = \dim W(\lambda)$, $r_\lambda = \dim R(\lambda)$ and $l_\lambda = \dim L(\lambda) = \dim(W(\lambda)/R(\lambda)) = w_\lambda - r_\lambda$. Then

$$\begin{aligned}\dim_{\mathbb{F}}(\mathcal{R}) &= \dim_{\mathbb{F}}(B) - \sum_{\lambda\in\Lambda} l_\lambda^2\\ &= \sum_{\lambda\in\Lambda} w_\lambda^2 - \sum_{\lambda\in\Lambda} l_\lambda^2\\ &= \sum_{\lambda\in\Lambda} r_\lambda(w_\lambda + l_\lambda).\end{aligned}$$

But it is evident from an easy induction in the poset Λ that the dimension of the space spanned by the elements $r(x,z)$ and $r(z,x)$ is at least equal to

$$\begin{aligned}&\sum_{\lambda\in\Lambda}(2w_\lambda r_\lambda - r_\lambda^2)\\ &\quad= \sum_{\lambda\in\Lambda} r_\lambda(2w_\lambda - r_\lambda)\\ &\quad= \sum_{\lambda\in\Lambda} r_\lambda(w_\lambda + l_\lambda).\end{aligned}$$

Comparing with $\dim(\mathcal{R})$, we obtain the statement (2).

We now turn to (3). We begin by showing

$$J \supseteq \mathcal{R} \iff R(\lambda) \subseteq JW(\lambda) \text{ for all } \lambda \in \Lambda. \tag{5.5}$$

First assume $J \supseteq \mathcal{R}$ and suppose $z \in R(\lambda)$; then take $x, y \in W(\lambda)$, such that $\phi_\lambda(x,y) \neq 0$. Since $\mathcal{R}$, and therefore J, contains an element $r(z,x)$ of the form above, we have $JW(\lambda) \ni r(z,x)y = \phi_\lambda(x,y)z$. Hence, $R(\lambda) \subseteq JW(\lambda)$ for each $\lambda \in \Lambda$.

Conversely, suppose $JW(\lambda) \supseteq R(\lambda)$ for each $\lambda \in \Lambda$. Since $J^* = J$, to show that $J \supseteq \mathcal{R}$, it will suffice to show that for any $\lambda \in \Lambda$, and $x \in W(\lambda), z \in R(\lambda)$, there is an element $r(z, x) \in J$, of the form above. Now by hypothesis, $z \in JW(\lambda)$; hence $C^\lambda(z \otimes x) = C^\lambda(ay \otimes x) \in aC^\lambda(y \otimes x) + B(< \lambda)$, for some $a \in J$ and $y \in W(\lambda)$. Hence, there is an element $a_1 = C^\lambda(z \otimes x) + b \in J$ where $b \in B(< \lambda)$. If $a_1 \notin \mathcal{R}$, then there is an element $\lambda' < \lambda$ such that $a_1 L(\lambda') \neq 0$, since $a_1 L(\mu) = 0$ for all $\mu \not< \lambda$.

By the cyclic nature of cell modules, if $JL(\mu) \neq 0$, then $JW(\mu) = W(\mu)$. Thus for any two elements $p, q \in W(\mu)$, since $p \in JW(\mu)$, the argument above shows that $C^\mu(p \otimes q) + b' \in J$ for some $b' \in B(< \mu)$. It follows that the argument in the proof of Lemma 5.3 may be applied to show that a_1 may be recursively modified by elements of J, to yield an element $a_0 = r(z, x) \in J \cap \mathcal{R}$ as required.

The statement (5.5) now follows from (2). To deduce (3), observe that if $\lambda \in \Lambda \setminus \Lambda^0$, then since $JL(\lambda) \neq 0$, there are elements $a \in J$ and $x \in W(\lambda)$ such that $ax \notin R(\lambda)$. But then by Lemma 5.3 (1), $W(\lambda) = B \cdot ax \subseteq J \cdot x \subseteq JW(\lambda)$, whence $JW(\lambda) \supseteq R(\lambda)$ always holds *a fortiori* for $\lambda \in \Lambda \setminus \Lambda^0$. In view of (5.5), this completes the proof of (3). □

Corollary 5.5. *Let notation be as in Theorem 5.4. Assume that for all $\lambda \in \Lambda$ such that $JL(\lambda) = 0$, $R(\lambda)$ is either zero or irreducible. Assume further that for any $\lambda \in \Lambda$ such that $JL(\lambda) = 0$ and $R(\lambda) \neq 0$, $JW(\lambda) \neq 0$. Then J contains the radical $\mathcal{R}$ of B.*

Proof. It follows from Theorem 5.4 that it suffices to show that for any λ such that $JL(\lambda) = 0$, $JW(\lambda) = R(\lambda)$. But by hypothesis, if $R(\lambda) \neq 0$ for some such λ, $JW(\lambda)$ is a non-zero submodule of $R(\lambda)$. By irreducibility, it follows that $JW(\lambda) = R(\lambda)$, whence the result.

6 The Classical Three-Dimensional Case

6.1 The Setup

Let $V = V(2)$, the classical three-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. In this section, we shall construct a quotient of the Brauer algebra $B_r(3)$ which is defined by adding a single relation to the defining relations of $B_r(3)$, and which maps surjectively onto $\mathrm{End}_{\mathfrak{sl}_2}(V^{\otimes r})$. For small r, we are able to show that our quotient is isomorphic to the endomorphism algebra. We shall make extensive use of the cellular structure of $B_r(3)$, as outlined in [GL96, Sect. 4]. In analogy with the Temperley–Lieb case above, where the case $r = 3$ (i.e. $V(1)^{\otimes 3}$) was critical, we start with the case $r = 4$, i.e. $V(2)^{\otimes 4}$.

Recall that given a commutative ring A, the Brauer algebra $B_r(\delta)$ over A may be defined as follows. It has generators $\{s_1, \dots, s_{r-1};\ e_1, \dots, e_{r-1}\}$, with relations $s_i^2 = 1$, $e_i^2 = \delta e_i$, $s_i e_i = e_i s_i = e_i$ for all i, $s_i s_j = s_j s_i$, $s_i e_j = e_j s_i$, $e_i e_j =$

e_je_i if $|i-j| \geq 2$, and $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$, $e_ie_{i\pm 1}e_i = e_i$ and $s_ie_{i+1}e_i = s_{i+1}e_i$, $e_{i+1}e_is_{i+1} = e_{i+1}s_i$ for all applicable i. We shall assume the reader is familiar with the diagrammatic representation of a basis of $B_r(\delta)$, and how basis elements are multiplied by concatenation of diagrams. In particular, the group ring $A\mathrm{Sym}_r$ is the subalgebra of $B_r(\delta)$ spanned by the diagrams with r "through strings," and the algebra contains elements $w \in \mathrm{Sym}_r$ which are appropriate products of the s_i.

In this section, we take $A = \mathbb{C}$ and $\delta = 3$. The algebra $B_r(3)$ acts on $V^{\otimes r}$ as follows. We take the same basis $\{v_i \mid i = 0, \pm 1\}$ as in Sect. 4.3. Then s_i acts by interchanging the ith and $(i+1)$st factors in the tensor $v_{j_1} \otimes \cdots \otimes v_{j_r}$. We define $w_0 \in V \otimes V$ as the specialisation at $q = 1$ of the element w_0 of Sect. 4.3, i.e. $w_0 = v_{-1,1} + v_{1,-1} - v_{0,0}$. Then the action of e_1 is obtained by putting $q = 1$ in Lemma 4.6, i.e.

$$e_1 v_{i,j} = \begin{cases} w_0, & \text{if } (i,j) = (1,-1) \text{ or } (-1,1), \\ -w_0, & \text{if } (i,j) = (0,0), \\ 0, & \text{if } i+j \neq 0. \end{cases}$$

The element e_i acts on the $i, i+1$ components similarly. In addition to the elements s_i and e_i, it will be useful to define the endomorphisms $e_{i,j} := (1,i)(2,i+1)\, e_1(1,i)(2,i+1)$, where we use the usual cycle notation for permutations in $B_r(\delta)$. The endomorphism $e_{i,j}$ acts on the ith and jth components of $V^{\otimes r}$ as e_1 and leaves the other components unchanged.

6.2 Cellular Structure

The Brauer algebra $B_r(\delta)$ was proved in [GL96, Sect. 4] to have a cellular structure. This facilitates discussion of its representation theory. We begin by reviewing briefly the cells and cell modules for $B_r(\delta)$. Our notation here differs slightly from that in *loc. cit.*

Given an integer $r \in \mathbb{Z}_{\geq 0}$, define $\mathcal{T}(r) := \{t \in \mathbb{Z} \mid 0 \leq t \leq r, \text{ and } r - t \in 2\mathbb{Z}\}$. For $t \in \mathbb{Z}_{\geq 0}$, let $\mathcal{P}(t)$ denote the set of partitions of t. Define $\Lambda(r) := \coprod_{t \in \mathcal{T}(r)} \mathcal{P}(t)$. This set is partially ordered by stipulating that $\lambda > \lambda'$ if $|\lambda| > |\lambda'|$ or $|\lambda| = |\lambda'|$, and $\lambda > \lambda'$ in the dominance order on partitions of $|\lambda|$. For any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p)$, denote by $|\lambda|$ the sum $\sum_i \lambda_i$ of its parts. Given $\lambda \in \Lambda(r)$, the corresponding set $M(\lambda)$ (cf. [GL96, (1.1) and Sect. 4]) is the set of pairs (S, τ), where S is an involution with $|\lambda|$ fixed points in Sym_r and τ is a standard tableau of shape λ.

If (S, τ) and (S', τ') are two elements of $M(\lambda)$, the basis element of $B_r(\delta)$ is in the notation of [GL96, (4.10)],

$$C^{\lambda}_{(S,\tau),(S',\tau')} := \sum_{w \in \mathrm{Sym}_{|\lambda|}} p_{w(\tau,\tau')}(w)[S, S', w],$$

where $w(\tau, \tau')$ is the element of $\mathrm{Sym}_{|\lambda|}$ corresponding to τ, τ' under the Robinson–Schensted correspondence, and for $u \in \mathrm{Sym}_{|\lambda|}$, $c_u = \sum_{w \in \mathrm{Sym}_{|\lambda|}} p_u(w) w$ is the corresponding Kazhdan–Lusztig basis element of $\mathbb{Z}\mathrm{Sym}_{|\lambda|}$. The cardinality $|M(\lambda)|$ is easily computed. Let $k(\lambda) = \frac{r-|\lambda|}{2}$, and let d_λ be the dimension of the representation (Specht module) of the symmetric group $\mathrm{Sym}_{|\lambda|}$ corresponding to λ. For any integer $t \geq 0$ denote by $t!!$ the product of the odd positive integers $2i + 1 \leq t$. Then we have, for any $\lambda \in \Lambda(r)$,

$$|M(\lambda)| = \binom{r}{|\lambda|} (2k(\lambda))!! d_\lambda := w_\lambda. \tag{6.1}$$

Now assume that the ground ring A is a field. We recall some facts from cellular theory.

6.1. *Maintain the notation above.*

(1) *For each $\lambda \in \Lambda(r)$, there is a left $B_r(\delta)$-module $W(\lambda)$, of dimension w_λ over A.*
(2) *The module $W(\lambda)$ has a bilinear form $\phi_\lambda : W(\lambda) \times W(\lambda) \to A$, which is invariant under the $B_r(\delta)$-action.*
(3) *Let $R(\lambda)$ be the radical of the form ϕ_λ. Then $L(\lambda) := W(\lambda)/R(\lambda)$ is either an irreducible $B_r(\delta)$-module, or is zero. The non-zero $L(\lambda)$ are pairwise non-isomorphic, and all irreducible $B_r(\delta)$-modules arise in this way.*
(4) *All composition factors $L(\mu)$ of $W(\lambda)$ satisfy $\mu \geq \lambda$.*
(5) *$B_r(\delta)$ is semisimple if and only if each form ϕ_λ is non-degenerate. Equivalently, the W_λ form a complete set of representatives of the isomorphism classes of simple $B_r(\delta)$-modules.*

We shall make use of the following facts.

Proposition 6.2. *Take $A = \mathbb{C}$ and assume that $\delta \neq 0$.*

(1) *The algebra $B_r(\delta)$ is quasi-hereditary; that is, each form ϕ_λ in the assertion 6.1 (2) is non-zero.*
(2) *The algebra $B_r(3)$ is semisimple if and only if $r \leq 4$.*

Proof. The statement (1) is immediate from [GL96, Corollary 4.14], while (2) follows from [RS, Theorem 2.3]. □

6.3 The Case r = 4

It is clear from dimension considerations that when $r \leq 3$, the surjection $\eta : B_r(3) \longrightarrow \mathrm{End}_{\mathfrak{sl}_2} V(2)^{\otimes r}$ (and its quantum analogue) is an isomorphism. The case $r = 4$ is therefore critical. In this subsection, we shall treat the classical case when $r = 4$.

In terms of Sect. 6.1, we now take $r = 4$ and $\delta = 3$. Our purpose is to identify the kernel of the surjection $\eta : B_4(3) \longrightarrow \mathrm{End}_{\mathfrak{sl}_2(\mathbb{C})} V^{\otimes 4}$. Define the element $\Phi \in B_4(3)$ by

$$\begin{aligned} \Phi &= Fe_2F - F - \frac{1}{4} Fe_2e_{1,4}F, \text{ where} \\ F &= (1 - s_1)(1 - s_3), \end{aligned} \tag{6.2}$$

where notation is as in Sect. 6.1.

The next statement summarises some of the properties of Φ.

Proposition 6.3. *Let $F, \Phi \in B_4(3)$ be the elements defined in (6.2). Then*

(1) $e_i\Phi = 0$ *for* $i = 1, 2, 3$.
(2) $\Phi^2 = -4\Phi$.
(3) Φ *acts as* 0 *on* $V^{\otimes 4}$. *That is,* $\Phi \in \mathrm{Ker}(\eta)$.

Proof. First note that

$$e_2Fe_2 = e_2 + e_2e_{1,4}. \tag{6.3}$$

To see this, note that $e_2Fe_2 = e_2(1 - s_1 - s_3 + s_1s_3)e_2 = e_2^2 - e_2s_1e_2 - e_2s_3e_2 + e_2s_1s_3e_2 = 3e_2 - 2e_2 + e_2e_{1,4}$.

To prove (1), note that it is trivial that $e_iF = 0$ for $i = 1, 3$, and hence that $e_i\Phi = 0$ for $i = 1, 3$. But the relation $e_2\Phi = 0$ now follows easily from (6.3) and the fact that $e_{1,4}^2 = 3e_{1,4}$.

It follows from (1) that $e_2F\Phi = 0$, since $F\Phi = 4\Phi$ (recall $F^2 = 4F$). Hence $\Phi^2 = -F\Phi = -4\Phi$ which proves (2).

For (3), note that F maps $V^{\otimes 4}$ onto $L(2) \otimes L(2)$, and hence that $\Phi(V^{\otimes 4}) \subseteq L(2) \otimes L(2)$. But by Corollary 4.8, e_2 acts injectively on $L(2) \otimes L(2)$, whence it follows from the fact that $e_2\Phi = 0$, just proved, that $\Phi(V^{\otimes 4}) = 0$. □

Theorem 6.4. *The kernel of $\eta : B_4(3) \longrightarrow \mathrm{End}_{\mathfrak{sl}_2} V(2)^{\otimes 4}$ is generated by the element Φ above.*

Proof. The set $\Lambda(4)$ has eight elements, ordered as follows:

$$(4) > (3, 1) > (2^2) > (2, 1^2) > (1^4) > (2) > (1^2) > (0).$$

The dimensions of the corresponding cell modules, which by the assertion 6.1 (5) and Proposition 6.2 (2) are simple in this case, are given respectively, by

$$1, 3, 2, 3, 1, 6, 6, 3.$$

Now since $B_4(3)$ is semisimple, it is isomorphic to a sum $\bigoplus_{j=1}^{8} M(j)$ of two-sided ideals, which are isomorphic to matrix algebras of size given in the list above (thus, e.g., $\dim M(1) = 1$, while $\dim M(7) = 36$). Moreover, the two-sided ideal $\mathcal{I}$ of $B_4(3)$ which is generated by the e_i is cellular, and is the sum of the matrix algebras $M(j)$ for $j \geq 6$.

Note that $B_4(3)/\mathcal{I} \cong \mathbb{C}\mathrm{Sym}_4$. Let $\mathcal{P}$ be the two-sided ideal of $B_4(3)$ generated by Φ. Then $\mathcal{P} + \mathcal{I} \ni F$, and $\frac{1}{4}F$ is an idempotent in $\mathbb{C}\mathrm{Sym}_4$ which generates a left ideal on which Sym_4 acts as $\mathrm{Ind}_K^{\mathrm{Sym}_4}(\varepsilon)$, where K is the subgroup of Sym_4 generated by s_1, s_3 and ε denotes the alternating representation. But it is easily verified that $\mathrm{Ind}_K^{\mathrm{Sym}_4}(\varepsilon)$ is isomorphic to the sum of the irreducible representations of Sym_4 which correspond to the partitions (2^2), $(2,1^2)$ and (1^4), each one occurring with multiplicity one. Here, we use the standard parametrisation in which the irreducible complex representations of Sym_n correspond to partitions of n, the trivial representation corresponding to the partition (n).

It follows that the two-sided ideal of $\mathbb{C}\mathrm{Sym}_4$ generated by F is the image of $\bigoplus_{j=3}^{5} M(j)$ under the surjection $B_4(3)/\mathcal{I} \longrightarrow \mathbb{C}\mathrm{Sym}_4$. It follows that $\mathcal{I} + \mathcal{P} = \bigoplus_{j\geq 3} M_j$, whence $\dim(\mathcal{I} + \mathcal{P}) = \dim \mathcal{I} + 14$.

But using the dimension formula for $\dim \mathrm{End}_{\mathfrak{sl}_2} V^{\otimes 4}$ in Sect. 2, the kernel N of η has dimension 14 in this case. Since $\mathcal{P} \subseteq N$, it follows that

$$\dim(\mathcal{I} + \mathcal{P}) \leq \dim(\mathcal{I} + N) \leq \dim \mathcal{I} + \dim \mathcal{P} \leq \dim \mathcal{I} + \dim N = \dim \mathcal{I} + 14,$$

with equality if and only if $\mathcal{I} \cap N = 0$ and $\mathcal{P} = N$.

Since we have proved equality, the theorem follows. □

6.4 The Case $r = 5$

This is the first case where $B := B_r(3)$ is not semisimple. We shall analyse this case to show how our methods yield non-trivial information on the algebras, such as the dimension of the radical. For this subsection only, we denote $B_5(3)$ by B.

In this case, the cells are again totally ordered; we write them as follows.

$$\begin{aligned}(5) &> (4,1) > (3,2) > (3,1^2) > (2^2,1) > (2,1^3) > (1^5) \\ &> (3) > (2,1) > (1^3) > (1).\end{aligned} \tag{6.4}$$

If $W(\lambda)$ denotes the cell module corresponding to λ, the dimensions of the $W(\lambda)$ above are respectively given by:

$$1, 4, 5, 6, 5, 4, 1, 10, 20, 10, 15.$$

Recall that $L(\lambda)$ is the irreducible head of $W(\lambda)$ for $\lambda \in \Lambda(5)$; write $l_\lambda := \dim L(\lambda)$. These integers are the dimensions of the simple B-modules.

We define the following two-sided ideals of B. Let $\mathcal{R}$ be the radical of $B_5(3)$, $\mathcal{I} = B(\leq (3))$ the ideal generated by the e_i, $\mathcal{P}$ the ideal generated by Φ, and N the kernel of $\eta : B \longrightarrow E := \mathrm{End}_{\mathfrak{sl}_2}(V^{\otimes 5})$.

We shall prove

Theorem 6.5. *Let $B = B_5(3)$ as above, let $\mathcal{R}$ be its radical, and maintain the above notation.*

(1) *The cell modules of B are all simple except for those corresponding to the partitions $(2,1)$ and (1^3), whose simple heads have dimension $15, 6$ respectively.*
(2) *The composition factors of $W(1^3)$ are $L(1^3)$ and $L(2,1^3)$.*
(3) *The composition factors of $W(2,1)$ are $L(2,1)$ and $L(2^2,1)$.*
(4) *The radical of B has dimension 239.*
(5) *The kernel N of $\eta : B \longrightarrow E$ is generated by Φ modulo the radical. That is, in the notation above, $N = \mathcal{P} + \mathcal{R}$.*

Proof. It is easily verified that $E \cong M_1(\mathbb{C}) \oplus M_4(\mathbb{C}) \oplus M_{10}(\mathbb{C}) \oplus M_{15}(\mathbb{C}) \oplus M_{15}(\mathbb{C}) \oplus M_6(\mathbb{C})$, where $M_n(\mathbb{C})$ denotes the algebra of matrices of size n over $\mathbb{C}$. Further, η induces a surjection $\overline{\eta} : \overline{B} := B/\mathcal{R} \cong \bigoplus_{\lambda \in \Lambda} M_{l_\lambda}(\mathbb{C}) \longrightarrow E$. We shall determine which of the simple components $M_{l_\lambda}(\mathbb{C})$ are in the kernel of $\overline{\eta}$.

Observe that since $B/\mathcal{I} \cong \mathbb{C}\mathrm{Sym}_5$, the cell modules $W(\lambda)$ ($|\lambda| = 5$) are all irreducible, and clearly $\mathcal{R} \subseteq \mathcal{I} \cap N$.

Now the element $F = (1 - s_1)(1 - s_3)$ generates the two-sided ideal of $\mathbb{C}\mathrm{Sym}_5$ which corresponds to the irreducible representations of Sym_5 which are constituents of $\mathrm{Ind}_K^{\mathrm{Sym}_5}(\varepsilon)$, where K is the subgroup generated by s_1 and s_3. An easy computation shows that these representations are precisely those which correspond to the partitions λ with $|\lambda| = 5$ and $\lambda \neq (5), (4,1)$. Let $\Lambda^1 = \{\lambda \in \Lambda \mid |\lambda| = 5, \lambda \neq (5), (4,1)\}$, and write $\Lambda^0 := \Lambda \setminus \Lambda^1$. It follows from the above that N acts non-trivially on the simple modules $W(\lambda)$ for $\lambda \in \Lambda^1$ (since $\Phi \in N$ does), and hence that $\mathrm{Ker}(\overline{\eta}) \supseteq \bigoplus_{\lambda \in \Lambda^1} M_{l_\lambda}(\mathbb{C})$. Using the number and dimensions of the matrix components of E, it follows, by comparing the sizes of the matrix algebras on both sides, that $W(1)$ is simple, one of the ten dimensional cell modules is simple, the other has head of dimension 6, and $W(2,1)$ has head of dimension 15.

Now the Gram matrix associated with the bilinear form $\phi_{(1^3)}$ on $W(1^3)$ is given by

$$\begin{bmatrix} 3 & 1 & -1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & 0 \\ -1 & 1 & 3 & 1 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 1 & 3 & 0 & 0 & 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & 3 & 1 & -1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 & 1 & 3 & 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 & -1 & 1 & 3 & 0 & 1 & -1 \\ 0 & -1 & -1 & 0 & 1 & 1 & 0 & 3 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 & -1 & 1 & 1 & 3 \end{bmatrix}.$$

Since this has rank 6, $W(1^3)$ is reducible. To understand the composition factors, note that the cell corresponding to the partition 1^3 contains just the longest element w_0 of Sym_3. The Kazhdan–Lusztig basis element $c_{w_0} = \sum_{w \in \mathrm{Sym}_3} \varepsilon(w) w$, and from this one sees easily that the element $\sum_{w\in\langle s_1,s_2,s_3\rangle} w$ of B acts trivially on $W(1^3)$, whence it follows that $W(1^3)$ has no submodule isomorphic to $L(5)$ or $L(4,1)$. Similarly, since $E(5) = \sum_{w\in\mathrm{Sym}_5} \varepsilon(w)w$ also acts trivially on $W(1^3)$ (note that $E(5)e_i = 0$ for all i), $W(1^3)$ has no submodule isomorphic to $L(1^5)$. It follows that $R(1^3) \cong L(2,1^3)$, proving (2).

Now consider $W(2,1)$. The corresponding cell of Sym_3 contains the elements r_1, r_2, r_1r_2 and r_2r_1, where the simple generators of Sym_3 are denoted by r_1, r_2 to avoid confusion with $s_1, s_2 \in B$. The corresponding Kazhdan–Lusztig basis elements are $1 - r_1$, $1 - r_2$, $(1 - r_1)(1 - r_2)$ and $(1 - r_2)(1 - r_1)$. In analogy with the previous case, one now verifies easily that $\sum_{w\in\langle s_1,s_2,s_4\rangle} w$ and $E(5)$ act trivially on $W(2,1)$, whence the latter cell module has no simple submodule isomorphic to $L(5), L(4,1), L(3,2)$ or $L(1^5)$. It follows by dimension that $R(2,1) \cong L(2^2,1)$, completing the proof of (1), (2) and (3).

Clearly $\dim \mathcal{R} = \dim B - \dim \overline{B} = 20^2 + 10^2 - (15^2 + 6^2) = 239$, which proves (4).

To prove (5) observe that since $F \in (\mathcal{P} + \mathcal{I})$, the argument concerning induced representations above shows that $B/(\mathcal{P} + \mathcal{I}) \cong M_1(\mathbb{C}) \oplus M_4(\mathbb{C})$. Hence

$$\dim(\overline{\mathcal{P}+\mathcal{I}}) \le \dim(\overline{N} + \overline{\mathcal{I}}) \le \dim \overline{N} + \dim \overline{\mathcal{I}},$$

with equality if and only if $\overline{N} \cap \overline{\mathcal{I}} = 0$ and $\overline{\mathcal{P}+\mathcal{R}} = \overline{N}$. But $\dim \overline{\mathcal{P}+\mathcal{I}} = \dim \overline{B} - 17$, and by the above argument, this is equal to $\dim \overline{N} + \dim \overline{\mathcal{I}}$, whence $\overline{\mathcal{P}+\mathcal{R}} = \overline{N}$, i.e. $N = \Phi + \mathcal{R}$. □

Remark 6.6. The results [DHW, (3.3) and (3.4)] may be applied to determine which of the modules $L(\lambda)$ for $\lambda \vdash 5$ are composition factors of the $W(\mu)$ for $\mu \vdash 3$. This would abbreviate the proof of Theorem 6.5 (2) and (3). We include our proof because it is complete and elementary.

6.5 *The General Classical Case*

Our objective in this section is to check some cases of our main conjecture below, and reduce it to a specific question about the action of Φ on certain cell modules of $B_r(3)$. To do this, we shall use the general results of Sect. 5 above about the radical of a cellular algebra.

In view of the results of the last subsection, we make the

6.7. Conjecture. *Let* $B = B_r(3)$, $E = \mathrm{End}_{\mathfrak{sl}_2(\mathbb{C})}(V(2)^{\otimes r})$ *and* $\eta : B \longrightarrow E$ *the natural surjection discussed above. The kernel* N *of* η *is generated by the element* $\Phi = Fe_2F - F - \frac{1}{4}Fe_2e_{14}F \in B$.

To make use of the theory in the last section, we shall develop more detail concerning the cellular structure of B_r. We maintain the notation above. In particular, $\mathcal{R}$ denotes the radical of B, $\mathcal{I}$ is the two-sided ideal generated by the e_i and $\mathcal{P}$ denotes the ideal generated by Φ.

We start with the following elementary observation.

Lemma 6.8. *Let $t \geq 4$ be an integer and consider the symmetric group* Sym_t *generated by simple transpositions $s_1, \ldots, s_{t-1}$. The two-sided ideal of* $\mathbb{C}\mathrm{Sym}_t$ *generated by $F = (1 - s_1)(1 - s_3)$ is the sum of the minimal ideals corresponding to all partitions with at least four boxes in the first two columns.*

This is an easy exercise, which may be proved by induction on t.

Theorem 6.9. *Let $\eta : B_r(3) \longrightarrow E := \mathrm{End}_{\mathfrak{sl}_2(\mathbb{C})}(V(2)^{\otimes r})$ be the surjection discussed above, and let $N = \mathrm{Ker}(\eta)$. Define $\Lambda^0 \subseteq \Lambda$ by $\Lambda^0 = \{(t), (t-1, 1), 1^3 \mid 0 \leq t \leq r;\quad t \equiv r \pmod 2\}$, and let $\Lambda^1 := \Lambda \setminus \Lambda^0$. Let Φ be the element of $B = B_r$ defined above. Then*

(1) *For $\lambda \in \Lambda^1$, there is an element $x_\lambda \in L(\lambda)$ such that $\Phi x_\lambda \neq 0$.*
(2) *N acts trivially on $L(\lambda)$ if and only if $\lambda \in \Lambda^0$*
(3) *$E \cong \bigoplus_{\lambda \in \Lambda^0} \overline{B}(\lambda)$.*
(4) *If $\mathcal{P}$ denotes the ideal of B generated by Φ, we have $\mathcal{P} + \mathcal{R} = N$, where $\mathcal{R}$ is the radical of B.*

Proof. Take $\lambda \in \Lambda^1$. If $t = |\lambda| \geq 4$, consider the subalgebra of B generated by the elements $\{s_i e_{t+1} e_{t+3} \ldots e_{t+2k-1} \mid 1 \leq i \leq t-1\}$, where $r = t + 2k$. This is isomorphic to $\mathbb{C}\mathrm{Sym}_t$, and Φ acts on the corresponding cell modules as $-F$. The statement (1) is now clear for this case, given Lemma 6.8.

Now suppose $t = |\lambda| \leq 3$. Now in analogy to the above argument, we consider the "leftmost" part of the diagrams, completed with $e_5 e_7 \ldots$ or $e_6 e_8 \ldots$ on the right according as t is odd or even. The cases $r = 4, 5$, which are known by Sects. 6.3 and 6.4 produce, when appropriately completed, elements $x_\lambda \in L(\lambda)$ as required. This proves (1).

To see (2), observe that (1) shows that N acts non-trivially on the simple modules $L(\lambda)$ for $\lambda \in \Lambda^1$, and so the set of λ such that N acts trivially in $L(\lambda)$ is contained in Λ^0. But $|\Lambda^0| = r + 1$, and it is easy to see that $V^{\otimes r}$ has $r + 1$ distinct simple components (as $\mathfrak{sl}_2$-module). It follows that N acts trivially in at least $r + 1$ of the simple modules $L(\lambda)$, and (2) is immediate (cf. Corollary 5.2), as is (3).

Since $\Phi \in \mathcal{P} + \mathcal{R}$, the latter is a two-sided ideal of B which acts non-trivially on $L(\lambda)$ for $\lambda \in \Lambda^1$. But $\Phi \in N$, so that $\mathcal{P} + \mathcal{R}$ acts trivially on $L(\lambda)$ for $\lambda \in \Lambda^0$. The statement (4) follows. □

Combined with the results of Sect. 5, Theorem 6.9 leads to the following criterion for the truth of Conjecture 6.7.

Corollary 6.10. *The conjecture 6.7 is equivalent to the following statement. For each $\lambda \in \Lambda^0$ (as above), the B_r-submodule of $W(\lambda)$ generated by $\Phi W(\lambda)$ contains $R(\lambda)$.*

Proof. It follows from Theorem 6.9 (4) that the conjecture is equivalent to the statement that $\mathcal{P} \supseteq \mathcal{R}$. By Theorem 5.4 (3), this is equivalent to the stated criterion.

□

Next, we show that the Conjecture is true for $r = 5$.

Proposition 6.11. *If* $r = 5$, $\mathcal{P} = \langle\Phi\rangle$ *contains the radical* $\mathcal{R}$ *of* B. *Hence, by Corollary 6.10 the conjecture is true for* $r = 5$.

Proof. In view of Theorem 6.5, it suffices to show that $\mathcal{P}W(\lambda) = R(\lambda)$ for $\lambda = (1^3)$ and $\lambda = (2,1)$. But again by Theorem 6.5, we have the situation of Corollary 5.5 here, whence it suffices to show simply that Φ acts non-trivially on the cell modules $W(1^3)$ and $W(2,1)$. This will require two calculations, which we now proceed to outline.
The case $W(1^3)$. In this case $M(\lambda) = \{((ij),\tau)\}$ where (ij) is a transposition in Sym_5 and τ is the unique standard tableau of shape 1^3. Thus, we may write a basis for $W(1^3)$ as $\{C_{ij} \mid 1 \le i < j \le 5\}$. Recalling that the Kazhdan–Lusztig basis element of $\mathbb{C}\mathrm{Sym}_3$ corresponding to (τ,τ) is $E(3) := \sum_{w\in\mathrm{Sym}_3} \varepsilon(w)w$, the following facts are easily verified using the diagrammatic representation of B_5.

$$s_1C_{45} = -C_{45};\;\; s_3C_{45} = C_{35};\;\; FC_{45} = 2(C_{45} - C_{35});$$
$$e_2C_{45} = 0;\;\; e_2C_{35} = -C_{23};\;\; e_{14}C_{23} = 0.$$

Using these equations, one calculates in straightforward fashion that

$$\Phi C_{45} = 2(C_{23} - C_{13} - C_{24} + C_{14} - C_{45} + C_{35}) \neq 0.$$

The case $W(2,1)$. In this case $M(\lambda) = \{((ij),\tau_k)\}$, where (ij) is a transposition in Sym_5 and τ_k is one of the two standard tableau of shape $(2,1)$. Explicitly,

$$\tau_1 = \begin{matrix} 1 & 3 \\ 2 & \end{matrix}, \qquad \tau_2 = \begin{matrix} 1 & 2 \\ 3 & \end{matrix}.$$

Thus, we may write a basis for $W(2,1)$ as $\{C_{ij,\tau_k} \mid 1 \le i < j \le 5,\;\; k = 1,2\}$. In this case, we need to recall that the Kazhdan–Lusztig cell representation of $\mathbb{C}\mathrm{Sym}_3$ which corresponds to $(2,1)$ may be thought of as having basis $\{c_{\tau_1}, c_{\tau_2}\}$ and action by $\mathrm{Sym}_3 = \langle r_1, r_2\rangle$ given by

$$r_1c_{\tau_1} = -c_{\tau_1};\;\; r_1c_{\tau_2} = c_{\tau_2} - c_{\tau_1};\;\; r_2c_{\tau_1} = c_{\tau_1} - c_{\tau_2};\;\; r_2c_{\tau_2} = -c_{\tau_2}.$$

With these facts, one verifies easily the following facts

$$s_1C_{45,\tau_1} = -C_{45,\tau_1};\;\; s_3C_{45,\tau_1} = C_{35,\tau_1};\;\; FC_{45,\tau_1} = 2(C_{45,\tau_1} - C_{35,\tau_1}).$$

Further,

$$e_2C_{45,\tau_1} = 0;\;\; e_2C_{35,\tau_1} = C_{23,\tau_1} - C_{23,\tau_2};$$
$$e_2FC_{45,\tau_1} = 2(C_{23,\tau_2} - C_{23,\tau_1});\;\; e_{14}C_{23,\tau_k} = 0 \text{ for } k = 1,2.$$

Using these equations, it is straightforward to calculate that

$$\begin{aligned}\Phi C_{45,\tau_1} &= 2(C_{23,\tau_2} - C_{13,\tau_2} - C_{24,\tau_2} + C_{14,\tau_2} \\ &\qquad - C_{23,\tau_1} + C_{13,\tau_1} + C_{24,\tau_1} - C_{14,\tau_1} - C_{45,\tau_1} + C_{35,\tau_1}) \neq 0. \quad (6.5)\end{aligned}$$

This completes the proof of the Proposition. □

A computer calculation has been done to verify the case $r = 6$.

Theorem 6.12. *Let η be the surjective homomorphism from $B_r := B_r(3)$ to $E_r :=$ $\mathrm{End}_{\mathfrak{sl}_2(\mathbb{C})}(V(2)^{\otimes r})$, and let $\Phi \in B_r$ be the element defined above. Then for $r \leq 6$, Φ generates the kernel of η.*

Proof. We have proved the result for $r \leq 5$. The case $r = 6$ was checked by a computer calculation, which verified that $\dim\langle\Phi\rangle$ is correct in that case. Since we know that $\Phi \in \mathrm{Ker}(\eta)$, the result follows. □

We are grateful to Derek Holt for doing this computation for us using the Magma computational algebra package, with an implementation of noncommutative Gröbner basis due to Allan Steel.

7 The Quantum Case

In this section, we develop the theme of Sect. 4.2 and consider the BMW algebra $BMW_r(q)$ over $\mathcal{A}_q$, and its specialisation $BMW_r(\mathcal{K})$. The results of the last section on the Brauer algebra all generalise to the present case, and we deduce some new ones through the technique of specialisation. One of the key observations is that $BMW_r(q)$ has the $\mathbb{C}$-algebra $BMW_r(1) \cong B_r(3)$ (cf. Lemma 4.2) as a specialisation.

7.1 Specialisation and Cell Modules

In analogy with the case of the Brauer algebra, of which it is a deformation, $BMW_r(q)$ has a cellular structure [X, Theorem 3.11] and is also quasi hereditary [X, Theorem 4.3]. For each partition $\lambda \in \Lambda(r)$, there is therefore a cell module $W_q(\lambda)$ of dimension w_λ for $BMW_r(q)$. Each cell module has a non-zero irreducible head $L_q(\lambda)$, and these irreducibles form a complete set of representatives of the isomorphism classes of simple $BMW_r(q)$-modules. Furthermore, $BMW_r(q)$ is semisimple if and only if all the cell modules are simple (see [X, Sect. 3]).

Recall that $BMW_r(q)$ is the $\mathcal{A}_q$-algebra defined by the presentation (4.14), where $\mathcal{A}_q$ is the localisation of $\mathbb{C}[q^{\pm 1}]$ at the multiplicative subset $\mathcal{S}$ generated by $[2]_q$, $[3]_q$ and $[3]_q - 1$. By Lemma 4.2, $BMW_r(q)$ may be thought of as an *integral form* of $BMW_r(\mathcal{K})$.

One may identify $BMW_r(q)$ with the $\mathcal{A}_q$-algebra generated by (r,r)-tangle diagrams, which satisfy the usual relations (cf. e.g., [X, Definition 2.5]). For each Brauer r-diagram T [GL96, Sect. 4], it is explained in [X, p. 285] how to construct an (r,r)-tangle diagram T_q by lifting each intersection in T to an appropriate crossing. The tangle diagrams obtained this way form a basis of $BMW_r(q)$, which we shall denote by $\mathcal{T}_q$.

The cell modules $W_q(\lambda)$ of $BMW_r(q)$ are parametrised by partitions $\lambda \in \Lambda(r)$. They may also be described diagramatically, in a similar way to the cell modules of the Brauer algebra $B_r(3)$ (cf. Sect. 6.2). We proceed to give this description. Let $t \in \mathcal{T}(r)$; that is, $0 \leq t \leq r$ and $r - t \in 2\mathbb{Z}$. For a partition λ of t, we take $M(\lambda)$ to be the set defined in Sect. 6.2 for the Brauer algebra, viz $M(\lambda)$ is the set of pairs (S, τ) where S is an involution in Sym_r with $|\lambda| = t$ fixed points, and τ is a standard tableau of shape λ. In analogy with Sect. 6.2, if (S,τ) and (S',τ') are two elements of $M(\lambda)$, we obtain the (cellular) basis element $C^{\lambda}_{(S,\tau),(S',\tau')}(q)$ of $BMW_r(q)$ by

$$C^{\lambda}_{(S,\tau),(S',\tau')}(q) = \sum_{w \in \mathrm{Sym}_t} P_{w(\tau,\tau'),w}(q)[S,S',w]_q, \tag{7.1}$$

where $C_v = \sum_{w \in \mathrm{Sym}_t} P_{v,w}(q)T_w$ is the Kazhdan–Lusztig basis element of the Hecke algebra $H_t(q^2)$, $[S,S',w]_q$ is the element of the basis $\mathcal{T}_q$ (i.e. dangle) corresponding to the Brauer diagram $[S,S',w]$, and all other notation is as in Sect. 6.2. Note that $P_{v,w}(q) \in \mathbb{Z}[q^{\pm 1}] \subset \mathcal{A}_q$, so that $C^{\lambda}_{(S,\tau),(S',\tau')}(q) \in BMW_r(q)$.

Now for each element $(S,\tau) \in M(\lambda)$, the arguments leading to [X, Corollary 3.13] describe how to associate a (r,t) dangle with (S,τ) which we denote by $(S,\tau)_q$. These form an $\mathcal{A}_q$-basis of $W_q(\lambda)$, with the action of $BMW_r(q)$ given by concatenation, using the relations in [X, Definition 2.5] and the action of the Hecke algebra $H_t(q^2)$ on its cell modules (which have basis $\{\tau\}$). The next statement is a general result about cellular algebras, adapted to our situation.

Proposition 7.1. *Let $\phi : \mathcal{A}_q \longrightarrow R$ be a homomorphism of commutative rings with* 1*, and denote by BMW_r^{ϕ} the specialisation $R \otimes_{\phi} BMW_r(q)$. Then*

(1) *There is a natural bijection between the $\mathcal{A}_q$-basis $\{(S,\tau)_q\}$ of $W_q(\lambda)$ and an R-basis of the specialised cell module $W^{\phi}(\lambda)$.*
(2) *If $a \in BMW_r(q)$, the matrix of $1 \otimes a \in BMW_r^{\phi}$ with respect to the basis in* (1) *is obtained by applying ϕ to the entries of the matrix of a.*
(3) *The Gram matrix of the canonical form on $W^{\phi}(\lambda)$ is obtained from that of $W_q(\lambda)$ by applying ϕ to the entries of the latter.*
(4) *If $W^{\phi}(\lambda)$ is simple, so is $W_q(\lambda)$.*
(5) *We have $\mathrm{rank}_{\mathcal{A}_q} L_q(\lambda) \geq \mathrm{rank}_R L^{\phi}(\lambda)$, where $L_q(\lambda)$ is the simple head of $W_q(\lambda)$, etc.*
(6) *For any pair $\mu \geq \lambda \in \Lambda(r)$, the multiplicity $[W_q(\lambda) : L_q(\mu)] \leq [W^{\phi}(\lambda) : L^{\phi}(\mu)]$.*

Proof. The bijection of (1) arises from any $\mathcal{A}_q$-basis $\{\beta\}$ of $W_q(\lambda)$, by taking $\beta \mapsto 1 \otimes \beta$. Given this, the assertion (2) is clear, as is (3). If $\Delta(\lambda)$ is the determinant of the Gram matrix of $W_q(\lambda)$ (i.e. the discriminant), the discriminant $\Delta^{\phi}(\lambda)$ of $W^{\phi}(\lambda)$

is given by $\Delta^{\phi}(\lambda) = \phi(\Delta(\lambda))$. If $W^{\phi}(\lambda)$ is simple, then $\Delta^{\phi}(\lambda) \neq 0$, whence $\Delta_q(\lambda) \neq 0$. This implies that if ϕ_q is the inclusion of $\mathcal{A}_q$ in $\mathcal{K}$, then $W^{\phi_q}(\lambda)(= W_{\mathcal{K}}(\lambda))$ is simple as $BMW_r(\mathcal{K})$ module. It follows that $W_q(\lambda)$ has no non-trivial $BMW_r(q)$-submodules, whence (4). Finally, note that $\mathrm{rank}_{\mathcal{A}_q}(L_q(\lambda))$ equals the rank of the Gram matrix of the form. Since this cannot increase on specialisation, (5) follows. To see (6), note that any composition series of $W_q(\lambda)$ specialises (under the functor $R \otimes_{\phi} -$) to a chain of submodules of $W^{\phi}(\lambda)$. But by (3), the specialisation of $L_q(\mu)$ has $L^{\phi}(\mu)$ as a subquotient, from which (6) follows. □

7.2 An Element of the Quantum Kernel

We next consider some elements of $BMW_r(q)$ which will play an important role in the remainder of this work, and which will be used to define the Temperley–Lieb analogue of the title. Let $f_i = -g_i - (1-q^{-2})e_i + q^2$, and set

$$F_q = f_1 f_3. \tag{7.2}$$

We also define $e_{14} = g_3^{-1} g_1 e_2 g_1^{-1} g_3$ and $e_{1234} = e_2 g_1 g_3^{-1} g_2 g_1^{-1} g_3$.

The next two results are quantum analogues of Proposition 6.3.

Lemma 7.2. *The following identities hold in $BMW_4(q)$ (and hence in $BMW_r(q)$).*

$$f_i = \frac{(g_i - q^2)(g_i - q^{-4})}{q^{-2} + q^{-4}}, \tag{7.3}$$

$$e_i f_i = 0, \quad f_i^2 = (q^2 + q^{-2}) f_i, \quad i = 1, 2, 3, \tag{7.4}$$

$$e_2 F_q e_2 = \tilde{a} e_2 - d e_{1234} + a e_2 e_{14}, \tag{7.5}$$

$$e_2 F_q e_2 e_{14} = e_{14} e_2 F_q e_2 = (q^2 + q^{-2})^2 e_2 e_{14}, \tag{7.6}$$

$$e_2 F_q e_{1234} = e_{1234} F_q e_2 = -d e_2 + a e_{1234} + q^{-4} \tilde{a} e_2 e_{14}, \tag{7.7}$$

where

$$a = 1 + (1-q^{-2})^2, \quad \tilde{a} = 1 + (1-q^2)^2, \quad d = (q-q^{-1})^2 = q^2(a-1) = q^{-2}(\tilde{a}-1).$$

The first relation follows easily from the relations (4.15), and the others are straightforward consequences. Note that $f_i = (q^2 + q^{-2}) d_i$, where d_i is the idempotent of Lemma 4.2. Alternatively, one may use the representation of elements of the BMW algebra by tangle diagrams, and the multiplication by composition of diagrams, to verify the above statements.

Define the following element of $BMW_4(q)$:

$$\Phi_q = a F_q e_2 F_q - b F_q - c F_q e_2 e_{14} F_q + d F_q e_{1234} F_q, \tag{7.8}$$

where

$$b = 1 + (1 - q^2)^2 + (1 - q^{-2})^2,$$
$$c = \frac{1 + (2 + q^{-2})(1 - q^{-2})^2 + (1 + q^2)(1 - q^{-2})^4}{([3]_q - 1)^2}.$$

Proposition 7.3. *The elements F_q, Φ_q have the following properties:*

(1) $F_q^2 = (q^2 + q^{-2})^2 F_q$.
(2) $e_i \Phi_q = \Phi_q e_i = 0$ *for* $i = 1, 2, 3$.
(3) $\Phi_q^2 = -(q^2 + q^{-2})^2 (1 + (1 - q^2)^2 + (1 - q^{-2})^2) \Phi_q$.
(4) Φ_q *acts as* 0 *on* $V_q^{\otimes 4}$.

Proof. Part (1) immediately follows from the second relation in (7.4).

The fact that $e_1 \Phi_q = e_3 \Phi_q = 0$ follows from the first relation of (7.4) in Lemma 7.2. Now

$$e_2 \Phi_q = a e_2 F_q e_2 F_q - b e_2 F_q - c e_2 F_q e_2 e_{14} F_q + d e_2 F_q e_{1234} F_q.$$

Using the relations (7.5)–(7.7), we readily obtain $e_2 \Phi_q = 0$. It can be similarly shown that $\Phi_q e_i = 0$ for $i = 1, 2, 3$. Thus by part (2), we see that $\Phi_q F_q e_2 = 0$, and therefore that $\Phi_q^2 = -b \Phi_q F_q = -(q^2 + q^{-2})^2 b \Phi_q$.

The proof of part (4) proceeds in much the same way as in the classical case. Note that $\Phi_q(V_q^{\otimes 4}) \subset L(2)_q \otimes L(2)_q$. Thus by Lemma 4.7, $\Phi_q(V_q^{\otimes 4}) \cong e_2 \Phi_q(V_q^{\otimes 4})$ as $\mathcal{U}_q(\mathfrak{sl}_2)$-modules. Since $e_2 \Phi_q = 0$ by part (2), the proof is complete. □

7.3 A Regular Form of Quantum $\mathfrak{sl}_2$

In this subsection, we consider the quantised universal enveloping algebra of $\mathfrak{sl}_2$ over the ring $\mathcal{A}_q$ and its representations. By "regular form", we shall understand an $\mathcal{A}_q$-lattice in a $\mathcal{K}$-representation of $\mathcal{U}_q(\mathfrak{sl}_2)$. Denote by $\mathcal{U}_{\mathcal{A}_q}$ the $\mathcal{A}_q$-algebra generated by $e, f, k^{\pm 1}$ and $h := \frac{k - k^{-1}}{q - q^{-1}}$, subject to the usual relations, and call it the *regular form* of $\mathcal{U}_q(\mathfrak{sl}_2)$. Recall (Definition 4.1) that we have homomorphisms ϕ_1 and ϕ_q from $\mathcal{A}_q$ to $\mathbb{C}, \mathcal{K}$, respectively; the resulting specialisation $\mathbb{C} \otimes_{\phi_1} \mathcal{U}_{\mathcal{A}_q}$ at ϕ_1 is isomorphic to the universal enveloping algebra of $\mathcal{U}(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ with

$$1 \otimes e \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad 1 \otimes f \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad 1 \otimes h \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 1 \otimes k \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The $\mathcal{A}_q$-span $V_q^{\text{reg}}(2)$ of the vectors $v_0, v_{\pm 1}$ (see Sect. 4.3) forms a $\mathcal{U}_{\mathcal{A}_q}$-module, which is an $\mathcal{A}_q$-lattice in $V_q(2)$. Denote the rth tensor power of $V_q^{\text{reg}}(2)$ over $\mathcal{A}_q$ by $V_q^{\text{reg}}(2)^{\otimes r}$; this has an $\mathcal{A}_q$-basis consisting of the elements $v_{i_1, i_2, \ldots, i_r}$. Then

$$\begin{aligned} \mathcal{K} \otimes_{\phi_q} V_q^{\text{reg}}(2)^{\otimes r} &= V_q(2)^{\otimes r}, \quad \text{as } \mathcal{U}_q(\mathfrak{sl}_2)\text{-module;} \\ \mathbb{C} \otimes_{\phi_1} V_q^{\text{reg}}(2)^{\otimes r} &\cong V(2)^{\otimes r}, \quad \text{as } \mathfrak{sl}_2\text{-module.} \end{aligned} \tag{7.9}$$

Remark 7.4. The vectors $1\otimes v_{i_1,i_2,\dots,i_r}$ form a basis for $\mathbb{C}\otimes_{\mathcal{A}_q} V_q^{\mathrm{reg}}(2)^{\otimes r}$. It follows that $1\otimes v\in\mathbb{C}\otimes_{\phi_1} V_q^{\mathrm{reg}}(2)^{\otimes r}$ is zero if and only if $v\in(q-1)V_q^{\mathrm{reg}}(2)^{\otimes r}$.

Denote by $E_q^{\mathrm{reg}}(r)$ the $\mathcal{A}_q$-algebra $\mathrm{End}_{\mathcal{U}_{\mathcal{A}_q}}(V_q^{\mathrm{reg}}(2)^{\otimes r})$. Recall (Theorem 4.4) that we have the surjection $\eta_q : BMW_r(\mathcal{K})\longrightarrow E_q(r)=\mathcal{K}\otimes_{\phi_q} E_q^{\mathrm{reg}}(r)$. The next result shows that η_q preserves the $\mathcal{A}_q$-structures.

Lemma 7.5. *We have $\eta_q(BMW_r(q))\subseteq E_q^{\mathrm{reg}}(r)$. In particular, if $[3]_q{}^{-1}e_i$, d_i and c_i are the idempotents of Lemma 4.3, then $\eta_q(g_i)$, $\eta_q(\frac{e_i}{[3]_q})$, $\eta_q(d_i)$ and $\eta_q(c_i)$ belong to $E_q^{\mathrm{reg}}(r)$ for all i.*

Proof. The formulae in Lemma 4.6 show explicitly that $\eta_q(\frac{e_i}{[3]_q})\in E_q^{\mathrm{reg}}(r)$. A similar computation shows that $\eta_q(d_i)\in E_q^{\mathrm{reg}}(r)$, as follows. Evidently it suffices to treat the case $i=1$. Write $\eta_q(d_1)(v_{k,l}):=x_{k,l}$; clearly we only need to show that $x_{k,l}\in V_q^{\mathrm{reg}}(2)^{\otimes 2}$ for $k,l=0,\pm1$. But one verifies easily that the following explicit formulae describe the action of d_i. Write $u_{-1}=q^{-2}v_{-1,0}-v_{0,-1}$, $u_0=-q^{-2}v_{-1,1}+(1-q^{-2})v_{0,0}+q^{-2}v_{1,-1}$ and $u_1=q^{-2}v_{0,1}-v_{1,0}$, and note that $u_i\in V_q^{\mathrm{reg}}(2)^{\otimes 2}$ for $i=0,\pm1$. Then, $x_{1,1}=x_{-1,-1}=0$, $x_{0,1}=\frac{1}{q^{-2}+q^2}u_1$, $x_{0,1}=-q^2x_{1,0}$, $x_{0,0}=\frac{1-q^{-2}}{q^{-2}+q^2}u_0$, $x_{-1,1}=-x_{1,-1}=\frac{1}{q^{-2}+q^2}u_0$, $x_{0,-1}=\frac{1}{q^{-2}+q^2}u_{-1}$, and $x_{-1,0}=-q^2x_{0,-1}$.

This shows that $\eta_q(d_i)\in E_q^{\mathrm{reg}}(r)$, and since $\eta_q(\frac{e_i}{[3]_q})+\eta_q(d_i)+\eta_q(c_i)=1$, it follows that $\eta_q(c_i)\in E_q^{\mathrm{reg}}(r)$. But $\eta_q(g_i)=q^{-4}\eta_q(\frac{e_i}{[3]_q})-q^{-2}\eta_q(d_i)+q^2\eta_q(c_i)$, whence the result. □

As a $\mathcal{U}_q(\mathfrak{sl}_2)$-module, $V_q(2)^{\otimes r}\cong\mathcal{K}\otimes_{\phi_q} V_q^{\mathrm{reg}}(2)^{\otimes r}$ is the direct sum of isotypic components $I_q(2l)$, where every irreducible $\mathcal{U}_q(\mathfrak{sl}_2)$-submodule of $I_q(2l)$ has highest weight $2l$. It follows from the $\mathcal{U}_q(\mathfrak{sl}_2)$ case of Theorem 8.5 in [LZ] that $I_q(2l)$ is an irreducible $\mathcal{U}_q(\mathfrak{sl}_2)\otimes_{\mathcal{K}} BMW_r(\mathcal{K})$-submodule of $V_q(2)^{\otimes r}$.

Lemma 7.6. (1) *$I_q^{\mathrm{reg}}(2l):=I_q(2l)\cap V_q^{\mathrm{reg}}(2)^{\otimes r}$ is a $BMW_r(q)\otimes_{\mathcal{A}_q}\mathcal{U}_{\mathcal{A}_q}$-submodule of $V_q^{\mathrm{reg}}(2)^{\otimes r}$.*
(2) *The specialisation $I(2l):=\mathbb{C}\otimes_{\phi_1} I_q^{\mathrm{reg}}(2l)$ of $I_q^{\mathrm{reg}}(2l)$ is isomorphic as a $\mathcal{U}(\mathfrak{sl}_2)$-module to the isotypic component of $V(2)^{\otimes r}$ with highest weight $2l$.*
(3) *$I(2l)$ is an irreducible $B_r(3)\otimes\mathcal{U}(\mathfrak{sl}_2)$-submodule of $V(2)^{\otimes r}$.*

Proof. By Lemma 7.5, $I_q^{\mathrm{reg}}(2l)$ is stable under the action of $BMW_r(q)$. Since it is evidently a $\mathcal{U}_{\mathcal{A}_q}$-module and the $\mathcal{U}_{\mathcal{A}_q}$ action commutes with the action of $BMW_r(q)$, part (1) follows.

In view of Remark 7.4, $I(2l)$ is a non-trivial subspace of $\mathbb{C}\otimes_{\phi_1} V_q^{\mathrm{reg}}(2)^{\otimes r}$. By part (1), $I(2l)$ is isomorphic to some $\mathcal{U}(\mathfrak{sl}_2)\otimes B_r(3)$ submodule of $V(2)^{\otimes r}$ whose $\mathfrak{sl}_2$-submodules all have highest weight $2l$. The $\mathfrak{sl}_2$ case of Theorem 3.13 in [LZ] states that the $\mathcal{U}(\mathfrak{sl}_2)$ isotypical component of $V^{\otimes r}$ with highest weight $2l$ is the unique irreducible $\mathcal{U}(\mathfrak{sl}_2)\otimes B_r(3)$ submodule with this $\mathfrak{sl}_2$ highest weight. This implies both parts (2) and (3). □

As a $BMW_r(\mathcal{K})$-module, $I_q(2l)$ is the direct sum of $\dim_{\mathcal{K}} V_q(2l)$ copies of a single irreducible $BMW_r(q)$-module, which we refer to as $L_q^{BMW}(2l)$. Similarly, $I(2l)$ is the direct sum of $\dim_{\mathbb{C}} V(2l)$ copies of an irreducible $B_r(3)$-module $L^{Br}(2l)$. Recall that both $\dim_{\mathcal{K}} V_q(2l)$ and $\dim_{\mathbb{C}} V(2l)$ are equal to $2l+1$.

Lemma 7.7. *With notation as above, the irreducible $BMW_r(q)$-module $L_q^{BMW}(2l)$ has the same dimension as that of the irreducible $B_r(3)$-module $L^{Br}(2l)$.*

Proof. If $l \neq l'$, $I(2l)$ and $I(2l')$ intersect trivially since they are isotypical components with different highest weights. Thus, $\sum_l \dim I_q(2l) = 3^r = \sum_l \dim I(2l)$. But the specialisation argument of Proposition 7.1 (5) shows that $\dim_{\mathcal{K}} I_q(2l) \geq \dim_{\mathbb{C}} I(2l)$, whence

$$\dim_{\mathcal{K}} I_q(2l) = \dim_{\mathbb{C}} I(2l).$$

Thus

$$\dim_{\mathcal{K}} L_q^{BMW}(2l) = \frac{\dim_{\mathcal{K}} I_q(2l)}{2l+1} = \frac{\dim_{\mathbb{C}} I(2l)}{2l+1} = \dim_{\mathbb{C}} L^{Br}(2l).$$

□

Denote by $\mathcal{R}(\mathcal{K})$ the radical of the BMW algebra $BMW_r(\mathcal{K})$ and let $\overline{BMW_r}(\mathcal{K}) = BMW_r(\mathcal{K})/\mathcal{R}(\mathcal{K})$ be its largest semisimple quotient. Then as explained in Sect. 5, $\overline{BMW_r}(\mathcal{K}) = \bigoplus_{\lambda\in\Lambda} \overline{B}_{\mathcal{K}}(\lambda)$ with $\overline{B}_{\mathcal{K}}(\lambda) \cong \mathrm{End}_{\mathcal{K}}(L_{\mathcal{K}}(\lambda))$, where $L_{\mathcal{K}}(\lambda)$ is the simple head of the cell module $W_{\mathcal{K}}(\lambda)$. As in Lemma 5.1, the surjective algebra homomorphism $\eta_q : BMW_r(\mathcal{K}) \to \mathrm{End}_{\mathcal{U}_q(\mathfrak{sl}_2)}(V_q^{\otimes r})$ induces a surjection $\overline{\eta}_q : \overline{BMW_r}(\mathcal{K}) \to \mathrm{End}_{\mathcal{U}_q(\mathfrak{sl}_2)}(V_q^{\otimes r})$.

Similarly, let $\overline{B}_r(3)$ denote the largest semi-simple quotient of the Brauer algebra. Then $\overline{B}_r(3) = \bigoplus_{\lambda\in\Lambda} \overline{B}(\lambda)$ with $\overline{B}(\lambda) \cong \mathrm{End}_{\mathbb{C}}(L(\lambda))$, where $L(\lambda)$ is the simple head of the cell module $W(\lambda)$. Let $\overline{\eta} : \overline{B}_r(3) \to \mathrm{End}_{\mathfrak{sl}_2}(V^{\otimes r})$ be the surjection induced by the map $\eta : B_r(3) \to \mathrm{End}_{\mathfrak{sl}_2}(V^{\otimes r})$.

Recall that in analogy with (5.2), we have

$$BMW_r(q) = \bigoplus_{\lambda\in\Lambda} B_{\mathcal{A}_q}(\{\lambda\}), \quad \text{where} \quad B_{\mathcal{A}_q}(\{\lambda\}) = \sum_{S,T\in M(\lambda)} \mathcal{A}_q C_{S,T}^{\lambda}. \tag{7.10}$$

Taking appropriate tensor products with $\mathcal{K}$ and $\mathbb{C}$, respectively, and writing $B_{\mathbb{C}}(\{\lambda\})$ for what was denoted $B(\{\lambda\})$ in Sects. 5 and 6, we obtain

$$BMW_r(\mathcal{K}) = \bigoplus_{\lambda\in\Lambda} B_{\mathcal{K}}(\{\lambda\}), \text{ where } B_{\mathcal{K}}(\{\lambda\}) = \sum_{S,T\in M(\lambda)} \mathcal{K} C_{S,T}^{\lambda}, \text{ and}$$
$$BMW_r(\mathbb{C}) = \bigoplus_{\lambda\in\Lambda} B_{\mathbb{C}}(\{\lambda\}), \text{ where } B_{\mathbb{C}}(\{\lambda\}) = \sum_{S,T\in M(\lambda)} \mathbb{C} C_{S,T}^{\lambda}. \tag{7.11}$$

Proposition 7.8. *Maintain the above notation. Then* $\overline{\eta}_q(\overline{B}_q(\lambda)) \neq 0$ *if and only if* $\overline{\eta}(\overline{B}(\lambda)) \neq 0$. *For such* λ, *we have* $\dim_{\mathcal{K}} \overline{B}_q(\lambda) = \dim_{\mathbb{C}} \overline{B}(\lambda)$.

Proof. Let $I_q^{\mathrm{reg}}(\lambda) := B_{\mathcal{A}_q}(\{\lambda\})(V_q^{\mathrm{reg}}(2)^{\otimes r})$, where $B_{\mathcal{A}_q}(\{\lambda\})$ is defined by equation (7.10). Set $I_q(\lambda) := \mathcal{K} \otimes_{\phi_q} I_q^{\mathrm{reg}}(\lambda)$ and $I(\lambda) := \mathbb{C} \otimes_{\phi_1} I_q^{\mathrm{reg}}(\lambda)$. Then

$$I(\lambda) = B_{\mathbb{C}}(\{\lambda\})(\mathbb{C} \otimes_{\phi_1} V_q^{\mathrm{reg}}(2)^{\otimes r}).$$

Note that $I_q(\lambda)$ is a $\mathcal{U}_q(\mathfrak{sl}_2)$-isotypic component of $V_q(2)^{\otimes r}$, and $I(\lambda)$ is isomorphic to an $\mathfrak{sl}_2$-isotypic component of $V(2)^{\otimes r}$. The $\mathcal{U}_q(\mathfrak{sl}_2)$-highest weight of $I_q(\lambda)$ is equal to the $\mathfrak{sl}_2$-highest weight of $I(\lambda)$.

If $\overline{\eta}_q(\overline{B}_q(\lambda)) = 0$, then $I_q(\lambda) = 0$. In this case, $I(\lambda) = 0$ and this is equivalent to $\overline{\eta}(\overline{B}(\lambda)) = 0$. If $\overline{\eta}_q(\overline{B}_q(\lambda)) \neq 0$, then $I_q(\lambda) \neq 0$, and it follows from Remark 7.4 that $I(\lambda) \neq 0$. Therefore, $\overline{\eta}(\overline{B}(\lambda)) \neq 0$.

With the first statement of the Proposition established, the second follows immediately from Lemma 7.7. □

Recall that Λ^0 denotes the set of all partitions with three or fewer boxes in the first two columns, and $\Lambda^1 = \Lambda(r)\backslash\Lambda_0$. We have the following analogue of Theorem 6.9.

Theorem 7.9. *Let* $N_{\mathcal{K}}$ *be the kernel of the surjective map* $\eta_q : BMW_r(\mathcal{K}) \to \mathrm{End}_{\mathcal{U}_q(\mathfrak{sl}_2)}(V_q(2)^{\otimes r})$. *Denote by* $\mathcal{P}_{\mathcal{K}}$ *the two-sided ideal of* $BMW_r(\mathcal{K})$ *generated by* Φ_q, *and by* $\mathcal{R}_{\mathcal{K}}$ *the radical of* $BMW_r(\mathcal{K})$.

(1) $\mathrm{End}_{\mathcal{U}_q(\mathfrak{sl}_2)}(V_q(2)^{\otimes r}) \cong \bigoplus_{\lambda\in\Lambda^0} \overline{B}_q(\lambda)$.
(2) *If* $\lambda \in \Lambda^1$, *then* $\Phi_q(L_q(\lambda)) \neq 0$.
(3) $N_{\mathcal{K}}$ *acts trivially on* $L_q(\lambda)$ *if and only if* $\lambda \in \Lambda^0$.
(4) $\mathcal{P}_{\mathcal{K}} + \mathcal{R}_{\mathcal{K}} = N_{\mathcal{K}}$.

Proof. Part (1) is an easy corollary of Proposition 7.8 in view of Theorem 6.9 (3).

For any $\lambda \in \Lambda^1$, $B_r(3)\Phi W(\lambda) = W(\lambda)$ by Theorem 6.9 (1) and the cyclic property of $W(\lambda)$. Since $B_r(3)\Phi W(\lambda) \cong \mathbb{C} \otimes_{\phi_1} BMW_r(\mathcal{K})\Phi_q W_{\mathcal{A}_q}(\lambda)$ and $W(\lambda) \cong \mathbb{C} \otimes_{\phi_1} W_{\mathcal{A}_q}(\lambda)$, it follows that $BMW_r(\mathcal{K})\Phi_q W_{\mathcal{A}_q}(\lambda) = W_{\mathcal{A}_q}(\lambda)$ since $\mathcal{K} \otimes_{\mathcal{A}_q} W_{\mathcal{A}_q}(\lambda)$ and $W(\lambda)$ have the same dimensions. This implies part (2).

The proof of parts (3) and (4) is essentially the same as that of Theorem 6.9 (2), (4), and will be omitted. □

Remark 7.10. (1) Although Theorem 7.9 has been stated over $\mathcal{K}$, it is clear that the statements (1)–(4) hold integrally, i.e. if we replace $\mathcal{K}$ by $\mathcal{A}_q$ and all $\mathcal{K}$ vector spaces by the corresponding free $\mathcal{A}_q$-modules.

(2) Note that (2) and (3) of Theorem 7.9 imply that $\Phi_q(L_q(\lambda)) = 0$ if and only if $\lambda \in \Lambda^0$. This is because $\Phi_q \in N_q$ implies (by (3)) that Φ_q acts trivially on $L_q(\lambda)$ for $\lambda \in \Lambda^0$, while (3) shows that $\Phi_q(L_q(\lambda)) \neq 0$ for $\lambda \in \Lambda \setminus \Lambda^0$.

The final result of this section is that to determine whether Φ_q generates N_q, it suffices to check the classical case.

Proposition 7.11. *With notation as in Theorem 7.9, if* $\langle\Phi\rangle = \mathcal{P}$ *contains* $\mathcal{R}$*, the radical of* $B_r(3)$*, then* $\langle\Phi_q\rangle_{BMW_r(\mathcal{K})} = \mathcal{P}_{\mathcal{K}}$ *contains the radical* $\mathcal{R}_{\mathcal{K}}$ *of* $BMW_r(\mathcal{K})$*.*

Proof. We have already noted that by [X, Theorem 3.11], $BMW_r(\mathcal{K})$ is cellular, with the canonical anti-involution being defined by $g_i^* = g_i$ and $e_i^* = e_i$. It follows that $\Phi_q^* = \Phi_q$, and hence that $\mathcal{P}_{\mathcal{K}}$ is a self-dual two-sided ideal of $BMW_r(\mathcal{K})$. Hence, we may apply Theorem 5.4 to deduce that $\mathcal{P}_{\mathcal{K}} \supseteq \mathcal{R}_{\mathcal{K}}$ if and only if $\mathcal{P}_{\mathcal{K}} W_{\mathcal{K}}(\lambda) = R_{\mathcal{K}}(\lambda)$ for each $\lambda \in \Lambda^0$, where Λ^0 is as in Theorem 7.9.

Now we are given that $\mathcal{P} \supseteq \mathcal{R}$, whence $\mathcal{P}W(\lambda) = R(\lambda)$, where $W(\lambda) = W_{\mathbb{C}}(\lambda)$, etc. Write $\mathcal{P}_q := \langle\Phi_q\rangle_{BMW_r(q)}$.

Let $\lambda \in \Lambda^0$ and consider the $\mathcal{A}_q$-submodule $\mathcal{P}_q W_{\mathcal{A}_q}(\lambda)$ of $R_{\mathcal{A}_q}(\lambda)$. For any element $r \in R_{\mathcal{A}_q}(\lambda)$, since $1 \otimes_{\phi_1} r \in 1 \otimes_{\phi_1} \mathcal{P}_q W_{\mathcal{A}_q}(\lambda)$, there exist elements $r_0 \in \mathcal{P}_q W_{\mathcal{A}_q}(\lambda)$ and $w = (q-1)r_1 \in (q-1)W_{\mathcal{A}_q}(\lambda)$ such that $r = r_0 + w$. But $\mathcal{P}_q W_{\mathcal{A}_q}(\lambda) \subseteq R_{\mathcal{A}_q}(\lambda)$ by Theorem 7.9 (3), whence $w \in R_{\mathcal{A}_q}(\lambda)$, and so evidently $r_1 \in R_{\mathcal{A}_q}(\lambda)$, since r_1 has zero inner product with $W_{\mathcal{A}_q}(\lambda)$.

It follows that multiplication by $q - 1$ is an invertible endomorphism of the quotient $R_q(\lambda)/\mathcal{P}_q W_{\mathcal{A}_q}(\lambda)$, whence the latter is an $\mathcal{A}_q$-torsion module. Hence $\mathcal{K} \otimes_{\phi_q} (R_q(\lambda)/\mathcal{P}_q W_{\mathcal{A}_q}(\lambda)) = 0$, i.e. $R_{\mathcal{K}}(\lambda) = \mathcal{P}_{\mathcal{K}} W_{\mathcal{K}}(\lambda)$. □

Corollary 7.12. (1) *With the above notation, if* Φ *generates* N *then* Φ_q *generates* $N_{\mathcal{K}}$*.*
(2) *If* $r \leq 6$*, then* Φ_q *generates* $N_{\mathcal{K}}$ *as an ideal of* $BMW_r(\mathcal{K})$*.*

The first statement is evident from Proposition 7.11, while the second follows from the first, together with Theorem 6.12.

We end this section by noting that our results imply the quantum analogues of the results of Sect. 6.

Corollary 7.13. (1) *The algebra* $BMW_4(\mathcal{K})$ *is semi-simple.*
(2) *The cell modules of* $BMW_5(\mathcal{K})$ *are all simple except for those corresponding to the partitions* $(2, 1)$ *and* (1^3)*, whose simple heads have dimensions* $15, 6$*, respectively.*
(3) *The cell modules* $W_{\mathcal{K}}(1^3)$ *and* $W_{\mathcal{K}}(2, 1)$ *have two composition factors each.*
(4) *The radical of* $BMW_5(\mathcal{K})$ *has dimension* 239*.*

Proof. All statements are easy consequences of Proposition 7.1. For example, it follows from *loc. cit.* (5) and (6) that if $W^{\phi}(\lambda)$ has just two composition factors, then either $W_q(\lambda)$ is irreducible, or it also has two composition factors, whose dimensions are the same as those of $W^{\phi}(\lambda)$. This implies the statements (3)–(5) above. □

8 A BMW-Analogue of the Temperley–Lieb Algebra

Although implicit above, we complete this work with an explicit definition of our analogue of the Temperley–Lieb algebra, together with some of its properties, as well as some questions about it.

Definition 8.1. Let $\mathcal{A}_q$ be the ring $\mathbb{C}[q^{\pm 1}, [3]_q{}^{-1}, (q+q^{-1})^{-1}, (q^2+q^{-2})^{-1}]$. The $\mathcal{A}_q$-algebra $P_r(q)$ has generators $\{g_i^{\pm 1}, e_i \mid i = 1, \ldots, r-1\}$ and relations given by (4.14) together with $\Phi_q = 0$, where Φ_q is the word in the generators defined in (7.8). We reproduce the relations here for convenience.

$$\begin{aligned}
g_i g_j &= g_j g_i \text{ if } |i-j| \geq 2 \\
g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} \text{ for } 1 \leq i \leq r-1 \\
g_i - g_i{}^{-1} &= (q^2 - q^{-2})(1 - e_i) \text{ for all } i \\
g_i e_i &= e_i g_i = q^{-4} e_i \\
e_i g_{i-1}^{\pm 1} e_i &= q^{\pm 4} e_i \\
e_i g_{i+1}^{\pm 1} e_i &= q^{\pm 4} e_i
\end{aligned}$$

$$\Phi_q = aF_q e_2 F_q - bF_q - cF_q e_2 e_{14} F_q + dF_q e_{1234} F_q = 0,$$

where

$$\begin{aligned}
f_i &= -g_i - (1 - q^{-2}) e_i + q^2, \\
F_q &= f_1 f_3, \\
e_{14} &= g_3^{-1} g_1 e_2 g_1^{-1} g_3, \\
e_{1234} &= e_2 g_1 g_3^{-1} g_2 g_1^{-1} g_3, \\
a &= 1 + (1 - q^{-2})^2, \ \tilde{a} = 1 + (1 - q^2)^2, \\
d &= (q - q^{-1})^2 = q^2(a-1) = q^{-2}(\tilde{a} - 1), \\
b &= 1 + (1 - q^2)^2 + (1 - q^{-2})^2, \\
c &= \frac{1 + (2 + q^{-2})(1 - q^{-2})^2 + (1 + q^2)(1 - q^{-2})^4}{([3]_q - 1)^2}.
\end{aligned}$$

8.1 *Properties of $P_r(q)$*

Let $\phi_q : \mathcal{A}_q \hookrightarrow \mathcal{K}(= \mathbb{C}(q^{\frac{1}{2}}))$ be the inclusion map, and let $\phi_1 : \mathcal{A}_q \longrightarrow \mathbb{C}$ be defined by $\phi_1(q) = 1$. Write $P_r(\mathcal{K}) := \mathcal{K} \otimes_{\phi_q} P_r(q)$, and $P_r(\mathbb{C}) := \mathbb{C} \otimes_{\phi_1} P_r(q)$. Then there are surjective homomorphisms

$$\begin{aligned}
\eta_q &: P_r(\mathcal{K}) \longrightarrow \mathrm{End}_{\mathcal{U}_q(\mathfrak{sl}_2)} V_q(2)^{\otimes r}, \text{ and} \\
\eta &: P_r(\mathbb{C}) \longrightarrow \mathrm{End}_{\mathfrak{sl}_2(\mathbb{C})} V(2)^{\otimes r},
\end{aligned} \tag{8.1}$$

where $V(2)$ is the two-dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$-module and $V_q(2)$ is its quantum analogue.

Moreover, it follows from Theorem 7.9 (4) and that $\mathrm{Ker}(\eta_q)$ is the radical of $P_r(\mathcal{K})$, i.e. that $\mathrm{End}_{\mathcal{U}_q(\mathfrak{sl}_2)} V_q(2)^{\otimes r}$ is the largest semi-simple quotient of $P_r(\mathcal{K})$. A similar statement applies to $P_r(\mathbb{C})$.

8.2 Some Open Problems

We finish with some problems relating to $P_r(q)$.

(1) Determine whether $P_r(q)$ is generically semi-simple, in particular whether $P_r(\mathcal{K})$ is semi-simple. By Proposition 7.11, this is true provided that $P_r(\mathbb{C})$ is semi-simple. The latter algebra has been shown (Proposition 6.11) to be semi-simple for $r \leq 5$ and the case $r = 6$ has been verified by computer.

(2) A question equivalent to (1) is to determine whether $P_r(\mathcal{K})$ has dimension given by the formula (2.5). More explicitly, we know that

$$\dim_{\mathcal{K}} P_r(\mathcal{K}) \geq \binom{2r}{r} + \sum_{p=0}^{r-1} \binom{2r}{2p}\binom{2p}{p}\frac{3p-2r+1}{p+1}, \tag{8.2}$$

with equality if and only if the ideal of $BMW_r(\mathcal{K})$ which is generated by Φ_q contains the radical $\mathcal{R}(\mathcal{K})$ of $BMW_r(\mathcal{K})$.

We therefore ask whether equality holds in (8.2).

(3) Is $P_r(q)$ free as $\mathcal{A}_q$-module?

(4) Determine whether $P_r(q)$ has a natural cellular structure.

(5) Generalise the program of this work to higher dimensional representations of quantum $\mathfrak{sl}_2$.

Finally, we note that an affirmative answer to Conjecture 6.7 implies an affirmative answer to both (1) and (2) above.

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Graded Lie Algebras and Intersection Cohomology

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To Toshiaki Shoji on the occasion of his 60th birthday

Abstract Let ι be a homomorphism of the multiplicative group into a connected reductive algebraic group over $\mathbf{C}$. Let G^ι be the centralizer of the image ι. Let LG be the Lie algebra of G and let L_nG (n integer) be the summands in the direct sum decomposition of LG determined by ι. Assume that n is not zero. For any G^ι-orbit $\mathcal{O}$ in L_nG and any irreducible G^ι-equivariant local system $\mathcal{L}$ on $\mathcal{O}$ we consider the restriction of some cohomology sheaf of the intersection cohomology complex of the closure of $\mathcal{O}$ with coefficients in $\mathcal{L}$ to another orbit $\mathcal{O}'$ contained in the closure of $\mathcal{O}$. For any irreducible G^ι-equivariant local system $\mathcal{L}'$ on $\mathcal{O}'$ we would like to compute the multiplicity of $\mathcal{L}'$ in that restriction. We present an algorithm which helps in computing that multiplicity.

Keywords Perverse sheaf · Graded Lie algebra · Canonical basis

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1 Introduction

Let (G, ι) be a pair consisting of a reductive connected algebraic group G over $\mathbf{C}$ and a homomorphism of algebraic groups $\iota : \mathbf{C}^* \to G$. The centralizer G^ι of $\iota(\mathbf{C}^*)$ in G acts naturally (with finitely many orbits) on the n-eigenspace L_nG of $\mathrm{Ad}(\iota(\mathbf{C}^*))$ on the Lie algebra of G. (Here $n \in \mathbf{Z} - \{0\}$.) If $\mathcal{O}$ is a G^ι-orbit on L_nG and $\mathcal{L}$ is a G^ι-equivariant irreducible local system on $\mathcal{O}$, then the intersection cohomology complex $K = IC(\bar{\mathcal{O}}, \mathcal{L})$ is defined and we are interested in the problem of

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computing, for any G^ι-orbit $\mathcal{O}'$ contained in $\bar{\mathcal{O}}$ and any G^ι-equivariant irreducible local system $\mathcal{L}'$ on $\mathcal{O}'$, the multiplicity $m_{i;\mathcal{L},\mathcal{L}'}$ of $\mathcal{L}'$ in the local system obtained by restricting to $\mathcal{O}'$ the ith cohomology sheaf of K.

The main purpose of this paper is to give an algorithm to produce combinatorially a square matrix whose entries are polynomials with coefficients given by the multiplicities $m_{i;\mathcal{L},\mathcal{L}'}$. Note that we do not have a purely combinatorial proof of the fact that the algorithm does not break down. We can only prove that using geometry. But this will not prevent a computer from carrying out the algorithm.

The method of this paper relies heavily on [L5] where many of the needed geometric results are proved. Note that in [L5], another purely algebraic description of the multiplicities above was obtained, which, however, did not provide an algorithm for computing them.

While the existence of the algorithm above has an intrinsic interest, it also implies (by results in [CG], [L2, 10.7]) a solution of a problem in representation theory, namely that of computing the multiplicities with which simple modules of an affine Hecke algebra appear in a composition series of certain "standard modules."

At the same time, as a biproduct of the algorithm, we find a way to compute the dimensions of weight spaces in certain standard modules over an affine Hecke algebra, (see 5.6).

In Sect. 2, we describe the algorithm. In Sect. 4, we show (based on the geometric preliminaries in Sect. 3) that the algorithm in Sect. 1 is correct and it indeed leads to the desired matrix of multiplicities. In Sect. 5, we define among other things a partial order on the set of isomorphism classes of irreducible G^ι-equivariant local systems on the various orbits in L_nG. In 5.7, we give a formulation of our results in terms of a canonical basis and two PBW-bases which generalizes the theory of canonical bases [L4] in the plus part of a quantized enveloping algebra of type A_n.

Notation. The cardinal of a finite set S is denoted by $|S|$.

Let $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$ where v is an indeterminate.

Let $\mathbf{k}$ be an algebraically closed field of characteristic $p \geq 0$. All algebraic varieties are assumed to be over $\mathbf{k}$.

We fix a prime number l invertible in $\mathbf{k}$. Let $\bar{\mathbf{Q}}_l$ be an algebraic closure of the field of l-adic numbers. We will say "local system" instead of "$\bar{\mathbf{Q}}_l$-local system." If $\mathcal{F}$ is an irreducible local system (or its isomorphism class) over a subvariety Y of an algebraic variety X, we set $S_{\mathcal{F}} = Y$.

For a connected affine algebraic group H, let U_H be the unipotent radical of H, $\underline{H} = H/U_H$, LH the Lie algebra of H, and Z_H the centre of H.

2 An Algorithm

2.1 Throughout this paper we assume that we are given a connected reductive algebraic group G and a homomorphism of algebraic groups $\iota : \mathbf{k}^* \to G$. We assume that either $p = 0$ or p is sufficiently large (as in the last paragraph of [L5, 2.1 (a)]).

We set $G^\iota = \{g \in G; g\iota(t) = \iota(t)g \quad \forall t \in \mathbf{k}^*\}$, a connected reductive subgroup of G. We have $LG = \bigoplus_{n\in\mathbf{Z}} L_n G$, where

$$L_n G = \{x \in LG; \mathrm{Ad}(\iota(t))x = t^n x \quad \forall t \in \mathbf{k}^*\}.$$

More generally, for any closed connected subgroup H of G that is normalized by $\iota(\mathbf{k}^*)$, we set $H^\iota = H \cap G^\iota$; we have $LH = \bigoplus_{n\in\mathbf{Z}} L_n H$ where $L_n H = LH \cap L_n G$. For $n \in \mathbf{Z}$, the adjoint action of G on LG restricts to an action of G^ι on $L_n G$.

2.2 In the remainder of this paper, we fix a subset Δ of $\mathbf{Z}$ consisting of two non-zero integers whose sum is 0. We assume that either $p = 0$ or $\Delta \subset (-p, p)$.

We say that (G, ι) is *rigid* if for some/any $n \in \Delta$ there exists a homomorphism of algebraic groups $\gamma : SL_2(\mathbf{k}) \to G$ such that $\gamma\left(\begin{smallmatrix} t^n & 0 \\ 0 & t^{-n}\end{smallmatrix}\right) = \iota(t^2) \mod Z_G$ for any $t \in \mathbf{k}^*$. In this case, let C_G^ι be the nilpotent G-orbit in LG such that the corresponding unipotent class in G contains $\gamma(u)$ for any unipotent element $u \in SL_2(\mathbf{k}) - \{1\}$.

Let $\mathcal{P}$ be the variety of parabolic subgroups of G. Let $\mathcal{P}^\iota = \{P \in \mathcal{P}; \iota(\mathbf{k}^*) \subset P\}$. If $P \in \mathcal{P}^\iota$, then ι gives rise to a homomorphism

$\mathbf{k}^* \to \underline{P}$, $t \mapsto$ (image of $\iota(t)$ under $P \to \underline{P}$),

denoted again by ι. Hence, $L_n\underline{P}$, $\underline{P}^\iota$ are well defined in terms of this ι; we have $L_n\underline{P} = L_n P / L_n U_P$.

2.3 Let $\mathcal{T}_G^{\mathrm{cu}}$ (resp. $\mathcal{T}_G^{\mathrm{pr}}$) be the set of isomorphism classes of G-equivariant irreducible local systems on some nilpotent orbit in LG which are *cuspidal* (resp. *primitive*) in the sense of [L1, 2.2] (resp. [L5, 2.7]).

We have $\mathcal{T}_G^{\mathrm{cu}} \subset \mathcal{T}_G^{\mathrm{pr}}$. The classification of local systems in $\mathcal{T}_G^{\mathrm{pr}}$ can be deduced from the known classification of local systems in $\mathcal{T}_G^{\mathrm{cu}}$. For example, if G is simple of type E_8, then $\mathcal{T}_G^{\mathrm{pr}}$ consists of two objects: one is in $\mathcal{T}_G^{\mathrm{cu}}$ and one is $\bar{\mathbf{Q}}_l$ over the G-orbit $\{0\}$. Let

$$\mathcal{J}_G = \{(P, \mathcal{E}); P \in \mathcal{P}^\iota, \mathcal{E} \in \mathcal{T}_{\underline{P}}^{\mathrm{cu}}, (\underline{P}, \iota) \text{ is rigid}, C_{\underline{P}}^\iota = S_\mathcal{E}\}.$$

Now G^ι acts on $\mathcal{J}_G$ by $g : (P, \mathcal{E}) \mapsto (gPg^{-1}, \mathrm{Ad}(g)_!\mathcal{E})$. Let $\underline{\mathcal{J}}_G$ be the set of orbits of this action. Let K_G be the $\mathbf{Q}(v)$-vector space with basis $(\mathbf{I}_\mathcal{S})_{\mathcal{S}\in\underline{\mathcal{J}}_G}$.

For a connected affine algebraic group $\mathcal{G}$, let $X_\mathcal{G}$ be the variety of Borel subgroups of $\mathcal{G}$ and let $\mathrm{rk}(\mathcal{G})$ be the dimension of a maximal torus of $\mathcal{G}$. We set

$$e_\mathcal{G} = \sum_j \dim H^{2j}(X_\mathcal{G}, \bar{\mathbf{Q}}_l)v^{2j}, \quad \vartheta_\mathcal{G} = (1 - v^2)^{\mathrm{rk}(\mathcal{G})} e_\mathcal{G} \in \mathbf{Z}[v^2].$$

If $\mathcal{F} \in \mathcal{T}_G^{\mathrm{pr}}$ and $\mathcal{G}$ is the connected centralizer in G of some element in $S_\mathcal{F}$, we set

$$r_\mathcal{F} = v^{-\dim X_\mathcal{G}} e_\mathcal{G} \in \mathcal{A}.$$

2.4 Let $Q \in \mathcal{P}^\iota$. Associating with the $\underline{Q}^\iota$-orbit of $(P', \mathcal{E}) \in \mathcal{J}_{\underline{Q}}$, the G^ι-orbit of $(P, \mathcal{E}) \in \mathcal{J}_G$ (where P is the inverse image of P' under $Q \to \underline{Q}$ and $\underline{P}', \underline{P}$ are identified in the obvious way) defines a map $a_Q^G : \underline{\mathcal{J}}_{\underline{Q}} \to \underline{\mathcal{J}}_G$. We define a $\mathbf{Q}(v)$-linear map $f_Q^G : K_{\underline{Q}} \to K_G$ by $\mathbf{I}_{\mathcal{S}'} \mapsto \mathbf{I}_{a_Q^G(\mathcal{S}')}$ for any $\mathcal{S}' \in \underline{\mathcal{J}}_{\underline{Q}}$.

2.5 We define a map $\mu : \mathcal{J}_G \to \mathcal{T}_G^{\mathrm{pr}}$ by $(P, \mathcal{E}) \mapsto \mathcal{F}$ where $\mathcal{F}$ is as follows. We choose a Levi M of P and identify M with $\underline{P}$ in the obvious way. Then $\mathcal{E}$ becomes a local system $\mathcal{E}_M$ on a nilpotent M-orbit D in LM. Let C be the nilpotent G-orbit in LG that contains D, and let $\mathcal{F}$ be the unique G-equivariant local system on C such that $\mathcal{F}|_D = \mathcal{E}_M$. (See [L5, 2.7].) Then $\mathcal{F} \in \mathcal{T}_G^{\mathrm{pr}}$ is clearly independent of the choice of M.

For $\mathcal{F} \in \mathcal{T}_G^{\mathrm{pr}}$, let $\mathcal{J}_G^{\mathcal{F}} = \mu^{-1}(\mathcal{F})$. Then $\mathcal{J}_G^{\mathcal{F}}$ is G^ι-stable; let $\underline{\mathcal{J}}_G^{\mathcal{F}}$ be the set of G^ι-orbits on $\mathcal{J}_G^{\mathcal{F}}$. We have a partition $\underline{\mathcal{J}}_G = \bigsqcup_{\mathcal{F}} \underline{\mathcal{J}}_G^{\mathcal{F}}$ and a direct sum decomposition

$$K_G = \bigoplus_{\mathcal{F}} K_G^{\mathcal{F}},$$

where $\mathcal{F}$ runs over $\mathcal{T}_G^{\mathrm{pr}}$ and $K_G^{\mathcal{F}}$ is the subspace of K_G spanned by $\{\mathbf{I}_{\mathcal{S}}; \mathcal{S} \in \underline{\mathcal{J}}_G^{\mathcal{F}}\}$.

For $\mathcal{F} \in \mathcal{T}_G^{\mathrm{pr}}$, let $Y^{\mathcal{F}}$ be the set of all $((P, \mathcal{E}), (P', \mathcal{E}')) \in \mathcal{J}_G^{\mathcal{F}} \times \mathcal{J}_G^{\mathcal{F}^*}$ such that P, P' have a common Levi. Now G^ι acts diagonally on $Y^{\mathcal{F}}$; let $\underline{Y}^{\mathcal{F}}$ be the set of orbits. Define $\tau : \underline{Y}^{\mathcal{F}} \to \mathbf{Z}$ by

$$\Omega \mapsto \tau(\Omega) = \dim \frac{L_n U_{P'} + L_n U_P}{L_n U_{P'} \cap L_n U_P} - \dim \frac{L_0 U_{P'} + L_0 U_P}{L_0 U_{P'} \cap L_0 U_P},$$

where $((P, \mathcal{E}), (P', \mathcal{E}'))$ is any element of the G^ι-orbit Ω and $n \in \Delta$. (The fact that this definition is independent of the choice of n in Δ is seen as in [L5, 16.3].)

2.6 For $(P, \mathcal{E}) \in \mathcal{J}_G^{\mathcal{F}}$, we choose a Levi M of P that contains $\iota(\mathbf{k}^*)$; let $\tilde{P}$ be the unique parabolic subgroup with Levi M such that $P \cap \tilde{P} = M$. We have $(\tilde{P}, \tilde{\mathcal{E}}) \in \mathcal{J}_G^{\mathcal{F}}$ for a unique $\tilde{\mathcal{E}}$. Although $\tilde{P}$ is not uniquely defined by P, its G^ι-orbit is uniquely defined by the G^ι-orbit of $(P, \mathcal{E})$ (since M is uniquely defined by P up to the conjugation action of U_P^ι). Thus, $(P, \mathcal{E}) \mapsto (\tilde{P}, \tilde{\mathcal{E}})$ induces a well defined involution $\mathcal{S} \mapsto \tilde{\mathcal{S}}$ of $\underline{\mathcal{J}}_G^{\mathcal{F}}$.

Similarly, for $((P, \mathcal{E}), (P', \mathcal{E}')) \in Y^{\mathcal{F}}$, we choose a common Levi M of P and P' that contains $\iota(\mathbf{k}^*)$ and let $\tilde{P}$ be the unique parabolic subgroup with Levi M such that $P \cap \tilde{P} = M$. We have $((\tilde{P}, \tilde{\mathcal{E}}), (P', \mathcal{E}')) \in Y^{\mathcal{F}}$ for a unique $\tilde{\mathcal{E}}$. Again the G^ι-orbit of $((\tilde{P}, \tilde{\mathcal{E}}), (P', \mathcal{E}'))$ is uniquely defined by the G^ι-orbit of $((P, \mathcal{E}), (P', \mathcal{E}'))$ (since M is uniquely defined by P, P' up to the conjugation action of $U_{P \cap P'}^\iota$). Thus, $((P, \mathcal{E}), (P', \mathcal{E}')) \mapsto ((\tilde{P}, \tilde{\mathcal{E}}), (P', \mathcal{E}'))$ induces a well defined involution $\Omega \mapsto \tilde{\Omega}$ of $\underline{Y}^{\mathcal{F}}$. As in the proof of [L5, 16.4 (c)], we have

$$\tau(\Omega) + \tau(\tilde{\Omega}) = c_{\mathcal{F}} \tag{a}$$

where

$$c_{\mathcal{F}} = \dim L_n G - \dim L_0 G - \dim L_n \underline{P} + \dim L_0 \underline{P}$$

for any $(P, \mathcal{E}) \in \mathcal{J}_G^{\mathcal{F}}$. (If $(P, \mathcal{E}')$ is another pair in $\mathcal{J}_G^{\mathcal{F}}$, then there exists an isomorphism $\underline{P} \xrightarrow{\sim} \underline{P}'$ which is compatible with ι so that $c_{\mathcal{F}}$ depends only on $\mathcal{F}$).

2.7 Let $\mathcal{F} \in \mathcal{T}_G^{\mathrm{pr}}$. In (a) and (b) below, we give a "combinatorial" interpretation of the sets $\underline{\mathcal{J}}_G^{\mathcal{F}}$ and $\underline{Y}^{\mathcal{F}}$. We may assume that $S_{\mathcal{F}} \cap L_n G \neq \emptyset$; otherwise both our sets are empty.

Let M be the centralizer in G of some maximal torus of the connected centralizer in G^ι of some element in $S_{\mathcal{F}} \cap L_n G$. Then $\iota(\mathbf{k}^*) \subset M$ and M is independent of the choices (up to G^ι-conjugacy) since, by [L5, 14.5], $S_{\mathcal{F}} \cap L_n G$ is a single G^ι-orbit. Let $\underline{X} = \{P \in \mathcal{P}; M \text{ is a Levi of } P\}$. If $P \in \underline{X}$, then $(P, \mathcal{E}) \in \mathcal{J}_G^{\mathcal{F}}$ for a unique $\mathcal{E}$, see [L5, 11.6 (c)]. We have an imbedding $\underline{X} \to \mathcal{J}_G^{\mathcal{F}}$, $P \mapsto (P, \mathcal{E})$ and an imbedding $\underline{X} \to \mathcal{J}_G^{\mathcal{F}^*}$, $P \mapsto (P, \mathcal{E}^*)$. Let $N_G M$ be the normalizer of M in G. It is known that the conjugation action of $N_G M/M$ on $\underline{X}$ is simply transitive. Note that $(N_G M)^\iota / M^\iota$ is naturally a subgroup of $N_G M/M$. We show:

(a) *The map j_1 : (set of $(N_G M)^\iota/M^\iota$-orbits on $\underline{X}$) $\to \underline{\mathcal{J}}_G^{\mathcal{F}}$ induced by the imbedding $\underline{X} \to \mathcal{J}_G^{\mathcal{F}}$ is bijective;*

(b) *The map j_2 : (set of $(N_G M)^\iota/M^\iota$-orbits on $\underline{X} \times \underline{X}$) $\to \underline{Y}^{\mathcal{F}}$ (orbits for diagonal action) induced by the imbedding $\underline{X} \times \underline{X} \to Y^{\mathcal{F}}$ is bijective.*

Let $(P, \mathcal{E}) \in \mathcal{J}_G^{\mathcal{F}}$. We can find a Levi M' of P that contains $\iota(\mathbf{k}^*)$. Using [L5, 11.4], we see that there exists $g \in G^\iota$ such that $gM'g^{-1} = M$. Then $gPg^{-1} \in \underline{X}$. We see that j_1 is surjective.

Now let $P, P' \in \underline{X}$ be such that $P' = gPg^{-1}$ for some $g \in G^\iota$. Then M and $M' = g^{-1}Mg$ are Levi subgroups of P that contain $\iota(\mathbf{k}^*)$. There is a unique $u \in U_P$ such that $uMu^{-1} = M'$. For any $t \in \mathbf{k}^*$, we set $u' = \iota(t)u\iota(t)^{-1} \in U_P$; we have

$$M' = \iota(t)M'\iota(t)^{-1} = u'(\iota(t)M\iota(t)^{-1})u'^{-1} = u'Mu'^{-1}.$$

By the uniqueness of u, we have $u' = u$. Thus $u \in U_P^\iota$. Let $g' = gu$. Then $g' \in G^\iota$, $P' = g'Pg'^{-1}$, $M = g'^{-1}Mg'$. Thus, $g' \in (N_G M)^\iota$. We see that j_1 is injective. This proves (a).

Let $((P, \mathcal{E}), (P', \mathcal{E}')) \in Y^{\mathcal{F}}$. We can find a common Levi M' of P, P' that contains $\iota(\mathbf{k}^*)$. As in the proof of (a), we can find $g \in G^\iota$ such that $gM'g^{-1} = M$. Then $(gPg^{-1}, gP'g^{-1}) \in \underline{X} \times \underline{X}$. We see that j_2 is surjective.

Now let P, P_1, P', P_1' in $\underline{X}$ be such that $P' = gPg^{-1}$, $P_1' = gP_1g^{-1}$ for some $g \in G^\iota$. Then M and $M' = g^{-1}Mg$ are Levi subgroups of $P \cap P_1$ that contain $\iota(\mathbf{k}^*)$. There is a unique $u \in U_{P \cap P_1}$ such that $uMu^{-1} = M'$. As in the proof of (a), we see using the uniqueness of u that $u \in U_{P \cap P_1}^\iota$. Let $g' = gu$. Then $g' \in G^\iota$, $P' = g'Pg'^{-1}$, $P_1' = g'P_1g'^{-1}$, $M = g'^{-1}Mg'$. Thus, $g' \in (N_G M)^\iota$. We see that j_2 is injective. This proves (b).

2.8 Define a symmetric $\mathbf{Q}(v)$-bilinear form $(:) : K_G \times K_G \to \mathbf{Q}(v)$ by setting (for $\mathcal{S} \in \underline{\mathcal{J}}_G^{\mathcal{F}}, \mathcal{S}' \in \underline{\mathcal{J}}_G^{\mathcal{F}'}$):

$$(\mathbf{I}_{\mathcal{S}} : \mathbf{I}_{\mathcal{S}'}) = 0 \text{ if } \mathcal{F}' \neq \mathcal{F}^*,$$

$$(\mathbf{I}_{\mathcal{S}} : \mathbf{I}_{\mathcal{S}'}) = \frac{\vartheta_{G^\iota}}{\vartheta_{Z^0_{\underline{P}}}} \sum_{\substack{\Omega \in \underline{Y}^{\mathcal{F}} \\ \pi_1(\Omega)=\mathcal{S}, \pi_2(\Omega)=\mathcal{S}'}} (-v)^{\tau(\Omega)} \text{ if } \mathcal{F}' = \mathcal{F}^*;$$

here $\pi_1 : \underline{Y}^{\mathcal{F}} \to \underline{\mathcal{J}}_G^{\mathcal{F}}, \pi_2 : \underline{Y}^{\mathcal{F}} \to \underline{\mathcal{J}}_G^{\mathcal{F}'}$ are the obvious projections and $P \in \mathcal{P}$ is such that $(P, \mathcal{E}) \in \mathcal{S}$ for some $\mathcal{E}$.

Let $^- : \mathbf{Q}(v) \to \mathbf{Q}(v)$ be the $\mathbf{Q}$-algebra involution such that $\overline{v^m} = v^{-m}$ for all $m \in \mathbf{Z}$. Define a $\mathbf{Q}$-linear involution $\beta : K_G \to K_G$ by

$$\beta(\rho \mathbf{I}_{\mathcal{S}}) = \overline{\rho} \mathbf{I}_{\mathcal{S}}$$

for all $\rho \in \mathbf{Q}(v), \mathcal{S} \in \underline{\mathcal{J}}_G$. Define a $\mathbf{Q}(v)$-linear involution $\sigma : K_G^{\mathcal{F}} \to K_G^{\mathcal{F}}$ by $\sigma(\mathbf{I}_{\mathcal{S}}) = \mathbf{I}_{\tilde{\mathcal{S}}}$ (see 2.6) for all $\mathcal{S} \in \underline{\mathcal{J}}_G^{\mathcal{F}}$. From 1.6 (a), we see that for $\xi \in K_G^{\mathcal{F}}$, $\xi' \in K_G^{\mathcal{F}'}$, we have

$$\overline{(\beta(\xi) : \beta(\xi'))} = (-v)^{c_{\mathcal{F}}} (\sigma(\xi) : \xi'). \tag{a}$$

Let

$$\mathcal{R}_G = \{\xi \in K_G ; (\xi : K_G) = 0\}.$$

From (a) we see that $\beta(\mathcal{R}_G) \subset \mathcal{R}_G$. Clearly, if $Q \in \mathcal{P}^\iota$, then for $\xi \in K_{\underline{Q}}$ we have

$$f_Q^G(\beta(\xi)) = \beta(f_Q^G(\xi)). \tag{b}$$

Moreover,

$$f_Q^G(\mathcal{R}_{\underline{Q}}) \subset \mathcal{R}_G. \tag{c}$$

See 4.5 for a proof.

2.9 Let $n \in \Delta$. Let $P \in \mathcal{P}^\iota$ be such that $(\underline{P}, \iota)$ is rigid. We can find a Levi subgroup M of P such that $\iota(\mathbf{k}^*) \subset M$. Let s be the unique element in $[LM, LM]$ such that $[s, x] = mx$ for any $m \in \mathbf{Z}$, $x \in L_m M$. We have $LG = \bigoplus_{r \in (n/2)\mathbf{Z}} L^r G$ where $L^r G = \{x \in LG ; [s, x] = rx\}$ and $LG = \bigoplus_{r \in (n/2)\mathbf{Z}, t \in \mathbf{Z}} L_t^r G$ where $L_t^r G = L^r G \cap L_t G$. We say that P (as above) is n-good if

$$LU_P = \bigoplus_{r \in (n/2)\mathbf{Z}, t \in \mathbf{Z}; 2t/n < 2r/n} L_t^r G.$$

(This implies that

$LM = \bigoplus_{r \in (n/2)\mathbf{Z}, t \in \mathbf{Z}; 2t/n = 2r/n} L_t^r G$, $LP = \bigoplus_{r \in (n/2)\mathbf{Z}, t \in \mathbf{Z}; 2t/n \leq 2r/n} L_t^r G$.)
Note that the condition that P is n-good is independent of the choice of M.

Let $\mathfrak{P}_n$ be the set of all $P \in \mathcal{P}^\iota$ such that $(\underline{P}, \iota)$ is rigid and P is n-good. Let $\mathfrak{P}'_n = \{P \in \mathfrak{P}_n; P \neq G\}$. Now G^ι acts on $\mathfrak{P}_n, \mathfrak{P}'_n$ by conjugation. Let $\underline{\mathfrak{P}}_n, \underline{\mathfrak{P}}'_n$ be the sets of orbits of these actions. These are finite sets since G^ι acts with finitely many orbits on $\mathcal{P}^\iota$. We have $G \in \mathfrak{P}_n$ if and only if (G, ι) is rigid. Hence, $\underline{\mathfrak{P}}'_n = \underline{\mathfrak{P}}_n$ if (G, ι) is not rigid, $\underline{\mathfrak{P}}_n = \underline{\mathfrak{P}}'_n \sqcup \{G\}$ if (G, ι) is rigid.

2.10 Let $n \in \Delta$. For $\eta \in \underline{\mathfrak{P}}_n$, we set

$$d_\eta = \dim L_0 G - \dim L_0 P + \dim L_n P,$$

where $P \in \eta$. For $\eta, \eta' \in \underline{\mathfrak{P}}'_n$, we say that $\eta' \prec \eta$ if $d_{\eta'} < d_\eta$. We say that $\eta' \preceq \eta$ if either $\eta = \eta'$ or $\eta' \prec \eta$. Now $\preceq$ is a partial order on $\underline{\mathfrak{P}}'_n$.

2.11 Our goal is to define subsets $\mathcal{Z}_n^\eta$ of K_G (for $n \in \Delta$ and $\eta \in \underline{\mathfrak{P}}_n$). The definition of these subsets is inductive and is based on a number of lemmas which will be verified in Sect. 4 (where we assume, as we may, that $\mathbf{k}$ is an algebraic closure of a finite field). If G is a torus, then η must be $\{G\}$ and $\mathcal{Z}_n^\eta$ consists of the unique basis element of K_G. We now assume that G is not a torus, and that the subsets $\mathcal{Z}_n^\eta$ are already defined when G is replaced by any $\underline{P}$ with $P \in \mathfrak{P}'_n$.

Our definition is based on the following scheme:

(i) We first define $\mathcal{Z}_n^\eta$ in the case where $\eta \in \underline{\mathfrak{P}}'_n$ by

$$\mathcal{Z}_n^\eta = f_P^G(\mathcal{Z}_n^{\{\underline{P}\}}),$$

where $P \in \eta$ and f_P^G is as in 2.4. (Note that $\mathcal{Z}_n^{\{\underline{P}\}} \subset K_{\underline{P}}$ is defined by the inductive assumption.) We set

$$\mathcal{Z}'_n = \bigcup_{\eta \in \underline{\mathfrak{P}}'_n} \mathcal{Z}_n^\eta.$$

(ii) Using (i), we define elements W_n^ξ for any $\xi \in \mathcal{Z}'_n$ by a procedure similar to the definition of the "new" basis of a Hecke algebra. (See 2.13.)

(iii) Using (i) and (ii) for n and $-n$, we define $\mathcal{Z}_n^\eta$ for $\eta \in \underline{\mathfrak{P}}_n - \underline{\mathfrak{P}}'_n$. (See 2.18.)

Lemma 2.12 *Let* $n \in \Delta$.

(a) *If* $\eta \in \underline{\mathfrak{P}}'_n$ *and* $P \in \eta$, *the map* $\mathcal{Z}_n^{\{\underline{P}\}} \to \mathcal{Z}_n^\eta$ *given by* $\xi \mapsto f_P^G(\xi)$ *is bijective.*

(b) *The union* $\bigcup_{\eta \in \underline{\mathfrak{P}}'_n} \mathcal{Z}_n^\eta$ *is disjoint.*

(c) *In the setup of* (a), *let* $\xi' \in \mathcal{Z}'_n$ *(relative to* $\underline{P}$ *instead of* G*). Then* $f_P^G(\xi')$ *is an* $\mathcal{A}$*-linear combination of elements in various* $\mathcal{Z}_n^{\eta'}$ *(with* $\eta' \in \underline{\mathfrak{P}}'_n$, $\eta' \prec \eta$*) plus an element of* $\mathcal{R}_G$.

(d) *In the setup of* (a) *let* $\xi_0 \in \mathcal{Z}_n^{\{\underline{P}\}}$. *Then* $\beta(\xi_0) - \xi_0$ *is an* $\mathcal{A}$*-linear combination of elements in* $\mathcal{Z}'_n$ *(relative to* $\underline{P}$*) plus an element of* $\mathcal{R}_{\underline{P}}$.

(e) *The matrix with entries* $(\xi : \xi') \in \mathbf{Q}(v)$ *indexed by* $\mathcal{Z}'_n \times \mathcal{Z}'_n$ *is non-singular.*

See 4.6, 4.7 for a proof.

2.13 Let $n \in \Delta$. We show that for any $\xi \in \mathcal{Z}'_n$, we have

$$\beta(\xi) = \sum_{\xi_1 \in \mathcal{Z}'_n} a_{\xi,\xi_1}\xi_1 \mod \mathcal{R}_G, \tag{a}$$

where $a_{\xi,\xi_1} \in \mathcal{A}$ are uniquely determined and satisfy the following conditions (where η, η_1 are given by $\xi \in \mathcal{Z}_n^{\eta}, \xi_1 \in \mathcal{Z}_n^{\eta_1}$):

$$a_{\xi,\xi_1} \neq 0 \text{ implies } \eta_1 \prec \eta \text{ or } \xi = \xi_1;$$
$$a_{\xi,\xi_1} = 1 \text{ if } \xi = \xi_1.$$

We have $\xi = f_P^G(\xi_0)$, where $\xi_0 \in \mathcal{Z}_n^{\{P\}}$ (notation of 1.11 (i)). We express $\beta(\xi_0)-\xi_0$ as in 1.12 (d). Applying f_P^G and using 1.12 (c), 1.8 (d) and 1.8 (c), we deduce that (a) holds except perhaps for the uniqueness statement. To show the uniqueness, we note that the a_{ξ,ξ_1} are determined from the system of linear equations

$$(\beta(\xi) : \xi_2) = \sum_{\xi_1 \in \mathcal{Z}'_n} (\xi_1 : \xi_2)a_{\xi,\xi_1}$$

(with $\xi_2 \in \mathcal{Z}'_n$), whose matrix of coefficients is invertible by 1.12 (e).

Using the equality $\beta^2 = 1 : K_G \to K_G$ and the inclusion $\beta(\mathcal{R}_G) \subset \mathcal{R}_G$ (see 2.8), we see that for any ξ, ξ_1 in $\mathcal{Z}'_n$ we have

$$\sum_{\xi_2 \in \mathcal{Z}'_n} \overline{a_{\xi,\xi_2}} a_{\xi_2,\xi_1} = \delta_{\xi,\xi_1}.$$

Using a standard argument, we see that there is a unique family of elements $c_{\xi,\xi_1} \in \mathbf{Z}[v]$ (defined for $\xi, \xi_1 \in \mathcal{Z}'_n$) such that for any $\xi, \xi_1 \in \mathcal{Z}'_n$ with $\xi \in \mathcal{Z}_n^{\eta}$, $\xi_1 \in \mathcal{Z}_n^{\eta_1}$, we have

$$c_{\xi,\xi_1} = \sum_{\xi_2 \in \mathcal{Z}'_n} \overline{c_{\xi,\xi_2}} a_{\xi_2,\xi_1};$$
$$c_{\xi,\xi_1} \neq 0 \text{ implies } \eta_1 \prec \eta \text{ or} \xi = \xi_1;$$
$$c_{\xi,\xi_1} \neq 0, \xi \neq \xi_1 \text{ implies } c_{\xi,\xi_1} \in v\mathbf{Z}[v];$$
$$c_{\xi,\xi_1} = 1 \text{ if} \xi = \xi_1.$$

For $\xi \in \mathcal{Z}'_n$, we set $W_n^{\xi} = \sum_{\xi_1 \in \mathcal{Z}'_n} c_{\xi,\xi_1}\xi_1$. Then $\beta(W_n^{\xi}) = W_n^{\xi} \mod \mathcal{R}_G$.

2.14 Until the end of 2.16, we assume that (G, ι) is rigid. For any $\mathcal{F} \in \mathcal{T}_G^{\text{pr}}$ such that $S_{\mathcal{F}} = C_G^{\iota}$, we set $\mathcal{S}_{\mathcal{F}} = \mu^{-1}(\mathcal{F})$ which, by [L5, 11.9], is a single G^{ι}-orbit $\mathcal{J}_G$ (μ as in 2.5). Let

$$\mathcal{C}' = \{r_{\mathcal{F}}^{-1}\mathbf{I}_{\mathcal{S}_{\mathcal{F}}}; \mathcal{F} \in \mathcal{T}_G^{\text{pr}}, S_{\mathcal{F}} = C_G^{\iota}\} \subset K_G;$$

here $r_{\mathcal{F}} \in \mathcal{A}$ is as in 2.3.

2.15 Let $n \in \Delta$. For any $x \in K_G$, we define $Y_n(x) \in K_G$ by the conditions

$$(Y_n(x) : \mathcal{Z}'_n) = 0, \quad x = Y_n(x) + \sum_{\xi \in \mathcal{Z}'_n} \gamma_{\xi}\xi$$

with $\gamma_x \in \mathbf{Q}(v)$. The coefficients γ_ξ are determined from the system of linear equations

$$\sum_{\xi \in \mathcal{Z}'_n} (\xi : \xi')\gamma_\xi = (x : \xi')$$

with $\xi' \in \mathcal{Z}'_n$, whose matrix of coefficients is invertible by 1.12 (e). Let

$$J_{-n} = \{\xi_0 \in \mathcal{Z}'_{-n}; Y_n(W_{-n}^{\xi_0}) \notin \mathcal{R}_G\}$$

and let $\mathcal{C}_n$ be the image of the map $J_{-n} \to K_G$, $\xi_0 \mapsto Y_n(W_{-n}^{\xi_0})$. This can be regarded as a surjective map $h_n : J_{-n} \to \mathcal{C}_n$.

Lemma 2.16 *h_n is bijective.*

See 4.10 for a proof.

Lemma 2.17 *For $n \in \Delta$, the union $\mathcal{Z}'_n \cup \mathcal{C}_n \cup \mathcal{C}'$ is disjoint.*

See 4.11 for a proof.

2.18 If (G, ι) is not rigid, then $\underline{\mathfrak{P}}_n = \underline{\mathfrak{P}}'_n$ and the definition of the subsets $\mathcal{Z}_n^\eta$ ($\eta \in \underline{\mathfrak{P}}_n$) is complete. If (G, ι) is rigid and $n \in \Delta$, we set $\mathcal{Z}_n^{\{G\}} = \mathcal{C}_n \cup \mathcal{C}'$. By 2.17, this union is disjoint. The definition of the subsets $\mathcal{Z}_n^\eta$ ($\eta \in \underline{\mathfrak{P}}_n$) is complete.

We set $\mathcal{Z}_n = \mathcal{Z}'_n$ if (G, ι) is not rigid and $\mathcal{Z}_n = \mathcal{Z}'_n \cup \mathcal{Z}_n^{\{G\}}$ if (G, ι) is rigid. By 2.17, the last union is disjoint.

2.19 For $n \in \Delta$ and $\xi \in \mathcal{Z}_n$, we define an element W_n^ξ as follows. When $\xi \in \mathcal{Z}'_n$, this is already defined in 2.13. When (G, ι) is rigid, we set $W_n^\xi = W_{-n}^{h_n^{-1}(\xi)}$ if $\xi \in \mathcal{C}_n$ and $W_n^\xi = \xi$ if $\xi \in \mathcal{C}'$.

We now define a matrix $(c_{\xi,\xi'})$ with entries in $\mathbf{Q}(v)$ indexed by $\mathcal{Z}_n \times \mathcal{Z}_n$ by the following requirements:

When $\xi, \xi' \in \mathcal{Z}'_n$, then $c_{\xi,\xi'}$ are as in 2.13.
When $\xi \in \mathcal{Z}'_n, \xi' \notin \mathcal{Z}'_n$, then $c_{\xi,\xi'} = 0$.
When $\xi \notin \mathcal{Z}'_n, \xi' \notin \mathcal{Z}'_n$, then $c_{\xi,\xi'} = \delta_{\xi,\xi'}$.
When $\xi \notin \mathcal{Z}'_n$, then $c_{\xi,\xi'}$ for $\xi' \in \mathcal{Z}'_n$ are determined by the system of linear equations $(W_n^\xi : \xi'') = \sum_{\xi' \in \mathcal{Z}'_n} (\xi' : \xi'') c_{\xi,\xi'}$ (with $\xi'' \in \mathcal{Z}'_n$) whose matrix of coefficients has invertible determinant.

Note that for any $\xi \in \mathcal{Z}_n$ we have

$$W_n^\xi = \sum_{\xi' \in \mathcal{Z}_n} c_{\xi,\xi'} \xi'. \tag{a}$$

Lemma 2.18 *Let $n \in \Delta$. Let $\mathcal{S} \in \underline{\mathcal{J}}_G$. There exist $e_{\mathcal{S},\xi} \in \mathcal{A}$ (for $\xi \in \mathcal{Z}_n$) and $r \in \mathcal{R}_G$ such that $\mathbf{I}_\mathcal{S} = \sum_{\xi \in \mathcal{Z}_n} e_{\mathcal{S},\xi} \xi + r$.*

See 4.14 for a proof.

3 Geometric Preliminaries

3.1 In this section, we assume that $\mathbf{k}$ is an algebraic closure of the finite field F_p with $|F_p| = p$. For $q \in \{p, p^2, \dots\}$, let F_q be the subfield of $\mathbf{k}$ with $|F_q| = q$. If X is an algebraic variety, we denote by $\mathcal{D}(X)$ the bounded derived category of (constructible) $\bar{\mathbf{Q}}_l$-sheaves on X. For $K \in \mathcal{D}(X)$, let $\mathcal{H}^i K$ be the ith cohomology sheaf of K. For $n \in \mathbf{Z}$, let $\bar{\mathbf{Q}}_l(n/2)$ be as in the Introduction to [L8]. We write $K[[n/2]]$ instead of $K[n] \otimes \bar{\mathbf{Q}}_l(n/2)$. We fix a square root $\sqrt{p}$ of p in $\bar{\mathbf{Q}}_l$. If $q = p^e (e \in \mathbf{N})$, we set $\sqrt{q} = (\sqrt{p})^e$. We shall assume that Frobenius relative to F_q acts on $\bar{\mathbf{Q}}_l(n/2)$ as multiplication by $(\sqrt{q})^{-n}$.

For a connected affine algebraic group $\mathcal{G}$, we have

$$\vartheta_{\mathcal{G}}|_{v=-1/\sqrt{q}} = |L\mathcal{G}(F_q)|^{-1}|\mathcal{G}(F_q)|$$

for any F_q-rational structure on $\mathcal{G}$ such that $\underline{\mathcal{G}}$ is F_q-split; here $\vartheta_{\mathcal{G}}$ is as in 2.3.

Define $\omega : \mathbf{k} \to \mathbf{k}$ by $x \mapsto x^p - x$. Let $\mathcal{U}$ be a local system of rank 1 on $\mathbf{k}$ such that $\mathcal{U} \oplus \bar{\mathbf{Q}}_l$ is a direct summand of $\omega_! \bar{\mathbf{Q}}_l$. Let E, E' be two $\mathbf{k}$-vector spaces of the same dimension $< \infty$ and let $\sigma : E \times E' \to \mathbf{k}$ be a perfect bilinear pairing. Let $s : E \times E' \to E$, $s' : E \times E' \to E'$ be the projections. Recall that the Fourier–Deligne transform is the functor $\mathcal{D}(E) \to \mathcal{D}(E')$ given by $A \mapsto s'_!(s^*(A) \otimes \sigma^* \mathcal{U})[[\dim E/2]]$.

We fix a perfect symmetric bilinear pairing $\langle , \rangle : LG \times LG \to \mathbf{k}$, which is invariant under the adjoint action of G.

In this section, we fix $n \in \Delta$. For any G^ι-orbit $\mathcal{O}$ on $L_n G$, let $\bar{\mathcal{O}}$ be the closure of $\mathcal{O}$ in $L_n G$. The natural G^ι-action on $L_n G$ has only finitely many orbits [L5, 3.5]. Let $\overset{\circ}{L}_n G$ be the unique open G^ι-orbit on $L_n G$.

3.2 Let V be an algebraic variety with a given family $\mathfrak{f}$ of simple perverse sheaves with the following property: any complex in $\mathfrak{f}$ comes from a mixed complex on V relative to a rational structure on V over some F_q. Let $\mathcal{D}^{\mathfrak{f}}(V)$ be the subcategory of $\mathcal{D}(V)$ whose objects are complexes K such that for any j, any composition factor of ${}^pH^j(K)$ is in $\mathfrak{f}$. Let $\mathcal{K}^{\mathfrak{f}}(V)$ be the free $\mathcal{A}$-module with basis $\mathbf{B}^{\mathfrak{f}}(V)$ given by the isomorphism classes of simple perverse sheaves in $\mathfrak{f}$. Let K be an object of $\mathcal{D}^{\mathfrak{f}}(V)$ with a given mixed structure relative to a rational structure of V over some F_q. We set

$$gr(K) = \sum_A \sum_{j,h \in \mathbf{Z}} (-1)^j (\text{mult. of } A \text{ in } {}^pH^j(K)_h)(-v)^{-h} A \in \mathcal{K}^{\mathfrak{f}}(V),$$

where A runs over a set of representatives for the isomorphism classes in $\mathfrak{f}$ and the subscript h denotes the subquotient of pure weight h of a mixed perverse sheaf. (This agrees with the definition in [L7, 36.8] after the change of variable $v \mapsto (-v)^{-1}$.) Note that $gr(K[[m/2]]) = v^m gr(K)$ for $m \in \mathbf{Z}$.

Now let V_1 be another algebraic variety with a given family $\mathfrak{f}_1$ of simple perverse sheaves like $\mathfrak{f}$ for V. Then $\mathcal{D}^{\mathfrak{f}_1}(V_1)$ is defined. Assume that we are given a functor

$\Theta : \mathcal{D}(V) \to \mathcal{D}(V_1)$ which restricts to a functor $\mathcal{D}^{\mathfrak{f}}(V) \to \mathcal{D}^{\mathfrak{f}_1}(V_1)$. Assume also that Θ is a composition of functors of the form $a_!, a^*$ induced by various maps a between algebraic varieties. In particular, Θ preserves the triangulated structures and makes sense also on the mixed level. Define an $\mathcal{A}$-linear map $gr(\Theta) : \mathcal{K}^{\mathfrak{f}}(V) \to \mathcal{K}^{\mathfrak{f}_1}(V_1)$ by the following requirement: if $A \in \mathfrak{f}$ is regarded as a pure complex of weight 0 (relative to a rational structure of V over some F_q), then $gr(\Theta)(A) = gr(\Theta(A))$ where $\Theta(A)$ is regarded as a mixed complex on V_1 (with mixed structure defined by that of A). Note that $gr(\Theta)(A)$ does not depend on the choice of mixed structure. If $\Theta' : \mathcal{D}(V_1) \to \mathcal{D}(V_2)$ is another functor like Θ, then so is $\Theta'\Theta$ and we have

$$gr(\Theta'\Theta) = gr(\Theta')gr(\Theta). \tag{a}$$

3.3 For an algebraic variety V with a fixed rational structure over some F_q and a mixed complex K on V, we define a function $\chi_K : V(F_q) \to \bar{\mathbf{Q}}_l$ by

$$\chi_K(x) = \sum_j (-1)^j (\text{trace of Frobenius on } \mathcal{H}^j_x(K)).$$

3.4 Let $\tilde{\mathfrak{I}}_{L_nG}$ be the collection consisting of all irreducible G^ι-equivariant local systems on various G^ι-orbits in L_nG. Let $\mathfrak{I}_{L_nG}$ be the set of all isomorphism classes of irreducible G^ι-equivariant local systems on various G^ι-orbits in L_nG. For $\mathcal{L} \in \tilde{\mathfrak{I}}_{L_nG}$, $\kappa \in \mathfrak{I}_{L_nG}$, we write $\mathcal{L} \in \kappa$ instead of "κ is the isomorphism class of $\mathcal{L}$."

For $\mathcal{L} \in \kappa$ (as above), we say that $\mathcal{L}$ or κ is *cuspidal* (resp. *semicuspidal*) if (G, ι) is rigid, $S_\kappa = \overset{\circ}{L}_nG$ and there exists $\mathcal{F} \in \mathcal{T}^{\mathrm{cu}}_G$ (resp. $\mathcal{F} \in \mathcal{T}^{\mathrm{pr}}_G$) such that $\overset{\circ}{L}_nG \subset S_{\mathcal{F}}$, $\mathcal{L} \cong \mathcal{F}|_{S_\kappa}$.

On the other hand, if $\mathcal{F} \in \mathcal{T}^{\mathrm{cu}}_G$ and $S_{\mathcal{F}} \cap L_nG \neq \emptyset$, then (G, ι) is rigid and $\mathcal{F}|_{\overset{\circ}{L}_nG}$ is irreducible, cuspidal in $\mathfrak{I}_{L_nG}$. (See [L5, 4.4].)

We write $\mathcal{K}(L_nG)$, $\mathbf{B}(L_nG)$ instead of $\mathcal{K}^{\mathfrak{f}}(L_nG)$, $\mathbf{B}^{\mathfrak{f}}(L_nG)$ (see 3.2) where $\mathfrak{f}$ is the family of simple G^ι-equivariant perverse sheaves on L_nG. The notation $\mathcal{K}(L_nG)$, $\mathbf{B}(L_nG)$ agrees with that in [L5, 3.9].

For $\kappa \in \mathfrak{I}_{L_nG}$, we set $\underline{\kappa}^\bullet = IC(\bar{S}_\kappa, \mathcal{L})[[\dim S_\kappa/2]]$ (extended by 0 on $L_nG - \bar{S}_\kappa$) where $\mathcal{L} \in \kappa$. We have $\mathbf{B}(L_nG) = \{\underline{\kappa}^\bullet; \kappa \in \mathfrak{I}_{L_nG}\}$. We set

$$B(L_nG) = \{\underline{\kappa}; \kappa \in \mathfrak{I}_{L_nG}\}.$$

We define a $\mathbf{Z}$-linear involution $\beta : \mathcal{K}(L_nG) \to \mathcal{K}(L_nG)$ by $\beta(v^m\underline{\kappa}^\bullet) = v^{-m}\underline{\kappa}^\bullet$ for $m \in \mathbf{Z}$, $\kappa \in \mathfrak{I}_{L_nG}$.

We choose a rational structure for G over some F_q with Frobenius map $F : G \to G$ such that $\iota(t^q) = F(\iota(t))$ for any $t \in \mathbf{k}^*$, such that any G^ι-orbit in L_nG or $L_{-n}G$ is defined over F_q and such that any irreducible G^ι-equivariant local system over such an orbit admits a mixed structure. Then G^ι is defined over F_q. We assume as we may that G^ι is F_q-split, and any connected component of $\mathcal{P}^\iota$ is defined over F_q.

Let $\kappa \in \mathfrak{I}_{L_n G}$ and let $\mathcal{L} \in \kappa$. Let $i : S_\kappa \to L_n G$ be the inclusion. We choose a mixed structure for $\mathcal{L}$ which is pure of weight 0. Then $i_! \mathcal{L}[[\dim S_\kappa/2]]$ is naturally a mixed complex on $L_n G$ and

$$\underline{\kappa} := gr(i_! \mathcal{L}[[\dim S_\kappa/2]]) \in \mathcal{K}(L_n G)$$

is defined as in 3.2. It is independent of the choice of rational/mixed structures. Using the definitions and the purity statement in [L5, 18.2], we see that

$$\underline{\kappa}^\bullet = \sum_{\kappa' \in \mathfrak{I}_{L_n G}} f_{\kappa,\kappa'} \underline{\kappa'}, \tag{a}$$

where we have (in $\mathbf{Z}[v]$):

$$f_{\kappa,\kappa'} = \sum_{i'} (\text{mult. of } \kappa' \text{ in the local system } \mathcal{H}^{i'} IC(\bar{S}_\kappa, \mathcal{L})|_{S_{\kappa'}}) v^{\dim S_\kappa - \dim S_{\kappa'} - i'}$$

if $S_{\kappa'} \subset \bar{S}_\kappa$,

$$f_{\kappa,\kappa'} = 0 \text{ if } S_{\kappa'} \not\subset \bar{S}_\kappa.$$

In particular,

$$f_{\kappa,\kappa} = 1,$$
$$f_{\kappa,\kappa'} = 0 \mathit{if} S_{\kappa'} = S_\kappa, \kappa' \neq \kappa,$$
$$f_{\kappa,\kappa'} \in v\mathbf{Z}[v] \mathit{if} \kappa' \neq \kappa.$$

We see that the $B(L_n G)$ is an $\mathcal{A}$-basis of $\mathcal{K}(L_n G)$.

3.5 Induction Let $P \in \mathcal{P}^\iota$. Then $\mathcal{K}(L_n \underline{P})$ is defined as in 3.4 (in terms of $\underline{P}, \iota$ instead of G, ι). Now P^ι and its subgroup U_P^ι act freely on $G^\iota \times L_n P$ by $y : (g, x) \mapsto (gy^{-1}, \mathrm{Ad}(y)x)$; we form the quotients $E' = G^\iota \times_{U_P^\iota} L_n P$, $E'' = G^\iota \times_{P^\iota} L_n P$. Let $\pi : L_n P \to L_n \underline{P}$ be the canonical map. We have a diagram:

$$L_n \underline{P} \xleftarrow{a} E' \xrightarrow{b} E'' \xrightarrow{c} L_n G,$$

where $a(g, x) = \pi(x)$, $b(g, x) = (g, x)$, $c(g, x) = \mathrm{Ad}(g)x$.

Note that a is smooth with connected fibres of dimension $s = \dim L_0 P + \dim L_n U_P$, b is a principal $\underline{P}^\iota$-bundle and c is proper.

Let A be a simple $\underline{P}^\iota$-equivariant perverse sheaf on $L_n \underline{P}$. There is a well defined simple perverse sheaf $\tilde{A}$ on E'' such that

$$a^* A[[s/2]] = b^* \tilde{A}[[\dim \underline{P}^\iota/2]].$$

Moreover, if we regard A as a pure complex of weight zero (relative to a rational structure over some F_q), then $\tilde{A}$ is naturally pure of weight zero and $c_!\tilde{A}$ is naturally a mixed complex whose perverse cohomology sheaves are G^ι-equivariant. Hence $gr(c_!\tilde{A}) \in \mathcal{K}(L_nG)$ is well defined; it is independent of the choice of mixed structure for A. Now $A \mapsto gr(c_!\tilde{A})$ defines an $\mathcal{A}$-linear map

$$\mathrm{ind}_P^G : \mathcal{K}(L_n\underline{P}) \to \mathcal{K}(L_nG).$$

3.6 Now assume that $A = IC(L_n\underline{P}, \mathcal{L})[[\dim L_n\underline{P}/2]]$ where $\mathcal{L} \in \tilde{\mathfrak{I}}_{L_n\underline{P}}$ is cuspidal. Let

$$\dot{L}_nG = \{(gP^\iota, z) \in G^\iota/P^\iota \times L_nG; \mathrm{Ad}(g^{-1})z \in \pi^{-1}(\overset{\circ}{L}_n\underline{P})\}.$$

We have a diagram

$$\overset{\circ}{L}_n\underline{P} \xleftarrow{\tilde{a}} \{(g,z) \in G^\iota \times L_nG; \mathrm{Ad}(g^{-1})z \in \pi^{-1}(\overset{\circ}{L}_n\underline{P})\} \xrightarrow{\tilde{b}} \dot{L}_nG \xrightarrow{\tilde{c}} L_nG$$

with

$$\tilde{a}(g,z) = \pi(\mathrm{Ad}(g^{-1})z), \tilde{b}(g,z) = (gP^\iota, z), \tilde{c}(gP^\iota, z) = z.$$

Let $\dot{\mathcal{L}}$ be the local system on $\dot{L}_nG$ defined by $\tilde{a}^*\mathcal{L} = \tilde{b}^*\dot{\mathcal{L}}$. Using [L5, 4.4 (b)], we see as in [L5, 6.6] that $c_!\tilde{A} = \tilde{c}_!\dot{\mathcal{L}}[[\dim L_0U_P/2 + \dim L_nP/2]]$. If $\mathcal{L}$ is regarded as a pure local system of weight zero (relative to a rational structure over some F_q), then $\dot{\mathcal{L}}$, A, $\tilde{A}$ are naturally mixed of weight zero and in $\mathcal{K}(L_nG)$ we have

$$gr(c_!\tilde{A}) = gr\left(\tilde{c}_!\dot{\mathcal{L}}\left[\left[\frac{\dim L_0U_P}{2} + \frac{\dim L_nP}{2}\right]\right]\right) = v^{\dim L_0U_P + \dim L_nP} gr(c'_!\dot{\mathcal{L}}).$$

3.7 We now fix $P, P' \in \mathcal{P}^\iota$. Let $\mathcal{L} \in \tilde{\mathfrak{I}}_{L_n\underline{P}}$ (resp. $\mathcal{L}' \in \tilde{\mathfrak{I}}_{L_n\underline{P}'}$) be cuspidal. Let

$$A = IC(L_n\underline{P}, \mathcal{L})[[\dim L_n\underline{P}/2]] \in \mathcal{D}(L_n\underline{P}),$$
$$A' = IC(L_n\underline{P}', \mathcal{L}')[[\dim L_n\underline{P}'/2]] \in \mathcal{D}(L_n\underline{P}').$$

Let $\dot{L}_nG, \dot{\mathcal{L}}, \tilde{c}, c, \pi$ be as in 3.5, 3.6, and let $\dot{L}'_nG, \dot{\mathcal{L}}', \tilde{c}', c', \pi'$ be the analogous entities defined in terms of $P', \mathcal{L}'$.

Let $R = \{h \in G^\iota; hPh^{-1}$ and P' have a common Levi$\}$. For $h \in R$, we set $Q = hPh^{-1}$; we have isomorphisms

$$\underline{P} \xrightarrow{d} \underline{Q} \xleftarrow{e} (Q \cap P')/U_{Q\cap P'} \xrightarrow{f} \underline{P}'$$

(d is induced by $\mathrm{Ad}(h)$, e and f are induced by the inclusions $Q \cap P' \subset Q$, $Q \cap P' \subset P'$). Then $fe^{-1}d : \underline{P} \to \underline{P}'$ is an isomorphism compatible with the homomorphisms $\iota : \mathbf{k}^* \to \underline{P}, \iota : \mathbf{k}^* \to \underline{P}'$. It induces a Lie algebra isomorphism $L\underline{P} \xrightarrow{\sim} L\underline{P}'$ compatible with the gradings, hence an isomorphism $\overset{\circ}{L}_n\underline{P} \xrightarrow{\sim} \overset{\circ}{L}_n\underline{P}'$. This carries $\mathcal{L}$ to a local system ${}^h\mathcal{L}$ on $\overset{\circ}{L}_n\underline{P}'$. We set

$$\tau(h) = \dim \frac{L_n U_{P'} + \mathrm{Ad}(h)(L_n U_P)}{L_n U_{P'} \cap \mathrm{Ad}(h)(L_n U_P)} - \dim \frac{L_0 U_{P'} + \mathrm{Ad}(h)(L_0 U_P)}{L_0 U_{P'} \cap \mathrm{Ad}(h)(L_0 U_P)}.$$

Let R' be the set of all $h \in R$ such that ${}^h\mathcal{L} \cong \mathcal{L}'^*$. Note that R and R' are unions of (P'^ι, P^ι)-double cosets in G^ι, and that $\tau(h)$ depends only on the (P'^ι, P^ι)-double coset Ω that contains h; we shall write also τ_Ω instead of $\tau(h)$.

3.8 In the setup of 3.7, we choose a rational structure for G over some F_q with Frobenius map $F : G \to G$ as in 3.4. We assume as we may that $F(P) = P$, $F(P') = P'$ and that $F^*\mathcal{L} \cong \mathcal{L}$, $F^*\mathcal{L}' \cong \mathcal{L}'$.

For various varieties X connected with G which inherit an F_q-rational structure from G, we shall write X^F instead of $X(F_q)$.

We may assume that $\mathcal{L}, \mathcal{L}'$ have mixed structures such that all values of $\chi_{\mathcal{L}} : (\mathring{L}_n\underline{P})^F \to \bar{\mathbf{Q}}_l$, $\chi_{\mathcal{L}'} : (\mathring{L}_n\underline{P}')^F \to \bar{\mathbf{Q}}_l$ are roots of 1. Then $\mathcal{L}, \mathcal{L}', A, A'$ are pure of weight 0. Note that $\dot{\mathcal{L}}, \dot{\mathcal{L}}', \tilde{c}_!\dot{\mathcal{L}}, \tilde{c}'_!\tilde{\mathcal{L}}', \tilde{A}, \tilde{A}', c_!\tilde{A}, c'_!\tilde{A}'$ hence also $\tilde{c}_!\dot{\mathcal{L}} \otimes \tilde{c}'_!\tilde{\mathcal{L}}'$, $c_!A \otimes c'_!\tilde{A}'$ are naturally mixed complexes. We have

$$\chi_{c_!\tilde{A}} = (\sqrt{q})^{-\dim L_0U_P - \dim L_n P}\chi_{\tilde{c}_!\dot{\mathcal{L}}}, \quad \chi_{c'_!\tilde{A}'} = (\sqrt{q})^{-\dim L_0U_{P'} - \dim L_n P'}\chi_{\tilde{c}'_!\dot{\mathcal{L}}'}.$$

Lemma 3.9 *We have*

$$\sum_{x \in (L_nG)^F} \chi_{c_!\tilde{A} \otimes c'_!\tilde{A}'}(x) = \left(\frac{\vartheta_{G^\iota}}{\vartheta_{Z^0_{\underline{P}}}} \sum_\Omega \epsilon_\Omega(-v)^{\tau_\Omega}\right)\Bigg|_{v=-1/\sqrt{q}}. \qquad \text{(a)}$$

Here Ω runs over the P'^ι, P^ι-double cosets in G^ι such that $\Omega \subset R'$ and ϵ_Ω are roots of 1. *Note that* $\frac{\vartheta_{G^\iota}}{\vartheta_{Z^0_{\underline{P}}}} \in \mathbf{Z}[v^2]$.

Let N be the left hand side of (a). We have

$$N = (\sqrt{q})^{\varpi} \sum_{x \in (L_nG)^F} \chi_{\tilde{c}_!\dot{\mathcal{L}} \otimes \tilde{c}'_!\tilde{\mathcal{L}}'}(x) = (\sqrt{q})^{\varpi} \sum_{x' \in X^F} \chi_{\dot{\mathcal{L}} \boxtimes \dot{\mathcal{L}}'}(x'),$$

where

$$\varpi = -\dim L_0U_P - \dim L_nP - \dim L_0U_{P'} - \dim L_nP',$$

$$X = \dot{L}_nG \times_{L_nG} \dot{L}'_nG = \{(gP^\iota, g'P'^\iota, z) \in G^\iota/P^\iota \times G^\iota/P'^\iota \times L_nG;$$
$$\mathrm{Ad}(g^{-1})z \in \pi^{-1}(\mathring{L}_n\underline{P}), \mathrm{Ad}(g'^{-1})z \in \pi'^{-1}(\mathring{L}_n\underline{P}')\}.$$

We have a partition $X = \bigsqcup_\Omega X_\Omega$ into locally closed subvarieties indexed by the various (P'^ι, P^ι)-double cosets Ω in G^ι, where

$$X_\Omega = \{(gP^\iota, g'P'^\iota, z) \in G^\iota/P^\iota \times G^\iota/P'^\iota \times L_nG;$$
$$g'^{-1}g \in \Omega, \mathrm{Ad}(g^{-1})z \in \pi^{-1}(\mathring{L}_n\underline{P}), \mathrm{Ad}(g'^{-1})z \in \pi'^{-1}(\mathring{L}_n\underline{P}')\}.$$

(There are only finitely many such Ω since P'^{ι}, P^{ι} are parabolic subgroups of G^{ι}.) From our assumptions, we see that each Ω is defined over F_q. We have $N = \sum_{\Omega} N_{\Omega}$, where

$$N_{\Omega} = (\sqrt{q})^{\varpi} \sum_{(gP^{\iota}, g'P'^{\iota}, z) \in X_{\Omega}^F} \chi_{\dot{\mathcal{L}}}(gP^{\iota}, z)\chi_{\dot{\mathcal{L}}'}(g'P'^{\iota}, z).$$

We fix an Ω as above. Now $G^{\iota F}$ acts on $(\dot{L}_n G)^F$ by $\tilde{g} : (gP^{\iota}, z) \mapsto (\tilde{g}gP^{\iota}, \mathrm{Ad}(\tilde{g})z)$, and from the definitions we see that $\chi_{\dot{\mathcal{L}}}$ is constant on the orbits of this action. A similar property holds for $\chi_{\dot{\mathcal{L}}'}$. It follows that the function

$$X_{\Omega}^F \to \bar{\mathbf{Q}}_l, \quad (gP^{\iota}, g'P'^{\iota}, z) \mapsto \chi_{\dot{\mathcal{L}}}(gP^{\iota}, z)\chi_{\dot{\mathcal{L}}'}(g'P'^{\iota}, z)$$

is constant on the orbits of the $G^{\iota F}$-action on X_{Ω}^F given by

$$\tilde{g} : (gP^{\iota}, g'P'^{\iota}, z) \mapsto (\tilde{g}gP^{\iota}, \tilde{g}g'P'^{\iota}, \mathrm{Ad}(\tilde{g})z).$$

Since $\alpha : X_{\Omega}^F \to (G^{\iota}/P'^{\iota})^F$, $(gP^{\iota}, g'P'^{\iota}, z) \mapsto g'P'^{\iota}$ is compatible with the obvious actions of $G^{\iota F}$ and since the $G^{\iota F}$-action on $(G^{\iota}/P'^{\iota})^F$ is transitive, we see that for $y \in (G^{\iota}/P'^{\iota})^F$, the sum

$$\sum_{(gP^{\iota}, g'P'^{\iota}, z) \in \alpha^{-1}(y)} \chi_{\dot{\mathcal{L}}}(gP^{\iota}, z)\chi_{\dot{\mathcal{L}}'}(g'P'^{\iota}, z)$$

is independent of the choice of y. It follows that $N_{\Omega} = (\sqrt{q})^{\varpi}|(G^{\iota}/P'^{\iota})^F|N'_{\Omega}$, where

$$N'_{\Omega} = \sum_{\substack{(gP^{\iota}, z) \in (\dot{L}_n G)^F; \\ z \in \pi'^{-1}(\overset{\circ}{L}_n \underline{P}), g \in \Omega}} \chi_{\dot{\mathcal{L}}}(gP^{\iota}, z)\chi_{\mathcal{L}'}(\pi'(z)).$$

(We have used that $\chi_{\dot{\mathcal{L}}'}(P'^{\iota}, z) = \chi_{\mathcal{L}'}(\pi'(z))$ which follows from the definitions.) We set

$$Y_{\Omega} = \{(gP^{\iota}, z) \in G^{\iota}/P^{\iota} \times L_n P'; \mathrm{Ad}(g^{-1})z \in \pi^{-1}(\overset{\circ}{L}_n \underline{P}), g \in \Omega\}.$$

Define $\sigma : Y_{\Omega} \to L_n \underline{P}'$ by $\sigma(gP^{\iota}, z) = \pi'(z)$. Then $N'_{\Omega} = \sum_{\xi \in (\overset{\circ}{L}_n \underline{P}')^F} N''_{\Omega}(\xi)\chi_{\mathcal{L}'}(\xi)$, where

$$N''_{\Omega}(\xi) = \sum_{(gP^{\iota}, z) \in \sigma^{-1}(\xi)^F} \chi_{\dot{\mathcal{L}}}(gP^{\iota}, z).$$

Let $K_{\Omega} = \sigma_!(\dot{\mathcal{L}}|_{Y_{\Omega}})$. This is naturally a mixed complex over $L_n \underline{P}'$, and we have $N''_{\Omega}(\xi) = \chi_{K_{\Omega}}(\xi)$ for $\xi \in (\overset{\circ}{L}_n \underline{P}')^F$. If we assume that $hPh^{-1} \cap P'$ contains no Levi of hPh^{-1} for some/any $h \in \Omega$, then we have $K_{\Omega} = 0$ (see [L5, 8.4 (b)]); hence $\chi_{K_{\Omega}} = 0$ and $N''_{\Omega}(\xi) = 0$ for any $\xi \in (\overset{\circ}{L}_n \underline{P}')^F$. It follows that $N'_{\Omega} = 0$, hence $N_{\Omega} = 0$.

On the other hand, if we assume that $hPh^{-1} \cap P'$ contains no Levi of P' for some/any $h \in \Omega$, then we have again $N_\Omega = 0$. (This follows from the previous paragraph applied to P, P', Ω^{-1} instead of P', P, Ω.)

Assume that Ω is not as in the previous two paragraphs. Thus, setting $Q = hPh^{-1}$ for some $h \in \Omega^F$, the intersection $Q \cap P'$ contains a Levi of Q and also a Levi of P'; it follows that Q, P' have a common Levi. We have $F(Q) = Q$ and $Q \in \mathcal{P}^\iota$. We have isomorphisms

$$\underline{P} \xrightarrow{d} \underline{Q} \xleftarrow{e} (Q \cap P')/U_{Q\cap P'} \xrightarrow{f} \underline{P'}$$

(d is induced by $\mathrm{Ad}(h)$, and e and f are induced by the inclusions $Q \cap P' \subset Q$, $Q \cap P' \subset P'$). Then $fe^{-1}d : \underline{P} \to \underline{P'}$ is an isomorphism compatible with the homomorphisms $\iota : \mathbf{k}^* \to \underline{P}, \iota : \mathbf{k}^* \to \underline{P'}$. It induces a Lie algebra isomorphism $L\underline{P} \xrightarrow{\sim} L\underline{P'}$ compatible with the gradings, hence an isomorphism $\overset{\circ}{L}_n\underline{P} \xrightarrow{\sim} \overset{\circ}{L}_n\underline{P'}$. This carries $\mathcal{L}$ to a mixed local system ${}^h\mathcal{L}$ on $\overset{\circ}{L}_n\underline{P'}$. By [L5, 8.4 (a)] and its proof [L5, 8.8], in which Tate twists must be also taken in account, we see that $K_\Omega|_{\overset{\circ}{L}_n\underline{P'}} = {}^h\mathcal{L}[[-\delta_\Omega]]$ where

$$\delta_\Omega = \dim L_0U_{P'} - \dim(L_0U_{P'} \cap L_0U_Q) + \dim(L_nU_{P'} \cap L_nU_Q)$$

and $K_\Omega|_{L_n\underline{P'} - \overset{\circ}{L}_n\underline{P'}} = 0$. We see that

$$N'_\Omega = q^{\delta_\Omega} \sum_{\xi \in (\overset{\circ}{L}_n\underline{P'})^F} \chi_{{}^h\mathcal{L}}(\xi)\chi_{\mathcal{L}'}(\xi) = q^{\delta_\Omega} \sum_{\xi \in (\overset{\circ}{L}_n\underline{P'})^F} \chi_{{}^h\mathcal{L}\otimes\mathcal{L}'}(\xi). \tag{b}$$

If ${}^h\mathcal{L} \not\cong \mathcal{L}'^*$, then ${}^h\mathcal{L}\otimes\mathcal{L}'$ has no direct summand $\cong \bar{\mathbf{Q}}_l$; hence by an argument as in [L6, 23.5] we have $H_c^j(\overset{\circ}{L}_n\underline{P'}, {}^h\mathcal{L}\otimes\mathcal{L}') = 0$ for any j; it follows that the last sum is 0 so that $N'_\Omega = 0$. (To use [L6, 23.5] we need to know that the transitive action of $\underline{P'}^\iota/Z^0_{\underline{P'}}$ on $\overset{\circ}{L}_n\underline{P'}$ has isotropy groups with unipotent identity components; in fact, in our case the isotropy groups are finite as we can see from [L5, 4.4, 2.5 (a)].) It follows that $N_\Omega = 0$.

We now assume that ${}^h\mathcal{L} \cong \mathcal{L}'^*$. Then ${}^h\mathcal{L} \otimes \mathcal{L}'$ has a unique direct summand isomorphic to $\bar{\mathbf{Q}}_l$, and Frobenius acts on the stalk of this direct summand at any point in $\overset{\circ}{L}_n\underline{P'}^F$ as multiplication by a root of unity $\epsilon(h)$. By an argument in [L6, 24.14], we see that

$$\sum_{\xi \in \overset{\circ}{L}_n\underline{P'}^F} \chi_{{}^h\mathcal{L}\otimes\mathcal{L}'}(\xi) = \epsilon(h)|\underline{P'}^{\iota F}||Z^{0F}_{\underline{P'}}|^{-1}.$$

Hence, $N_\Omega = \epsilon(h)(\sqrt{q})^{\varpi+2\delta_\Omega}|(G^\iota/P'^\iota)^F||\underline{P'}^{\iota F}/Z^{0F}_{\underline{P'}}|$ that is,

$$N_\Omega = \epsilon(h)(\sqrt{q})^{\varpi+2\delta_\Omega-2\dim L_0U_{P'}}|G^{\iota F}||Z^{0F}_{\underline{P'}}|^{-1}. \tag{c}$$

We set

$$
\begin{aligned}
s_0 &= \dim L_0 U_P = \dim L_0 U_Q,\ s'_0 = \dim L_0 U_{P'},\\
s_n &= \dim L_n U_P = \dim L_n U_Q,\ s'_n = \dim L_n U_{P'},\\
t'_0 &= \dim(L_0 U_{P'} \cap L_0 U_Q),\ t'_n = \dim(L_n U_{P'} \cap L_n U_Q),\\
r_0 &= \dim(L_0 U_{P'} + L_0 U_Q),\ r_n = \dim(L_n U_{P'} + L_n U_Q).
\end{aligned}
$$

We have $r_0 + t'_0 = s_0 + s'_0$, $r_n + t'_n = s_n + s'_n$.

Since Q^ι, P'^ι are parabolic subgroups of G^ι with a common Levi, we have $s_0 = s'_0$. Let $z = \dim Z^0_{\underline{P}} = \dim Z^0_{\underline{P}'}$. Let $g = \dim G^\iota$. Since the action of $\underline{P}^\iota / Z^0_{\underline{P}}$ on $L_n \underline{P}$ has an open orbit with finite stabilizers, we have $\dim L_n \underline{P} = \dim \underline{P}^\iota - z$. Moreover, $\dim \underline{P}^\iota = \dim G^\iota - 2\dim U^\iota_P = g - 2s_0$. Thus, $\dim L_n \underline{P} = g - z - 2s_0$. Similarly, $\dim L_n \underline{P}' = g - z - 2s_0$. We have $\dim L_n P = \dim L_n \underline{P} + \dim L_n U_P = g - z - 2s_0 + s_n$. Similarly, $\dim L_n P' = g - z - 2s_0 + s'_n$. We see that the exponent of $\sqrt{q}$ in (c) is

$$
\begin{aligned}
&\varpi + 2\delta_\Omega - 2\dim L_n U_{P'}\\
&\quad = -s_0 - (g - z - 2s_0 + s_n) - s_0 - (g - z - 2s_0 + s'_n) - 2t'_0 + 2t'_n\\
&\quad = 2(s_0 - t'_0) - (s_n + s'_n - 2t'_n) - 2g + 2z\\
&\quad = (r_0 - t'_0) - (r_n - t'_n) - 2g + 2z\\
&\quad = -\tau_\Omega - 2g + 2z.
\end{aligned}
$$

Thus, we have

$$N_\Omega = \epsilon(h)(\sqrt{q})^{-\tau_\Omega} \frac{|G^{\iota F}| q^{-g}}{|Z^{0F}_{\underline{P}'}| q^{-z}}.$$

The lemma follows.

3.10 Let $\mathfrak{f}_2$ be the family $\{\bar{\mathbf{Q}}_l\}$ of simple perverse sheaves on a point. Write $\mathcal{K}(\text{point})$ instead of $\mathcal{K}^{\mathfrak{f}_2}(\text{point})$. We identify $\mathcal{K}(\text{point}) = \mathcal{A}$ in an obvious way. Define an $\mathcal{A}$-bilinear pairing

$$(:) : \mathcal{K}(L_n G) \times \mathcal{K}(L_n G) \to \mathcal{A}$$

by the requirement that if K, K' are simple G^ι-equivariant perverse sheaves on $L_n G$, we have

$$(K : K') = gr(\rho_! i^*(K \boxtimes K')) \in \mathcal{A}, \tag{a}$$

where K, K' are regarded as pure complexes of weight zero (relative to a rational structure over some F_q), $i : L_n G \to L_n G \times L_n G$ is the diagonal and $\rho : L_n G \to$ point is the obvious map (so that $\rho_! i^*(K \boxtimes K')$ is a mixed complex). Note that $(K : K')$ does not depend on the choices, hence it is well defined.

3.11 Let $P, P' \in \mathcal{P}^\iota$. Let $V = (G^\iota \times_{P^\iota} L_n P) \times (G^\iota \times_{P'^\iota} L_n P')$, $V_1 = L_n G \times L_n G$, $V_2 =$ point. Let $\mathfrak{f}$ be the family of simple perverse sheaves on V of the form $\tilde{A} \boxtimes \tilde{A}'$ where $\tilde{A}$ is defined in terms of a simple $\underline{P}^\iota$-equivariant perverse sheaf A on $L_n \underline{P}$ as in 3.5 and $\tilde{A}'$ is defined in a similar way in terms of A', a simple $\underline{P}'^\iota$-equivariant perverse sheaf on $L_n\underline{P}'$. Let $\mathfrak{f}_1$ be the family of simple $G^\iota \times G^\iota$-equivariant perverse sheaves on V_1. Let $\mathfrak{f}_2$ be as in 3.10. Let $\Theta = (c, c')_! : \mathcal{D}(V) \to \mathcal{D}(V_1)$ where $c : G^\iota \times_{P^\iota} L_n P \to L_n G$ is as in 3.5 and $c' : G^\iota \times_{P'^\iota} L_n P' \to L_n G$ is the analogous map. Let $\Theta' = \rho_! i^* : \mathcal{D}(V_1) \to \mathcal{D}(V_2)$ with ρ, i as in 3.10. Then 2.2 (a) is applicable. Thus, we have $gr(\rho_! i^*(c, c')_!) = gr(\rho_! i^*)gr((c, c')_!)$. If $A, A', \tilde{A}, \tilde{A}'$ are as above, then

$$(\mathrm{ind}_P^G(A) : \mathrm{ind}_{P'}^G(A')) = gr(\rho_! i^*)gr((c, c')_!)(\tilde{A} \boxtimes \tilde{A}'),$$

hence

$$(\mathrm{ind}_P^G(A) : \mathrm{ind}_{P'}^G(A')) = gr(\rho_! i^*(c, c')_!)(\tilde{A} \boxtimes \tilde{A}'). \tag{a}$$

We apply this to $A, A', \tilde{A}, \tilde{A}'$ as in 3.8. We choose an F_q-rational structure on G and mixed structures on A, A' as in 3.8. From (a) we see that

$$(\mathrm{ind}_P^G(A) : \mathrm{ind}_{P'}^G(A')) = \sum_{j,h\in\mathbf{Z}} d_{j,h}(-1)^j(-v)^{-h},$$

where $d_{j,h} = \dim H_c^j(L_n G, c_! \tilde{A} \otimes c'_! \tilde{A}')_h$ and the subscript h denotes the subquotient of pure weight h of a mixed vector space. Let $\{\lambda_{j,h;k}; k \in [1, d_{j,h}]\}$ be the eigenvalues of the Frobenius map on $H_c^j(L_n G, c_! \tilde{A} \otimes c'_! \tilde{A}')_h$. By the Grothendieck trace formula for the sth power of the Frobenius map ($s \in \mathbf{Z}_{>0}$) we have

$$\sum_{j,h\in\mathbf{Z}} \sum_{k\in[1,d_{j,h}]} (-1)^j \lambda_{j,h,k}^s = \sum_{x\in L_n G(F_{q^s})} \chi_{c_!\tilde{A}\otimes c'_!\tilde{A}'}(x)$$

(in the right hand side χ is taken relative to F_{q^s}). Using Lemma 3.9 and its proof (with q replaced by q^s), we deduce

$$\sum_{j,h\in\mathbf{Z}} \sum_{k\in[1,d_{j,h}]} (-1)^j \lambda_{j,h,k}^s = \sum_{\Omega\in R'} \left(\sum_{i\in[1,u]} q^{-sa_i} - \sum_{i'\in[1,u']} q^{-sb_{i'}} \right) \sum_{\Omega} \epsilon_\Omega^s (\sqrt{q})^{-s\tau_\Omega}.$$

Here we write

$$\vartheta_{G^\iota}/\vartheta_{Z^0_{\underline{P}}} = \sum_{i\in[1,u]} v^{2a_i} - \sum_{i'\in[1,u']} v^{2b_{i'}}$$

with $a_1, a_2, \ldots, a_u, b_1, b_2, \ldots, b_{u'}$ in $\mathbf{N}$. We can find some integer $m \geq 1$ such that $\epsilon_\Omega^m = 1$ for any $\Omega \in R'$. Then for any $s \in m\mathbf{Z}_{>0}$, we have

$$\begin{aligned} &\sum_{\substack{j,h\in\mathbf{Z} \\ j \text{ even}}} \sum_{k\in[1,d_{j,h}]} \lambda_{j,h,k}^s + \sum_{\Omega\in R'} \sum_{i'\in[1,u']} (\sqrt{q})^{-2sb_{i'}-s\tau_\Omega} \\ &= \sum_{\substack{j,h\in\mathbf{Z} \\ j \text{ odd}}} \sum_{k\in[1,d_{j,h}]} \lambda_{j,h,k}^s + \sum_{\Omega\in R'} \sum_{i\in[1,u]} (\sqrt{q})^{-2sa_i-s\tau_\Omega}. \end{aligned}$$

It follows that the multisets

$$\bigcup_{\substack{j,h\in\mathbf{Z}\\ j \text{ even}}} \bigcup_{k\in[1,d_{j,h}]} \{\lambda_{j,h,k}\} \cup \bigcup_{\Omega\in R'} \bigcup_{i'\in[1,u']} \{(\sqrt{q})^{-2b_{i'}-\tau_\Omega}\},$$

$$\bigcup_{\substack{j,h\in\mathbf{Z}\\ j \text{ odd}}} \bigcup_{k\in[1,d_{j,h}]} \{\lambda_{j,h,k}\} \cup \bigcup_{\Omega\in R'} \bigcup_{i\in[1,u]} \{(\sqrt{q})^{-2a_i-\tau_\Omega}\}$$

coincide. Hence for any $h \in \mathbf{Z}$, these two multisets contain the same number of elements ξ of weight h (i.e., such that any complex absolute value of ξ is $(\sqrt{q})^h$). Since $\lambda_{j,h,k}$ has weight h, we see that for any h we have

$$\begin{aligned}&\sum_{j\in 2\mathbf{Z}} d_{j,h} + |\{(\Omega, i') \in R' \times [1, u']; -2b_{i'} - \tau_\Omega = h\}|\\ &= \sum_{j\in 2\mathbf{Z}+1} d_{j,h} + |\{(\Omega, i) \in R' \times [1, u]; -2a_i - \tau_\Omega = h\}|.\end{aligned}$$

It follows that

$$\begin{aligned}&\sum_{j,h\in\mathbf{Z};\, j \text{ even}} d_{j,h}(-v)^{-h} + \sum_{\Omega\in R'} \sum_{i'\in[1,u']} v^{2b_{i'}}(-v)^{\tau_\Omega}\\ &= \sum_{j,h\in\mathbf{Z};\, j \text{ odd}} d_{j,h}(-v)^{-h} + \sum_{\Omega\in R'} \sum_{i\in[1,u]} v^{2a_i}(-v)^{\tau_\Omega}.\end{aligned}$$

Equivalently,

$$\sum_{j,h\in\mathbf{Z}} d_{j,h}(-1)^j(-v)^{-h} = \frac{\vartheta_{G^\iota}}{\vartheta_{Z^0_{\underline{P}}}} \sum_{\Omega\in R'} (-v)^{\tau_\Omega},$$

that is,

$$(\mathrm{ind}_P^G(A) : \mathrm{ind}_{P'}^G(A')) = \frac{\vartheta_{G^\iota}}{\vartheta_{Z^0_{\underline{P}}}} \sum_{\Omega\in R'} (-v)^{\tau_\Omega}. \tag{b}$$

3.12 The perfect pairing $\langle,\rangle : LG \times LG \to \mathbf{k}$ (see 3.1) restricts to a perfect pairing $L_nG \times L_{-n}G \to \mathbf{k}$ denoted again by $\langle,\rangle$. Note that $\langle \mathrm{Ad}(g)x, \mathrm{Ad}(g)x'\rangle = \langle x, x'\rangle$ for $x \in L_nG$, $x' \in L_{-n}G$, $g \in G^\iota$. The Fourier–Deligne transform $\mathcal{D}(L_nG) \to \mathcal{D}(L_{-n}G)$ (defined as in 3.1 in terms of $\langle,\rangle$) takes a simple G^ι-equivariant perverse sheaf A on L_nG to a simple G^ι-equivariant perverse sheaf $\Phi_n^G(A)$ (or $\Phi_n(A)$) on $L_{-n}G$. Moreover $A \mapsto \Phi_n(A)$ defines a bijection $\mathbf{B}(L_nG) \xrightarrow{\sim} \mathbf{B}(L_{-n}G)$, and this extends uniquely to an isomorphism $\Phi_n : \mathcal{K}(L_nG) \xrightarrow{\sim} \mathcal{K}(L_{-n}G)$ of $\mathcal{A}$-modules. From [L5, 3.14 (a)], we see that this is the inverse of the isomorphism $\Phi_{-n} : \mathcal{K}(L_{-n}G) \xrightarrow{\sim} \mathcal{K}(L_nG)$ defined like Φ_n in terms of $-n$ instead of n. Let

$$\kappa \mapsto \dot{\kappa}, \quad \mathfrak{I}_{L_nG} \to \mathfrak{I}_{L_{-n}G}$$

be the bijection, such that for any $\kappa \in \mathfrak{I}_{L_nG}$ we have

$$\Phi_n(\underline{\kappa}^\bullet) = \underline{\dot{\kappa}}^\bullet. \tag{a}$$

The inverse of this bijection is denoted again by $\kappa \mapsto \dot{\kappa}$.

For any simple G^ι-equivariant perverse sheaf A on L_nG, the restriction of the G^ι-action on LG_n to Z_G (a subgroup of G^ι) is the trivial action of Z_G. Then Z_G acts naturally by automorphisms on A and this action is via scalar multiplication by a character $c_A : Z_G \to \mathbf{k}^*$ (trivial on the identity component of Z_G). From the definitions, we see that

$$c_{\Phi_n(A)} = c_A. \tag{b}$$

Now assume that (G, ι) is rigid and that $\mathcal{L} \in \tilde{\mathfrak{I}}_{L_nG}$ is cuspidal (so that $S_{\mathcal{L}} = \overset{\circ}{L}_nG$). Let $A = IC(L_nG, \mathcal{L})[[\dim L_nG/2]]$. According to [L5, 10.6 (e)], we have $\Phi_n(A) = IC(L_{-n}G, \mathcal{L}')[[\dim L_{-n}G/2]]$ where $\mathcal{L}' \in \tilde{\mathfrak{I}}_{L_{-n}G}$ is cuspidal (so that $S_{\mathcal{L}} = \overset{\circ}{L}_{-n}G$). We can find $\mathcal{F} \in \mathcal{T}_G^{\mathrm{cu}}, \mathcal{F}' \in \mathcal{T}_G^{\mathrm{cu}}$ such that $\mathcal{L} = \mathcal{F}|_{\overset{\circ}{L}_nG}$, $\mathcal{L}' = \mathcal{F}'|_{\overset{\circ}{L}_{-n}G}$. We show:

$$\mathcal{F} = \mathcal{F}'. \tag{c}$$

From (b), we see that the natural action of Z_G on any stalk of $\mathcal{L}$ and $\mathcal{L}'$ is through the same character of Z_G. Now Z_G also acts naturally on each stalk of $\mathcal{F}$ and $\mathcal{F}'$ through some character of Z_G (in the adjoint action of G on LG, Z_G acts trivially). Since $\mathcal{L} = \mathcal{F}|_{\overset{\circ}{L}_nG}$, $\mathcal{L}' = \mathcal{F}'|_{\overset{\circ}{L}_{-n}G}$, the character of Z_G attached to $\mathcal{F}$ is the same as the character of Z_G attached to $\mathcal{F}'$. But from the classification of cuspidal local systems, it is known that an object in $\mathcal{T}_G^{\mathrm{cu}}$ is completely determined by the associated character of Z_G. This proves (c).

3.13 Let $P \in \mathcal{P}^\iota$. Define an $\mathcal{A}$-linear map $\widetilde{\mathrm{Ind}}_{LP}^{LG} : \mathcal{K}(L_n\underline{P}) \to \mathcal{K}(L_nG)$ as in [L5, 6.2]. We show:

$$\widetilde{\mathrm{Ind}}_{LP}^{LG}(\xi) = \mathrm{ind}_P^G(\xi) \text{ for any } \xi \in \mathcal{K}(L_n\underline{P}). \tag{a}$$

Let A be a simple $\underline{P}^\iota$-equivariant perverse sheaf on $L_n\underline{P}$. Let $\tilde{A}$ and $c : E'' \to L_nG$ be as in 3.5. We regard A as a pure complex of weight zero (relative to a rational structure over some F_q). Then $\tilde{A}$ is naturally pure of weight zero and, by Deligne [D], $c_!\tilde{A}$ is pure of weight zero. Using [BBD], we deduce that for any j, ${}^pH^j(c_!\tilde{A})$ is pure of weight j. Hence, the definition of $\mathrm{ind}_P^G(A)$ becomes

$$\mathrm{ind}_P^G(A) = \sum_{A_1} \sum_{j \in \mathbf{Z}} (\text{mult. of } A_1 \text{ in } {}^pH^j(c_!\tilde{A}))v^{-j} A_1, \tag{b}$$

where A_1 runs over the set of simple G^ι-equivariant perverse sheaves on L_nG (up to isomorphism). On the other hand, since $c_!\tilde{A}$ has weight zero, we have (by [BBD]) $c_!\tilde{A} \cong \bigoplus_{j \in \mathbf{Z}} {}^pH^j(c_!\tilde{A})[-j]$ in $\mathcal{D}(L_nG)$; hence $\widetilde{\mathrm{Ind}}_{LP}^{LG}(A)$ is equal to the right hand side of (b). We see that $\mathrm{ind}_P^G(A) = \widetilde{\mathrm{Ind}}_{LP}^{LG}(A)$. This proves (a).

From (a), we see that a number of results proved in [L5] for $\widetilde{\mathrm{Ind}}_{LP}^{LG}$ imply corresponding results for ind_P^G (see (c)–(f) below).

(c) *The elements* $\mathrm{ind}_P^G(A) \in \mathcal{K}(L_n G)$ *with* $P \in \mathcal{P}^\iota$ *and* A *as in* 3.6 *span the* $\mathbf{Q}(v)$-*vector space* $\mathbf{Q}(v) \otimes_{\mathcal{A}} \mathcal{K}(L_n G)$.
(We use [L5, 13.3, 17.3].)

(d) *Let* $P, P' \in \mathcal{P}^\iota$. *Assume that* $P \subset P'$. *Let* Q *be the image of* P *under* $P' \to \underline{P'}$. *Note that* Q *is a parabolic subgroup of* $\underline{P'}$ *containing* $\iota(\mathbf{k}^*)$ *and* $\underline{Q} = \underline{P}$. *Then* $\mathrm{ind}_P^G : \mathcal{K}(L_n \underline{P}) \to \mathcal{K}(L_n G)$ *is equal to the composition*

$$\mathcal{K}(L_n \underline{P}) \xrightarrow{\mathrm{ind}_Q^{\underline{P'}}} \mathcal{K}(L_n \underline{P'}) \xrightarrow{\mathrm{ind}_{P'}^G} \mathcal{K}(L_n G).$$

(We use [L5, 6.4].)

(e) *Let* $P \in \mathcal{P}^\iota$. *For any* $\xi \in \mathcal{K}(L_n \underline{P})$, *we have* $\mathrm{ind}_P^G(\Phi_n^{\underline{P}}(\xi)) = \Phi_n^G(\mathrm{ind}_P^G(\xi)) \in \mathcal{K}(L_{-n} G)$ (*here* ind_P^G *in the left hand side is defined in terms of* $-n$ *instead of* n).
(We use [L5, 10.5].)

(f) *Let* $P \in \mathcal{P}^\iota$, $\xi \in \mathcal{K}(L_n \underline{P})$. *We have* $\mathrm{ind}_P^G(\beta(\xi)) = \beta(\mathrm{ind}_P^G(\xi))$.

The proof is along the lines of the proof of the analogous equality [L7, 36.9 (c)], which is based on the relative hard Lefschetz theorem [BBD].

3.14 Let $\mathbf{Z}((v))$ be the ring of power series $\sum_{j\in\mathbf{Z}} a_j v^j$ ($a_j \in \mathbf{Z}$) such that $a_j = 0$ for $j \ll 0$. We have naturally $\mathcal{A} \subset \mathbf{Z}((v))$ and $\mathbf{Z}((v))$ becomes an $\mathcal{A}$-algebra.

Let $\{,\} : \mathcal{K}(L_n G) \times \mathcal{K}(L_n G) \to \mathbf{Z}((v))$ be the $\mathcal{A}$-bilinear pairing defined in [L5, 3.11]. We show:

$$(\xi : \xi') = \vartheta_{G^\iota}\{\xi, \xi'\}|_{v\mapsto -v} \text{ for any } \xi, \xi' \in \mathcal{K}(L_n G). \tag{a}$$

If $\xi = \mathrm{ind}_P^G(A), \xi' = \mathrm{ind}_{P'}^G(A')$ with P, P', A, A' as in 3.7, then (a) follows by comparing 2.11 (b) with the analogous formula for $\{\widetilde{\mathrm{Ind}}_{LP}^{LG}(A), \widetilde{\mathrm{Ind}}_{LP'}^{LG}(A')\}$ in [L5, 15.3]. (We use also 2.13 (a).) This implies (a) in general, in view of 2.13 (c).

For $\kappa \in \mathfrak{I}_{L_n G}$, $\mathcal{L} \in \kappa$, we define $\kappa^* \in \mathfrak{I}_{L_n G}$ by $\mathcal{L}^* \in \kappa^*$. Then $\underline{\kappa^*}^\bullet$ is the Verdier dual of $\underline{\kappa}^\bullet$. We show that for κ, κ' in $\mathfrak{I}_{L_n G}$, we have

$$(\underline{\kappa}^\bullet : \underline{\kappa'}^\bullet) \in \delta_{\kappa',\kappa^*} + v\mathbf{Z}[v]. \tag{b}$$

$$(\underline{\kappa} : \underline{\kappa'}) \in \delta_{\kappa',\kappa^*} + v\mathbf{Z}[v]. \tag{c}$$

Using (a) and [L5, 3.11 (d)], we see that to prove (b) it is enough to verify the following statement.

If $f \in \mathcal{A}$ and $\vartheta_{G^\iota}^{-1} f \in \delta + v\mathbf{Z}[[v]]$ with $\delta \in \mathbf{Z}$, then $f \in \delta + v\mathbf{Z}[v]$.
This is clear since $\vartheta_{G^\iota}^{-1} \in 1 + v\mathbf{Z}[[v]]$.

Now (c) follows from (b) using the fact that the transition matrix from $(\underline{\kappa})$ to $(\underline{\kappa}^\bullet)$ is uni-triangular with off-diagonal entries in $v\mathbf{Z}[v]$ (see 2.4 (a)).

3.15 Let $x \in L_nG$. Let $\mathcal{O}$ be the G^ι-orbit of x in L_nG. Let $P = P(x)$ be the parabolic subgroup associated with x in [L5, 5.2]. Recall that $\iota(\mathbf{k}^*) \subset P$. We show:

(a) *The adjoint action of* U_P^ι *on* $x + L_nU_P$ *is transitive.*

Let S be the orbit of x under this action. Since S is an orbit of an action of a unipotent group on the affine space $x + L_nU_P$, it is closed in $x + L_nU_P$. Hence, it is enough to show that $\dim S = \dim L_nU_P$ or that $\dim U_P^\iota - \dim(U_P^\iota)_x = \dim L_nU_P$, where $(U_P^\iota)_x$ is the stabilizer of x in U_P^ι. The last equality is proved in the course of the proof of [L5, 5.9]. This proves (a).

Let $\pi : L_nP \to L_n\underline{P}$ be the canonical map. We have $x \in \pi^{-1}(\overset{\circ}{L}_n\underline{P})$, see [L5, 5.3 (b)]. We show:

(b) *The adjoint action of* P^ι *on* $\pi^{-1}(\overset{\circ}{L}_n\underline{P})$ *is transitive.*

Let $y \in \pi^{-1}(\overset{\circ}{L}_n\underline{P})$. To show that y is in the P^ι-orbit of x, we may replace y by a P^ι-conjugate. Hence, we may assume that $y \in x + L_nU_P$. In that case, we may use (a). This proves (b).

Let $L_n\underline{P} \xleftarrow{a} E' \xrightarrow{b} E'' \xrightarrow{c} L_nG$ be as in 3.5 (defined in terms of the present P). Let $E_1'' = G^\iota \times_{P^\iota} \pi^{-1}(\overset{\circ}{L}_n\underline{P})$, an open subset of E''. We show:

(c) $c : E'' \to L_nG$ *restricts to an isomorphism* $E_1'' \to \mathcal{O}$.

Using (b), we see that G^ι acts transitively on E_1''; hence, $c(E_1'')$ is a single G^ι-orbit. It contains x, hence it equals $\mathcal{O}$. We see that $E_1'' \subset c^{-1}(\mathcal{O})$. By the proof of [L5, 6.8 (b)], c restricts to an isomorphism $c^{-1}(\mathcal{O}) \xrightarrow{\sim} \mathcal{O}$. In particular, $c^{-1}(\mathcal{O})$ is a single G^ι-orbit. Since E_1'' is a G^ι-orbit contained in $c^{-1}(\mathcal{O})$, we must have $E_1'' = c^{-1}(\mathcal{O})$ and (c) follows.

Now let $\mathcal{L}'$ be an irreducible $\underline{P}^\iota$-equivariant local system on $\overset{\circ}{L}_n\underline{P}$. Define $\kappa' \in \mathfrak{I}_{L_n\underline{P}}$ by $\mathcal{L}' \in \kappa'$. Then $\underline{\kappa}' \in \mathcal{K}(L_n\underline{P})$ is defined as in 3.4. Let $\tilde{\mathcal{L}}'$ be the local system on E_1'' whose inverse image under the obvious map $G^\iota \times_{U_P^\iota} \pi^{-1}(\overset{\circ}{L}_n\underline{P}) \to E_1''$ coincides with the inverse image of $\mathcal{L}'$ under

$$G^\iota \times_{U_P^\iota} \pi^{-1}(\overset{\circ}{L}_n\underline{P}) \to \overset{\circ}{L}_n\underline{P}_n, (g, x) \mapsto \pi(x).$$

Let $\mathcal{L}$ be the irreducible G^ι-equivariant local system on $\mathcal{O}$ corresponding to $\tilde{\mathcal{L}}'$ under the isomorphism $E_1'' \to \mathcal{O}$ induced by c. (See (c).) Define $\kappa \in \mathfrak{I}_{L_nG}$ by $\mathcal{L} \in \kappa$. Then $\underline{\kappa} \in \mathcal{K}(L_nG)$ is defined as in 3.4. We show:

(d) $\underline{\kappa} = \mathrm{ind}_P^G(\underline{\kappa}')$.

We choose an F_q-rational structure on G as in 3.4, so that x is F_q-rational; hence, P is defined over F_q and a mixed structure for $\mathcal{L}'$ which is pure of weight 0. Let $i' : \overset{\circ}{L}_n\underline{P}_n \to L_n\underline{P}$, $i : \mathcal{O} \to L_nG$, $i_1 : E_1'' \to E''$ be the inclusions. Let $(A_r)_{r \in [1,m]}$ be a set of representatives for the simple $\underline{P}^\iota$-equivariant perverse sheaves on $L_n\underline{P}$. Now $A = i'_!\mathcal{L}'[[\dim \overset{\circ}{L}_n\underline{P}/2]]$ is naturally a mixed complex on $L_n\underline{P}$, and we have

$$\underline{\kappa}' = \sum_{j,h} \sum_{r \in [1,m]} (-1)^j (\text{mult. of } A_r \text{ in } {}^pH_j(A)_h)(-v)^{-h} A_r \in \mathcal{K}(L_n\underline{P}).$$

We attach to each A_r a simple perverse sheaf $\tilde{A}_r$ on E'' by

$$a^* A_r[[s/2]] = b^* \tilde{A}_r[[\dim \underline{P}^\iota/2]],$$

where s is as in 3.5. Let $\tilde{A} = i_{1!}\tilde{\mathcal{L}}'[[\dim E_1''/2]] \in \mathcal{D}(E'')$. From the definitions we have $a^* A[[s/2]] = b^* \tilde{A}[[\dim \underline{P}^\iota/2]]$. Since a, b are smooth morphisms with connected fibres of dimension s, $\dim \underline{P}^\iota$, we deduce that

$$a^*({}^p H^j(A)_h)[[s/2]] = b^*({}^p H^j(\tilde{A})_h)[[\dim \underline{P}^\iota/2]]$$

and

$$\text{mult. of } A_r \text{ in } {}^p H_j(A)_h = \text{mult. of } \tilde{A}_r \text{ in } {}^p H_j(\tilde{A})_h$$

for any r, j, h. Hence,

$$\underline{\kappa}' = \sum_{j,h} \sum_{r \in [1,m]} (-1)^j (\text{mult. of } \tilde{A}_r \text{ in } {}^p H_j(\tilde{A})_h)(-v)^{-h} A_r.$$

By the definition of ind_P^G, we have (in $\mathcal{K}(L_n G)$):

$$\begin{aligned}\mathrm{ind}_P^G(\underline{\kappa}') &= \sum_{j,h} \sum_{r \in [1,m]} (-1)^j (\text{mult. of } \tilde{A}_r \text{ in } {}^p H_j(\tilde{A})_h)(-v)^{-h} gr(c_!(\tilde{A}_r)) \\ &= gr(c_!\xi),\end{aligned}$$

where

$$\xi = \sum_{j,h} \sum_{r \in [1,m]} (-1)^j (\text{mult. of } \tilde{A}_r \text{ in } {}^p H_j(\tilde{A})_h)(-v)^{-h} \tilde{A}_r = gr(\tilde{A}) \in \mathcal{K}^{\mathfrak{f}}(E'')$$

and $\mathfrak{f}$ is the family of simple perverse sheaves on E'' of the form $\tilde{A}_r$ $(r \in [1, m])$. Thus,

$$\mathrm{ind}_P^G(\underline{\kappa}') = gr(c_!(gr(i_{1!}\tilde{\mathcal{L}}'[[\dim E_1''/2]]))).$$

Let $\mathfrak{f}_1$ be the family of simple G^ι-equivariant perverse sheaves on E_1''. Let $\mathfrak{f}_0$ be the family of simple G^ι-equivariant perverse sheaves on $L_n G$. Applying 2.2 (a) to $\Theta = i_{1!} : \mathcal{D}^{\mathfrak{f}_1}(E_1'') \to \mathcal{D}^{\mathfrak{f}}(E'')$, $\Theta' = c_! : \mathcal{D}^{\mathfrak{f}}(E'') \to \mathcal{D}^{\mathfrak{f}_0}(L_n G)$, we obtain $gr(c_! i_{1!}) = gr(c_!) gr(i_{1!})$. We see that

$$\mathrm{ind}_P^G(\underline{\kappa}') = gr(c_! i_{1!}\tilde{\mathcal{L}}'[[\dim \mathcal{O}/2]]).$$

(Recall that $\dim E_1'' = \dim \mathcal{O}$.) From the definitions, we have $c_! i_{1!}\tilde{\mathcal{L}}' = i_! \mathcal{L}$. Hence,

$$\mathrm{ind}_P^G(\underline{\kappa}') = gr(i_! \mathcal{L}[[\dim \mathcal{O}/2]]) = \underline{\kappa}.$$

This proves (d).

3.16 We preserve the setup of 3.15. Let $\mathcal{O}_2$ be a $\underline{P}^\iota$-orbit in $L_n\underline{P} - \overset{\circ}{L}_n\underline{P}$. Let $E_2'' = G^\iota \times_{P^\iota} \pi^{-1}(\mathcal{O}_2)$, a subset of E''. We show:

$$\textit{The image of } E_2'' \textit{under } c{:}E'' \to L_nG \textit{is contained in } \bar{\mathcal{O}} - \mathcal{O}. \tag{a}$$

Let $y \in c(E'')$. We show that $y \in \bar{\mathcal{O}}$. We have $y = \mathrm{Ad}(g)\eta$ for some $g \in G^\iota$ and $\eta \in L_nP$. Replacing y by $\mathrm{Ad}(g^{-1})y$, we may assume that $y \in L_nP$. By [L5, 5.9], L_nP is contained in the closure of the P^ι-orbit of x in L_nP which is clearly contained in $\bar{\mathcal{O}}$. Thus $y \in \bar{\mathcal{O}}$. We see that $c(E'') \subset \bar{\mathcal{O}}$. In particular, $c(E_2'') \subset \bar{\mathcal{O}}$. By the proof of 2.15 (c), we have $E_1'' = c^{-1}(\mathcal{O})$. Since $E_1'' \cap E_2'' = \emptyset$, we have $c^{-1}(\mathcal{O}) \cap E_2'' = \emptyset$; hence $c(E_2'') \cap \mathcal{O} = \emptyset$. Thus, $c(E_2'') \subset \bar{\mathcal{O}} - \mathcal{O}$ and (a) is proved.

Now let $\kappa'' \in \mathfrak{I}_{L_n\underline{P}}$ be such that $S_{\kappa''} = \mathcal{O}_2$. Then $\underline{\kappa}'' \in \mathcal{K}(L_n\underline{P})$ is defined as in 3.4. From the definitions, we see that $\mathrm{ind}_P^G(\underline{\kappa}'') \in \sum_\kappa \mathcal{A}\underline{\kappa}$ where κ runs over the elements of $\mathfrak{I}_{L_nG}$ such that S_κ is contained in the closure of $c(E_2'')$. Using this and (a), we see that

$$\mathrm{ind}_P^G(\underline{\kappa}'') \in \sum_{\kappa; S_\kappa \subset \bar{\mathcal{O}} - \mathcal{O}} \mathcal{A}\underline{\kappa}. \tag{b}$$

3.17 Let $\mathcal{L}', \mathcal{L}'' \in \tilde{\mathfrak{I}}_{L_nG}$. Define $\kappa', \kappa'' \in \mathfrak{I}_{L_nG}$ by $\mathcal{L}' \in \kappa'$, $\mathcal{L}'' \in \kappa''$. Assume that $S_{\mathcal{L}'} \cap S_{\mathcal{L}''} = \emptyset$. We show:

$$(\underline{\kappa}' : \underline{\kappa}'') = 0. \tag{a}$$

We choose an F_q-rational structure on G as in 3.4 and mixed structures on $\mathcal{L}', \mathcal{L}''$ which makes them pure of weight 0. Let $i' : S_{\mathcal{L}'} \to L_nG$, $i'' : S_{\mathcal{L}''} \to L_nG$. Let $V = S_{\mathcal{L}'} \times S_{\mathcal{L}''}$, $V_1 = L_nG \times L_nG$, $V_2 =$ point. Let $\mathfrak{f}_1, \mathfrak{f}_2$ be as in 3.11. Let $\mathfrak{f}$ be the family of simple perverse sheaves on V consisting of $\mathcal{L}' \boxtimes \mathcal{L}''$. Let $\Theta = (i', i'')_! : \mathcal{D}(V) \to \mathcal{D}(V_1)$. Let $\Theta' = \rho_! i^* : \mathcal{D}(V_1) \to \mathcal{D}(V_2)$ where ρ, i are as in 3.10. Then 2.2 (a) is applicable. Thus, we have $gr(\rho_! i^*(i', i'')_!) = gr(\rho_! i^*) gr((i', i'')_!)$. Hence

$$gr(\rho_! i^*) gr((i', i'')_!)(\mathcal{L}' \boxtimes \mathcal{L}'') = gr(\rho_! i^*(i', i'')_!)(\mathcal{L}' \boxtimes \mathcal{L}'').$$

The right hand side is zero since $i^*(i', i'')_!(\mathcal{L}' \boxtimes \mathcal{L}'') = 0$ (by our assumption $S_{\mathcal{L}'} \cap S_{\mathcal{L}''} = \emptyset$). Thus, $gr(\rho_! i^*) gr((i', i'')_!)(\mathcal{L}' \boxtimes \mathcal{L}'') = 0$. Using this and 2.10 (a), we see that (a) holds.

4 Computation of Multiplicities

4.1 For any $n \in \Delta$, let $G^\iota \backslash L_nG$ be the (finite) set of G^ι-orbits on L_nG. We define a map

$$\mathfrak{P}_n \to G^\iota \backslash L_nG \tag{a}$$

as follows. Let $P \in \mathfrak{P}_n$. Let $M, L_t^r G$ be as in 2.9. We have $L_0^0 = LG'$ for a well defined connected reductive subgroup G' of G. Now G' acts on $L_n^n G$ (by restriction of the adjoint action of G on LG), and there is a unique open G'-orbit $\mathcal{O}_0$ for this action. Since $G' \subset G^\iota$, there is a unique G^ι-orbit $\mathcal{O}$ on $L_n G$ that contains $\mathcal{O}_0$. Now (a) associates $\mathcal{O}$ with P. Clearly (a) factors through a map

$$\underline{\mathfrak{P}}_n \to G^\iota \backslash L_n G. \tag{b}$$

This is a bijection. Its inverse associates with the G^ι-orbit of $x \in L_n G$, the G^ι-orbit of the parabolic subgroup associated with x in [L5, 5.2].

Another parametrization of $G^\iota \backslash L_n G$ was given by Vinberg [V] (see also Kawanaka [K]).

For any $\eta \in \underline{\mathfrak{P}}_n$, let $\mathcal{O}_\eta$ be the G^ι-orbit in $L_n G$ corresponding to η under (b). Note that

$$\dim \mathcal{O}_\eta = d_\eta, \tag{c}$$

where d_η is as in 2.10. This follows easily from [L5, 5.4 (a), 5.9].

4.2 In the remainder of this section, we assume that $\mathbf{k}$ is as in 3.1. From the bijection 3.1 (b), we see that $B(L_n G) = \bigsqcup_{\eta \in \underline{\mathfrak{P}}_n} B_n^\eta$, where

$$B_n^\eta := \{\underline{\kappa}; \kappa \in \mathfrak{I}_{L_n G}, S_\kappa = \mathcal{O}_\eta\}.$$

4.3 We set ${}^{\mathbf{Q}(v)}\mathcal{K}(L_n G) = \mathbf{Q}(v) \otimes_{\mathcal{A}} \mathcal{K}(L_n G)$. For $n \in \Delta$, we define a $\mathbf{Q}(v)$-linear map

$$t_n = t_n^G : K_G \to {}^{\mathbf{Q}(v)}\mathcal{K}(L_n G)$$

by sending the basis element $\mathbf{I}_{\mathcal{S}}$ to $\operatorname{ind}_P^G(A)$ where $(P, \mathcal{E}) \in \mathcal{S}$ and

$$A = IC(L_n \underline{P}, \mathcal{E}|_{\overset{\circ}{L_n \underline{P}}})[[\dim L_n \underline{P}/2]].$$

The pairing ${}^{\mathbf{Q}(v)}\mathcal{K}(L_n G) \times {}^{\mathbf{Q}(v)}\mathcal{K}(L_n G) \to \mathbf{Q}(v)$ obtained from the pairing (:) in 3.10 by linear extension will be denoted again by (:). From 2.11 (b), we see that the equality

$$(t_n(\xi) : t_n(\xi')) = (\xi : \xi') \tag{a}$$

holds when ξ, ξ' run through a basis of K_G; hence, it holds for any ξ, ξ' in K_G. Since t_n is surjective (see 2.13 (c)) and the pairing (:) on ${}^{\mathbf{Q}(v)}\mathcal{K}(L_n G)$ is non-degenerate (see 2.14 (b)), we see that $\ker t_n = \mathcal{R}_G$ so that t_n induces an isomorphism

$$\tilde{t}_n : K_G/\mathcal{R}_G \to {}^{\mathbf{Q}(v)}\mathcal{K}(L_n G). \tag{b}$$

4.4 Let $n \in \Delta$. We extend $\beta : \mathcal{K}(L_n G) \to \mathcal{K}(L_n G)$ to a $\mathbf{Q}$-linear endomorphism of ${}^{\mathbf{Q}(v)}\mathcal{K}(L_n G)$ (denoted again by β) by $\rho \otimes \xi \mapsto \bar{\rho} \otimes \beta(\xi)$. We show that

$$\beta(t_n(\xi)) = t_n(\beta(\xi)) \tag{a}$$

for any $\xi \in K_G$. We may assume that $\xi = \mathbf{I}_{\mathcal{S}}$ for some $\mathcal{S} \in \underline{\mathcal{J}}_G$. Then we have $\beta(\xi) = \xi$. It is enough to show that $\beta(\operatorname{ind}_P^G(A)) = \operatorname{ind}_P^G(A)$ where P, A are as in 4.3. By 2.13 (f), we have $\beta(\operatorname{ind}_P^G(A)) = \operatorname{ind}_P^G(\beta(A))$, and it remains to note that $\beta(A) = A$.

We extend $\Phi_n : \mathcal{K}(L_nG) \to \mathcal{K}(L_{-n}G)$ to a $\mathbf{Q}(v)$-linear isomorphism ${}^{\mathbf{Q}(v)}\mathcal{K}(L_nG) \to {}^{\mathbf{Q}(v)}\mathcal{K}(L_{-n}G)$ (denoted again by Φ_n). We show that

$$\Phi_n(t_n(\xi)) = t_{-n}(\xi) \tag{b}$$

for any $\xi \in K_G$. We may assume that $\xi = \mathbf{I}_{\mathcal{S}}$ for some $\mathcal{S} \in \underline{\mathcal{J}}_G$. It is then enough to show that if $(P, \mathcal{E}) \in \mathcal{S}$ and

$$A = IC(L_n\underline{P}, \mathcal{E}|_{\overset{\circ}{L_n\underline{P}}})[[\dim L_n\underline{P}/2]],$$
$$A' = IC(L_{-n}\underline{P}, \mathcal{E}|_{\overset{\circ}{L_{-n}\underline{P}}})[[\dim L_{-n}\underline{P}/2]],$$

then $\Phi_n^G(\mathrm{ind}_P^G(A)) = \mathrm{ind}_P^G(A')$. Using 2.13 (e), we see that it is enough to show that $\mathrm{ind}_P^G(\Phi_n^{\underline{P}}(A)) = \mathrm{ind}_P^G(A')$. Hence, it is enough to show that $\Phi_n^{\underline{P}}(A) = A'$. This follows from 2.12 (c) (applied to $\underline{P}$ instead of G).

4.5 Let $n \in \Delta$. Let $Q \in \mathcal{P}^\iota$. For any $\xi \in K_{\underline{Q}}$, we have

$$t_n^G(f_Q^G(\xi)) = \mathrm{ind}_Q^G(t_n^{\underline{Q}}(\xi)), \tag{a}$$

where f_Q^G is as in 2.4 and ind_Q^G is extended by $\mathbf{Q}(v)$-linearity. We may assume that $\xi = \mathbf{I}_{\mathcal{S}'}$ for some $\mathcal{S}' \in \underline{\mathcal{J}}_{\underline{Q}}$; then the result follows from 2.13 (d).

We show that 1.8 (c) holds. Using 3.3 (b), we see that it is enough to show that $t_n(f_Q^G(\mathcal{R}_{\underline{Q}})) = 0$. Using (a), it is enough to show that $\mathrm{ind}_Q^G(t_n^{\underline{Q}}(\mathcal{R}_{\underline{Q}})) = 0$. This follows from $t_n^{\underline{Q}}(\mathcal{R}_{\underline{Q}}) = 0$ (see 4.3).

4.6 In Sect. 2, we tried to associate with any $n \in \Delta$ and $\eta \in \underline{\mathfrak{P}}_n$ a subset $\mathcal{Z}_n^\eta$ of K_G. We will go again through the definitions (with the help of results in Sect. 3) and we will add the requirement that

(a) *for any $\eta \in \underline{\mathfrak{P}}_n$, t_n restricts to a bijection $\mathcal{Z}_n^\eta \xrightarrow{\sim} B_n^\eta$.*

We may assume that G is not a torus and that the subsets $\mathcal{Z}_n^\eta$ are already defined when G is replaced by any $\underline{P}$ with $P \in \mathfrak{P}'_n$. (If G is a torus, then η must be $\{G\}$ and we define $\mathcal{Z}_n^\eta$ to be the subset consisting of the unique basis element of K_G.)

Assume first that $n \in \Delta$ and $\eta \in \underline{\mathfrak{P}}'_n$ (see 2.9). We define $\mathcal{Z}_n^\eta$ as in 2.11. We show that (a) holds for our η. Let $\xi \in \mathcal{Z}_n^\eta$. With notation in 1.11 (i), we have $\xi = f_P^G(\xi')$ for some $\xi' \in \mathcal{Z}_n^{\{\underline{P}\}}$, where $P \in \eta$. Using 3.5 (a), $t_n^G(f_P^G(\xi'))$ is equal to $\mathrm{ind}_P^G(t_n^{\underline{P}}(\xi'))$ which by the induction hypothesis belongs to $\mathrm{ind}_P^G(B_n^{\{\underline{P}\}})$. Thus, $t_n(f_P^G(\xi')) \in \mathrm{ind}_P^G(B_n^{\{\underline{P}\}})$. By 2.15 (d) and its proof, ind_P^G maps $B_n^{\{\underline{P}\}}$ into B_n^η and in fact defines a bijection $a : B_n^{\{\underline{P}\}} \xrightarrow{\sim} B_n^\eta$ (we use the isomorphism in [L5, 5.8]). Thus, we have $t_n(f_P^G(\xi')) \in B_n^\eta$. We consider the diagram

$$\begin{array}{ccc} \mathcal{Z}_n^{\{\underline{P}\}} & \xrightarrow{a_1} & \mathcal{Z}_n^\eta \\ {\scriptstyle a_2}\downarrow & & \downarrow{\scriptstyle a_3} \\ B_n^{\{\underline{P}\}} & \xrightarrow{a} & B_n^\eta, \end{array}$$

where a_1 defined by $\mathrm{ind}_{\underline{P}}^{G}$, a_2 is defined by $t_n^{\underline{P}}$ and a_3 is defined by t_n^G. This diagram is commutative by 3.5 (a). By the induction hypothesis, a_2 is a bijection. We have just seen that a is a bijection. It follows that a_3a_1 is a bijection. Hence a_1 is injective. By the definition of $\mathcal{Z}_n^\eta$, a_1 is surjective. Thus, a_1 is a bijection. In particular, 1.12 (a) holds. Since a_3a_1 is a bijection, we see that a_3 is a bijection. Thus (a) holds in our case.

4.7 Let $n \in \Delta$. Define $\mathcal{Z}'_n$ as in 2.11. Let $L'_nG = \{x \in L_nG; P(x) \neq G\}$. Let

$$I_n = \{\kappa \in \mathfrak{I}_{L_nG}; S_\kappa \subset L'_nG\}, \quad \underline{I}_n = \{\underline{\kappa}; \kappa \in I_n\}.$$

Now t_n defines a bijection $\mathcal{Z}'_n \xrightarrow{\sim} \underline{I}_n$.

We show that 1.12 (b) holds. Let η, η' be two distinct elements of $\underline{\mathfrak{P}}'_n$. Then $\mathcal{Z}_n^\eta, \mathcal{Z}_n^{\eta'}$ are disjoint since their images B_n^η, $B_n^{\eta'}$ under t_n are disjoint. (A local system in B_n^η has a support different from that of a local system in $B_n^{\eta'}$ since 3.1 (b) is a bijection.)

We show that 1.12 (c) holds. Using 3.5 (a) and 3.6 (a), we see that this follows from 2.16 (b). (We use also 3.1 (c).)

We show that 1.12 (d) holds. Using 3.4 (a) and 3.6 (a), we see that it is enough to prove the following statement (for G instead of $\underline{P}$). If (G, ι) is rigid and $\kappa \in \mathfrak{I}_{L_nG} - I_n$, then $\beta(\kappa) - \kappa \in \sum_{\kappa' \in I_n} \mathcal{A}\kappa'$. This is immediate from the definitions.

We show that 1.12 (e) holds. Using 3.3 (a), we see that it is enough to show that the matrix with entries $(t_n(\xi) : t_n(\xi'))$ indexed by $\mathcal{Z}'_n \times \mathcal{Z}'_n$ is non-singular. It is also enough to show that the matrix with entries $(\underline{\kappa} : \underline{\kappa}')$ indexed by $I_n \times I_n$ is non-singular. This follows from 2.14 (c) since I_n is stable under $\kappa \mapsto \kappa^*$.

4.8 Let $[\underline{I}_n]$ be the $\mathcal{A}$-submodule of $\mathcal{K}(L_nG)$ with basis $\underline{I}_n$. Now $L_nG - L'_nG$ is empty (resp. is $\overset{\circ}{L}_nG$) if (G, ι) is not rigid (resp. rigid). Hence L'_nG is a closed subset of L_nG. This, together with 2.4 (a) shows that $\{\underline{\kappa}^\bullet; \kappa \in I_n\}$ is an $\mathcal{A}$-basis of $[\underline{I}_n]$.

4.9 For $\xi \in \mathcal{Z}'_n$, we define W_n^ξ as in 2.13. We have $t_n(\xi) = \underline{\kappa}$ where $\kappa \in I_n$. We show:

(a) $t_n(W_n^\xi) = \underline{\kappa}^\bullet$.

Let $y = t_n(W_n^\xi)$. Applying t_n to the equality $\beta(W_n^\xi) = W_n^\xi \mod \mathcal{R}_G$ in 2.13 and using 3.4 (a), we see that $\beta(y) = y$. Applying t_n to the equality $W_n^\xi = \sum_{\xi_1 \in \mathcal{Z}'_n} c_{\xi,\xi_1}\xi_1$ in 2.13, we obtain $y = \sum_{\xi_1 \in \mathcal{Z}'_n} c_{\xi,\xi_1} t_n(\xi_1)$. We see that $y = \sum_{\kappa'} \tilde{f}_{\kappa,\kappa'}\underline{\kappa}'$ where κ' runs over the elements in $\mathfrak{I}_{L_nG}$ and

$$\tilde{f}_{\kappa,\kappa'} \neq 0 \text{ implies } \dim S_{\kappa'} < \dim S_\kappa \text{ or } \xi = \xi';$$
$$\tilde{f}_{\kappa,\kappa'} \neq 0,\ \kappa \neq \kappa' \text{ implies } \tilde{f}_{\kappa,\kappa'} \in v\mathbf{Z}[v];$$
$$\tilde{f}_{\kappa,\kappa'} = 1 \text{ if } \kappa = \kappa'.$$

These conditions together with $\beta(y) = y$ determine y uniquely. Since $\underline{\kappa}^\bullet$ satisfies the same conditions as y (see 3.4), we see that $y = \underline{\kappa}^\bullet$. This proves (a).

4.10 Until the end of 4.11, we assume that (G, ι) is rigid. For $n \in \Delta$, we define J_{-n} as in 2.15. Let $C = \mathbf{Q}(v) \otimes_{\mathcal{A}} [\underline{I_n}]$. Let $\xi_0 \in \mathcal{Z}'_{-n}$. Then $t_{-n}(\xi_0) = \underline{\kappa}$, where $\kappa \in I_{-n}$. We have $\xi_0 \in J_{-n}$ if and only if $t_n(W_{-n}^{\xi_0}) \notin C$, that is, if and only if $\Phi_n(t_n(W_{-n}^{\xi_0})) \notin \Phi_n(C)$ that is (using 3.4 (b)), if and only if $t_{-n}(W_{-n}^{\xi_0}) \notin \Phi_n(C)$ that is (using 3.9 (a) with n replaced by $-n$), if and only if $\underline{\kappa}^\bullet \notin \Phi_n(C)$. Now $\{\underline{\kappa_1}^\bullet; \kappa_1 \in I_n\}$ is a $\mathbf{Q}(v)$-basis of C. By 2.12 (a), $\{\underline{\dot{\kappa}_1}^\bullet; \kappa_1 \in I_n\}$ is a $\mathbf{Q}(v)$-basis of $\Phi_n(C)$. Hence the condition that $\underline{\kappa}^\bullet \in \Phi_n(C)$ is equivalent to the condition that $\kappa = \dot{\kappa}_1$ for some $\kappa \in I_n$. We see that $\xi_0 \in J_{-n}$ if and only if $\dot{\kappa} \notin I_n$.

Assume now that $\xi_0 \in J_{-n}$. Since $\dot{\kappa} \notin I_n$, we have $S_{\dot{\kappa}} = \overset{\circ}{L}_n G$. Define h_n as in 2.15. Let $z = t_n h_n(\xi_0)$. From the definitions, we have $t_n(W_{-n}^{\xi_0}) - z \in C$ and $(z : C) = 0$. As we have seen earlier, we have $\Phi_n(t_n(W_{-n}^{\xi_0})) = \underline{\kappa}^\bullet$. Hence $t_n(W_{-n}^{\xi_0}) = \Phi_{-n}(\underline{\kappa}^\bullet) = \underline{\dot{\kappa}}^\bullet$. Thus, we have $\underline{\dot{\kappa}}^\bullet - z \in C$. Since $S_{\dot{\kappa}} = \overset{\circ}{L}_n G$, we have $\underline{\dot{\kappa}}^\bullet = \underline{\dot{\kappa}} \mod C$. It follows that $\underline{\dot{\kappa}} - z \in C$. Using $S_{\dot{\kappa}} = \overset{\circ}{L}_n G$ and 2.17 (a), we see that $(\underline{\dot{\kappa}} : C) = 0$. Since $(z : C) = 0$ we see that $(\underline{\dot{\kappa}} - z : C) = 0$. Since $(:)$ is non-degenerate on C (see 4.7) and $\underline{\dot{\kappa}} - z \in C$, we see that $\underline{\dot{\kappa}} - z = 0$. Thus

(a) *$t_n h_n$ is the map $\xi_0 \mapsto \underline{\dot{\kappa}}$ where $\kappa \in I_{-n}$ is given by $t_{-n}(\xi_0) = \underline{\kappa}$.*

We show that 2.16 holds. It is enough to show that h_n is injective. It is also enough to show that $t_n h_n$ is injective. Let $\xi'_0 \in J_{-n}$. Define $\kappa' \in I_{-n}$ by $t_{-n}(\xi'_0) = \underline{\kappa}'$. Assume that $\underline{\dot{\kappa}} = \underline{\dot{\kappa}}'$. Then $\dot{\kappa} = \dot{\kappa}'$ and $\kappa = \kappa'$. Since $t_{-n} : \mathcal{Z}'_{-n} \to I_{-n}$ is bijective, it follows that $\xi_0 = \xi'_0$. Thus 2.16 is proved.

Let $\mathcal{C}_n = h_n(J_{-n})$ (see 2.15). The previous proof shows that the map $\mathcal{C}_n \to t_n(\mathcal{C}_n)$ (restriction of t_n) is a bijection. We see that

$$t_n(\mathcal{C}_n) = \{\underline{\kappa}'; \kappa' \in \mathfrak{I}_{L_n G}, S_{\kappa'} = \overset{\circ}{L}_n G, \dot{\kappa}' \in I_{-n}\}.$$

4.11 Define $\mathcal{C}'$ as in 2.14. If $\mathcal{F} \in \mathcal{T}_G^{\mathrm{pr}}, S_{\mathcal{F}} = C_G^\iota$ and $r_{\mathcal{F}}, \mathcal{S}_{\mathcal{F}}$ are as in 2.14, then for $n \in \Delta$ we have $t_n(r_{\mathcal{F}}^{-1}\mathbf{I}_{\mathcal{S}_{\mathcal{F}}}) = \underline{\kappa} = \underline{\kappa}^\bullet$, where $\kappa \in \mathcal{K}(L_n G)$ is $\mathcal{F}|_{\overset{\circ}{L}_n G}$. (The last two equalities follow from [L5, 11.13].) Replacing n by $-n$, we have similarly $t_{-n}(r_{\mathcal{F}}^{-1}\mathbf{I}_{\mathcal{S}_{\mathcal{F}}}) = \underline{\kappa}' = \underline{\kappa}'^\bullet$ where $\kappa' \in \mathcal{K}(L_{-n} G)$ is $\mathcal{F}|_{\overset{\circ}{L}_{-n} G}$. Using 3.4 (b), we have

$$\Phi_n(\underline{\kappa}^\bullet) = \Phi_n(t_n(r_{\mathcal{F}}^{-1}\mathbf{I}_{\mathcal{S}_{\mathcal{F}}})) = t_{-n}(r_{\mathcal{F}}^{-1}\mathbf{I}_{\mathcal{S}_{\mathcal{F}}}) = \underline{\kappa}'^\bullet.$$

Thus, $\kappa' = \dot{\kappa}$ so that $\dot{\kappa} \notin I_{-n}$.

We show that the map $\mathcal{C}' \to t_n(\mathcal{C}')$ (restriction of t_n) is a bijection. It is enough to note that the map $\mathcal{F} \mapsto \mathcal{F}|_{\overset{\circ}{L}_n G}$ is a bijection from $\mathcal{T}_G^{\mathrm{pr}}$ to the set of semicuspidal objects in $\mathfrak{I}_{L_n G}$. This follows from the fact that if $x \in \overset{\circ}{L}_n G$, the centralizer of x in G and the centralizer of x in G^ι have the same group of components.

We now show that 2.17 holds. First, we show that $\mathcal{C}_n \cap \mathcal{C}' = \emptyset$. It is enough to show that if $\underline{\kappa} \in t_n(\mathcal{C}_n)$ and $\underline{\tilde{\kappa}} \in t_n(\mathcal{C}')$, then $\kappa \neq \tilde{\kappa}$. From our assumption we have $\dot{\kappa} \in I_{-n}$, $\dot{\tilde{\kappa}} \notin I_{-n}$ (see above). Thus, $\kappa \neq \tilde{\kappa}$, as required.

Next we show that $\mathcal{Z}'_n \cap (\mathcal{C}_n \cup \mathcal{C}') = \emptyset$. It is enough to show that if $\underline{\kappa} \in t_n(\mathcal{Z}'_n)$ and $\underline{\tilde{\kappa}} \in t_n(\mathcal{C}_n \cup \mathcal{C}')$, then $\kappa \neq \tilde{\kappa}$. From our assumption we have $S_{\underline{\kappa}} \neq \overset{\circ}{L}_n G$, $S_{\underline{\tilde{\kappa}}} = \overset{\circ}{L}_n G$. Thus, $\kappa \neq \tilde{\kappa}$, as required. This proves 2.17. We see also that the map $\mathcal{Z}'_n \cup \mathcal{C}_n \cup \mathcal{C}' \to t_n(\mathcal{Z}'_n \cup \mathcal{C}_n \cup \mathcal{C}')$ (restriction of t_n) is a bijection.

4.12 If (G, ι) is not rigid, then the definition of the subsets $\mathcal{Z}^\eta_n$ ($\eta \in \underline{\mathfrak{P}}_n$) is complete. If (G, ι) is rigid, then $\underline{\mathfrak{P}}_n - \underline{\mathfrak{P}}'_n = \{G\}$. For $n \in \Delta$, we set $\mathcal{Z}^{\{G\}}_n = \mathcal{C}_n \cup \mathcal{C}'$. By 2.17, this union is disjoint. The definition of the subsets $\mathcal{Z}^\eta_n$ ($\eta \in \underline{\mathfrak{P}}_n$) is complete. Define $\mathcal{Z}_n$ as in 2.18. Note that the map $\mathcal{Z}_n \to t_n(\mathcal{Z}_n)$ (restriction of t_n) is a bijection.

We show that $t_n(\mathcal{Z}_n) = B(L_n G)$. Let $\kappa \in \mathfrak{I}_{L_n G}$. If $\kappa \in I_n$, then $\underline{\kappa} \in t_n(\mathcal{Z}'_n)$. If $\kappa \in \mathfrak{I}_{L_n G} - I_n$ and $\dot{\kappa} \in I_{-n}$, then $\underline{\kappa} \in t_n(\mathcal{C}_n)$. If $\kappa \in \mathfrak{I}_{L_n G} - I_n$ and $\dot{\kappa} \in \mathfrak{I}_{L_{-n} G} - I_{-n}$, then by [L5, 12.3] we have $\underline{\kappa} \in t_n(\mathcal{C}')$. We see that

(a) t_n *restricts to a bijection* $\mathcal{Z}_n \xrightarrow{\sim} B(L_n G)$.

4.13 For $n \in \Delta$ and $\xi \in \mathcal{Z}_n$, we define an element W^ξ_n as in 2.19. We show:

(a) *If* $t_n(\xi) = \underline{\kappa}$ *with* $\kappa \in \mathfrak{I}_{L_n G}$, *then* $t_n(W^\xi_n) = \underline{\kappa}^\bullet$.

When $\xi \in \mathcal{Z}'_n$, this follows from 3.9 (a). Next assume that $\xi \in \mathcal{C}_n$. Define $\xi_0 \in J_{-n}$ by $h_n(\xi_0) = \xi$. Define $\kappa_0 \in \mathfrak{I}_{L_{-n} G}$ by $t_{-n}(\xi_0) = \underline{\kappa}_0$. Using the definition 3.4 (b) and 3.9 (a) (for $-n$ instead of n), we have

$$t_n(W^\xi_n) = t_n(W^{\xi_0}_{-n}) = \Phi_{-n}(t_{-n}(W^{\xi_0}_{-n})) = \Phi_{-n}\underline{\kappa}^\bullet_0 = \underline{\dot{\kappa}_0}^\bullet.$$

By 3.10 (a), we have $\underline{\kappa} = t_n(\xi) = \underline{\dot{\kappa}_0}$. Thus (a) holds in our case.

Finally, assume that $\xi \in \mathcal{C}'$. In this case we have $t_n(W^\xi_n) = t_n(\xi) = \underline{\kappa} = \underline{\kappa}^\bullet$, see 4.11. This proves (a).

Let $\kappa \in \mathfrak{I}_{L_n G}$. Let $\xi \in \mathcal{Z}_n$ be such that $t_n(\xi) = \underline{\kappa}$. Let $c_{\xi,\xi'}$ be as in 2.19 ($\xi' \in \mathcal{Z}_n$). Applying t_n to both sides of 1.19 (a) and using (a) we obtain

$$\underline{\kappa}^\bullet = \sum_{\xi' \in \mathcal{Z}_n} c_{\xi,\xi'} t_n(\xi').$$

Comparing this with 2.4 (a), we see that for any κ, κ' in $\mathfrak{I}_{L_n G}$ we have

$$f_{\kappa,\kappa'} = c_{\xi,\xi'}, \tag{b}$$

where $\xi, \xi' \in \mathcal{Z}_n$ are defined by $t_n\xi = \kappa$, $t_n\xi' = \kappa'$. Note that (b) provides a method to compute explicitly the matrix of multiplicities $(f_{\kappa,\kappa'})$.

4.14 We prove 2.18. Let $\mathcal{S} \in \underline{\mathcal{J}}^D$. Let $(P, \mathcal{E}) \in \mathcal{S}$, $\mathcal{L} = \mathcal{E}|_{\overset{\circ}{L}_n \underline{P}}$. Let $\tilde{c}$, $\dot{\mathcal{L}}$, A be as in 3.6. We regard $\dot{\mathcal{L}}$ as a pure local system of weight 0. By [L5, 21.1 (b)] and its proof, $\mathcal{H}^i \tilde{c}_! \dot{\mathcal{L}}$ is pure of weight i and is 0 unless $i \in 2\mathbf{N}$. It follows that in $\mathcal{K}(L_n G)$ we have $\mathrm{ind}^G_P(A) = \sum_{\kappa \in \mathfrak{I}_{L_n G}} \tilde{e}_{\mathcal{S},\kappa} \underline{\kappa}$, where $\tilde{e}_{\mathcal{S},\kappa} \in \mathcal{A}$ is equal to a power of v times $\sum_i (\text{mult. of } \mathcal{L}' \text{ in } \mathcal{H}^i \tilde{c}_! \dot{\mathcal{L}}) v^{-i}$. (Here $\mathcal{L}' \in \kappa$.) From the definitions we have for any $\xi \in \mathcal{Z}_n$

(a) $e_{\mathcal{S},\xi} = \tilde{e}_{\mathcal{S},\kappa}$,

where $\kappa = t_n(\xi)$. Hence 2.18 holds.

5 Further Results

5.1 In this section, we assume that $\mathbf{k}$ is as in 3.1.

To any $n \in \Delta$ and any $\kappa \in \mathfrak{I}_{L_n G}$, we shall associate a G^ι-orbit $\mathcal{S}_\kappa \in \underline{\mathcal{J}}_G$, an element $r_\kappa \in \mathcal{A} - \{0\}$ and an element $L_\kappa \in \mathcal{K}(L_n G)$ such that $L_\kappa = t_n(r_\kappa^{-1}\mathbf{I}_{\mathcal{S}_\kappa})$. We may assume that these objects are already defined when G is replaced by $\underline{P}$ with $P \in \mathfrak{P}'_n$.

(i) Assume first that κ is semicuspidal. There is a unique $\mathcal{F} \in \mathcal{J}_G^{\mathrm{pr}}$ such that $S_\mathcal{F} \cap L_n\Gamma = \overset{\circ}{L}_n G$ and $\mathcal{F}|_{\overset{\circ}{L}_n G} \in \kappa$. Let $\mathcal{S}_k = \mathcal{S}_\mathcal{F}$ (see 2.18), $r_\kappa = r_\mathcal{F}$ (see 2.3) and $L_\kappa = \underline{\kappa} = \underline{\kappa}^\bullet$. These elements satisfy the required condition (see 4.11).

(ii) Next we assume that $\kappa \in I_n$. We have $\underline{\kappa} \in B_n^\eta$ for a unique $\eta \in \underline{\mathfrak{P}}'_n$. Let $P \in \eta$. Then $P \neq G$. Let $a : B_n^{\{\underline{P}\}} \xrightarrow{\sim} B_n^\eta$ be the bijection in 4.6. Let $\underline{\kappa}_1 = a^{-1}(\underline{\kappa})$. Now $\mathcal{S}_{\kappa_1}, r_{\kappa_1}, L_{\kappa_1} \in \mathcal{K}(L_n\underline{P})$ are already defined from the induction hypothesis. Let $r_\kappa = r_{\kappa_1}$, $L_k = \mathrm{ind}_P^G(L_{\kappa_1})$, $\mathcal{S}_\kappa = a_P^G(\mathcal{S}_{\kappa_1})$. These elements satisfy the required condition.

(iii) Next we assume that $\kappa \notin I_n$ and κ is not semicuspidal. By 4.12, we have $\dot{\kappa} \in I_{-n}$. Now $\mathcal{S}_{\dot{\kappa}}, r_{\dot{\kappa}}, L_{\dot{\kappa}} \in \mathcal{K}(L_{-n}G)$ are defined as in (ii). Let $\mathcal{S}_\kappa = \mathcal{S}_{\dot{\kappa}}$, $r_\kappa = r_{\dot{\kappa}}$, $L_\kappa = \Phi_{-n}(L_{\dot{\kappa}})$. These elements satisfy the required condition.

This completes the definition of $\mathcal{S}_\kappa, r_\kappa, L_\kappa$.

5.2 For $n \in \Delta$, we shall define a partial order $\leq$ on $\mathfrak{I}_{L_n G}$. We may assume that $\leq$ is already defined when G is replaced by $\underline{P}$ with $P \in \mathfrak{P}'_n$.

(i) Assume that at least one of κ, κ' is semicuspidal. Then $\kappa \leq \kappa'$ if and only if $\kappa = \kappa'$.

(ii) Assume that $S_\kappa \neq S_{\kappa'}$ and neither κ nor κ' is semicuspidal. Then $\kappa \leq \kappa'$ if and only if $S_\kappa \subset \bar{S}_{\kappa'} - S_{\kappa'}$.

(iii) Assume that $S_\kappa = S_{\kappa'}$ and $\kappa \in I_n$ (hence also $\kappa' \in I_n$). We have $\underline{\kappa}, \underline{\kappa}' \in B_n^\eta$ for a unique $\eta \in \underline{\mathfrak{P}}'_n$. Let $P \in \eta$; then $P \neq G$. Let $a : B_n^{\{\underline{P}\}} \xrightarrow{\sim} B_n^\eta$ be the bijection in 4.6. Let $\underline{\kappa}_1 = a^{-1}(\underline{\kappa}), \underline{\kappa}'_1 = a^{-1}(\underline{\kappa}')$. We say that $\kappa \leq \kappa'$ if and only if $\kappa_1 \leq \kappa'_1$ (which is known by the inductive assumption applied to $\underline{P}$).

(iv) Assume that $S_\kappa = S_{\kappa'}$, $\kappa \notin I_n$ (hence $\kappa' \notin I_n$) and neither κ nor κ' is semicuspidal. By 4.12, we have $\dot{\kappa} \in I_{-n}$, $\dot{\kappa}' \in I_{-n}$. We say that $\kappa \leq \kappa'$ if and only if $\dot{\kappa} \leq \dot{\kappa}'$ which is known from (ii) (if $S_{\dot{\kappa}} \neq S_{\dot{\kappa}'}$) or (iii) (if $S_{\dot{\kappa}} = S_{\dot{\kappa}'}$).

This completes the definition of $\leq$. We write $\kappa' < \kappa$ instead of $\kappa' \leq \kappa$, $\kappa' \neq \kappa$.

5.3 Example In this subsection, we assume that G is the group of automorphisms of a 4-dimensional $\mathbf{k}$-vector space V preserving a fixed non-degenerate symplectic form. We fix a direct sum decomposition $V = V_{-1} \oplus V_1$ where V_1, V_{-1} are Lagrangian subspaces. For any $t \in \mathbf{k}^*$ define $\iota(t) \in G$ by $\iota(t)x = tx$ for $x \in V_1$, $\iota(t)x = t^{-1}x$ for $x \in V_{-1}$. Let $\Delta = \{2, -2\}$. Note that (G, ι) is rigid. We identify G^ι with $GL(V_1)$ by $g \mapsto g|_{V_1}$. The grading of LG defined by ι has non-zero components in degrees $-2, 0, 2$ and L_2G (resp. $L_{-2}G$) may be identified as a representation of G^ι with S^2V_1 (resp. $S^2V_1^*$) where S^2 stands for the second symmetric power. For $n \in \Delta$, the set $\mathfrak{I}_{L_n G}$ consists of five objects $\kappa_{0,n}, \kappa_{2,n}, \tilde{\kappa}_{2,n}, \kappa_{3,n}, \tilde{\kappa}_{3,n}$ where $\kappa_{i,n}$ represents the local system $\bar{\mathbf{Q}}_l$ on the G^ι-orbit of dimension i $(i = 0, 2, 3)$ and $\tilde{\kappa}_{i,n}$ represents a non-trivial local system of rank 1 on the G^ι-orbit of dimension i $(i = 2, 3)$. The effect of the Fourier–Deligne transform is as follows.

$$\Phi_n(\kappa^\bullet_{0,n}) = \kappa^\bullet_{3,-n}, \Phi_n(\kappa^\bullet_{2,n}) = \tilde{\kappa}^\bullet_{3,-n}, \Phi_n(\kappa^\bullet_{3,n}) = \kappa^\bullet_{0,-n}, \Phi_n(\tilde{\kappa}^\bullet_{3,n}) = \kappa^\bullet_{2,-n},$$
$$\Phi_n(\tilde{\kappa}^\bullet_{2,n}) = \tilde{\kappa}^\bullet_{2,n}.$$

The partial order in 5.2 is $\kappa_{0,n} < \kappa_{2,n} < \kappa_{3,n} < \tilde{\kappa}_{3,n}$; $\tilde{\kappa}_{2,n}$ is isolated. We have

$$L_{\kappa_{0,n}} = \kappa^\bullet_{0,n}, L_{\kappa_{2,n}} = \kappa^\bullet_{2,n} + \kappa^\bullet_{0,n}, L_{\kappa_{3,n}} = \kappa^\bullet_{3,n}, L_{\tilde{\kappa}_{3,n}} = \tilde{\kappa}^\bullet_{3,n} + \kappa^\bullet_{3,n},$$
$$L_{\tilde{\kappa}_{2,n}} = \tilde{\kappa}^\bullet_{2,n}.$$

5.4 We show that for any $n \in \Delta$ and any $\kappa \in \mathfrak{I}_{L_nG}$, we have

(a) $L_\kappa \in \underline{\kappa} + \sum_{\kappa' \in \mathfrak{I}_{L_nG}; \kappa' < \kappa} \mathcal{A}\, \underline{\kappa}'$.

We may assume that (a) is already known when G is replaced by $\underline{P}$ with $P \in \mathfrak{P}'_n$.

(i) Assume that κ is semicuspidal. Then $L_\kappa = \kappa$ and (a) is clear.

(ii) Assume that $\kappa \in I_n$. Let P, a, κ_1 be as in 4.1 (ii). By the induction hypothesis, we have $L_{\kappa_1} \in \underline{\kappa}_1 + \sum_{\kappa'_1; \kappa'_1 < \kappa_1} \mathcal{A}\, \underline{\kappa}'_1$. Applying ind_P^G and using 2.15 (d), we obtain
$L_\kappa \in \underline{\kappa} + \sum_{\kappa'_1 \in X} \mathcal{A}\, \underline{a(\kappa'_1)} + \sum_{\kappa'_1 \in Y} \mathcal{A}\, \mathrm{ind}_P^G(\underline{\kappa}'_1)$,
where
$X = \{\kappa'_1; \kappa'_1 < \kappa_1, S_{\kappa'_1} = S_{\kappa_1}\}, Y = \{\kappa'_1; \kappa'_1 < \kappa_1, S_{\kappa'_1} \neq S_{\kappa_1}\}$.
For $\kappa'_1 \in X$, we have $S_{a(\kappa'_1)} = S_\kappa$ and $a(k'_1) < \kappa$. For $\kappa'_1 \in Y$, we have $S_{\kappa'_1} \subset \bar{S}_{\kappa_1} - S_{\kappa_1}$. By 2.16 (b), for any $\kappa'_1 \in Y$, $\mathrm{ind}_P^G(\underline{\kappa}'_1)$ is an $\mathcal{A}$-linear combination of elements $\underline{\kappa}'$ where $S_{\kappa'} \subset \bar{S}_\kappa - S_\kappa$ (hence $\kappa' < \kappa$). Hence L_κ satisfies (a).

(iii) Assume that $\kappa \notin I_n$ and κ is not semicuspidal. By 4.12, we have $\dot{\kappa} \in I_{-n}$. By (ii), we have
$L_{\dot{\kappa}} \in \underline{\dot{\kappa}} + \sum_{\kappa'; \kappa' < \dot{\kappa}} \mathcal{A}\, \underline{\kappa}'$.

Using $\underline{\kappa}_0 \in \underline{\kappa}^\bullet_0 + \sum_{\kappa'_0 < \kappa_0} \mathcal{A}\, \underline{\kappa}'^\bullet_0$ for $\kappa_0 \in I_{-n}$, we deduce $L_{\dot{\kappa}} \in \underline{\dot{\kappa}}^\bullet + \sum_{\kappa'; \kappa' < \dot{\kappa}} \mathcal{A}\, \underline{\kappa}'^\bullet$. Applying Φ^G_{-n}, we obtain

$L_\kappa \in \underline{\kappa}^\bullet + \sum_{\kappa'; \kappa' < \dot{\kappa}} \mathcal{A}\, \underline{\dot{\kappa}}'^\bullet$.

Using $\underline{\kappa}^\bullet_1 \in \underline{\kappa}_1 + \sum_{\kappa'_1 < \kappa_1} \mathcal{A}\, \underline{\kappa}'_1$ for $\kappa_1 \in I_n$, we see that it is enough to show that for any κ' such that $\kappa' < \dot{\kappa}$ we have $\dot{\kappa}' < \kappa$. If $S_\kappa = S_{\dot{\kappa}'}$, then this follows from 4.2 (iv). If $S_\kappa \neq S_{\dot{\kappa}'}$, then $S_{\dot{\kappa}'} \subset \bar{S}_\kappa - S_\kappa$ (we have $S_\kappa = \overset{\circ}{L}_nG$), hence again $\dot{\kappa}' < \kappa$ (using 4.2 (ii)).

This completes the proof of (a).

From (a), we deduce

(b) *The set $\{L_\kappa; \kappa \in \mathfrak{I}_{L_nG}\}$ is an $\mathcal{A}$-basis of $\mathcal{K}(L_nG)$.*

5.5 We show that:

(a) *the map $\mathfrak{I}_{L_nG} \to \underline{\mathcal{J}}_G, \kappa \mapsto \mathcal{S}_\kappa$ is injective.*

Let $\kappa, \kappa' \in \mathfrak{I}_{L_nG}$ be such that $\mathcal{S}_\kappa = \mathcal{S}_{\kappa'}$. Then $t_n(\mathcal{S}_\kappa) = t_n(\mathcal{S}_{\kappa'})$ hence $r_\kappa L_\kappa = r_{\kappa'} L_{\kappa'}$. Using now 4.4 (b), we deduce that $\kappa = \kappa'$ as desired.

5.6 Let $\xi \in \mathcal{Z}_n$ and let $\kappa = t_n(\xi)$. By [L3, 3.36], the numbers $\tilde{e}_{\mathcal{S},\kappa}|_{v=1}$ (for various $\mathcal{S}$, see 4.14) are the dimensions of the various "weight spaces" of a standard module over an affine Hecke algebra. Using 3.14 (a) we see that the dimensions of these weight spaces are given by the numbers $e_{\mathcal{S},\xi}|_{v=1}$ (for various $\mathcal{S}$), hence are computable from the algorithm in Sect. 2.

5.7 In this subsection, we shall summarize some of the results of this paper in terms of the vector space $\bar{K}_G = K_G/\mathcal{R}_G$. (Any text marked as ♠ . . . ♠ applies only in the case where (G,ι) is rigid.) Note that the pairing (:) on K_G induces a pairing $\bar{K}_G \times \bar{K}_G \to \mathbf{Q}(v)$ denoted again by (:). Also $\beta : K_G \to K_G$ induces an involution $\bar{K}_G \to \bar{K}_G$ denoted again by β. Let $\pi : K_G \to \bar{K}_G$ be the obvious map.

Since for $n \in \Delta$, $B(L_nG)$ is a $\mathbf{Q}(v)$-basis of ${}^{\mathbf{Q}(v)}\mathcal{K}(L_nG)$, we see (using 3.12 (a) and 3.3 (b)) that π restricts to a bijection of $\mathcal{Z}_n$ onto a basis $\overline{\mathcal{Z}}_n$ of $\bar{K}_G$. We say that $\overline{\mathcal{Z}}_n$ is a *PBW-basis* of $\bar{K}_G$. ♠ The last bijection restricts to bijections of $\mathcal{Z}'_n, \mathcal{C}_n, \mathcal{C}'$ onto subsets $\overline{\mathcal{Z}}'_n, \overline{\mathcal{C}}_n, \overline{\mathcal{C}}'$ of $\overline{\mathcal{Z}}_n$. ♠ In the case where (G,ι) is not rigid, we set $\overline{\mathcal{Z}}'_n = \overline{\mathcal{Z}}_n$.

Let $\mathcal{M}_n$ be the $\mathbf{Z}[v]$-submodule of $\bar{K}_G$ with basis $\overline{\mathcal{Z}}_n$. For any $u \in \overline{\mathcal{Z}}_n$ let $\bar{W}^u_n = \pi(W^\xi_n)$, where $\xi \in \mathcal{Z}_n$ is given by $\pi(\xi) = u$. From 3.13 (a) and 2.4 (a) we see that $\{\bar{W}^n_u; u \in \overline{\mathcal{Z}}_n\}$ is a $\mathbf{Z}[v]$-basis of $\mathcal{M}_n$ and that for any $u \in \overline{\mathcal{Z}}_n$ we have $\bar{W}^u_n - u \in v\mathcal{M}_n$. Define a bijection $u \mapsto \dot{u}$ of $\overline{\mathcal{Z}}_n$ onto $\overline{\mathcal{Z}}_{-n}$ as follows. Let $\xi \in \mathcal{Z}_n$ be such that $\pi(\xi) = u$; let $\kappa \in \mathfrak{I}_{L_nG}$ be such that $t_n(\xi) = \underline{\kappa}$ (see 3.12 (a)). Let $\xi' \in \mathcal{Z}_{-n}$ be such that $t_{-n}\xi' = \dot{\underline{\kappa}}$ (see 3.12). Then $\dot{u} = \pi(\xi')$. The inverse of the bijection $u \mapsto \dot{u}$ is denoted again by $u \mapsto \dot{u}$.

For u, ξ, κ, ξ' as above, we have $\bar{W}^u_n = \pi(W^\xi_n)$, $t_n(W^\xi_n) = \underline{\kappa}^\bullet$, $\bar{W}^{\dot{u}}_{-n} = \pi(W^{\xi'}_{-n})$, $t_{-n}(W^{\xi'}_{-n}) = \dot{\underline{\kappa}}^\bullet$. By 2.12 (a), we have $\Phi_n(\underline{\kappa}^\bullet) = \dot{\underline{\kappa}}^\bullet$, hence $t_{-n}(W^{\xi'}_{-n}) = \Phi_n(t_n(W^\xi_n))$ and this equals $t_{-n}(W^\xi_n)$ (see 3.4 (b)). Thus, $t_{-n}(W^{\xi'}_{-n} - W^\xi_n) = 0$. Since $\ker t_{-n} = \mathcal{R}_G$, we see that $W^{\xi'}_{-n} - W^\xi_n \in \mathcal{R}_G$. Applying π, we deduce

(a) $\bar{W}^{\dot{u}}_{-n} = \bar{W}^u_n$.

Moreover, from the proof in 4.10, we see that

(b) ♠ $u \in \overline{\mathcal{C}}_n \implies \dot{u} \in \overline{\mathcal{Z}}_{-n}$. ♠

From (a) we see that the basis $(\bar{W}^u_n)$ of $\bar{K}_G$ coincides with the basis $(\bar{W}^u_{-n})$. We call this the *canonical basis* of $\bar{K}_G$. It follows that $\mathcal{M}_n = \mathcal{M}_{-n}$. We shall write $\mathcal{M}$ instead of $\mathcal{M}_n = \mathcal{M}_{-n}$. We have

(c) $\bar{W}^u_n - u \in v\mathcal{M}$, $\bar{W}^{\dot{u}}_{-n} - \dot{u} \in v\mathcal{M}$.

Combining with (a), we see that

(d) $u - \dot{u} \in v\mathcal{M}$ for any $u \in \overline{\mathcal{Z}}_n$.

Let $\pi' : \mathcal{M} \to \mathcal{M}/v\mathcal{M}$ be the obvious map. From (a), (c), we see that there exists a $\mathbf{Z}$-basis X of $\mathcal{M}/v\mathcal{M}$ such that π' restricts to bijections $(\bar{W}^u_n) = (\bar{W}^u_{-n}) \xrightarrow{\sim} X$, $\overline{\mathcal{Z}}_n \xrightarrow{\sim} X$, $\overline{\mathcal{Z}}_{-n} \xrightarrow{\sim} X$. ♠ Moreover, X can be partitioned as $X = X_0 \sqcup X_n \sqcup X_{-n} \sqcup X'$ so that π' restricts to bijections $\overline{\mathcal{C}}_n \xrightarrow{\sim} X_n$, $\overline{\mathcal{C}}_{-n} \xrightarrow{\sim} X_{-n}$, $\overline{\mathcal{C}}' \xrightarrow{\sim} X'$, $\overline{\mathcal{Z}}'_n \xrightarrow{\sim} X_0 \cup X_{-n}$, $\overline{\mathcal{Z}}'_{-n} \xrightarrow{\sim} X_0 \cup X_n$. ♠ We set

(e) $\tilde{X} = \overline{\mathcal{Z}}'_n \cup \overline{\mathcal{Z}}'_{-n}$ (if (G,ι) is not rigid), $\tilde{X} = \overline{\mathcal{Z}}'_n \cup \overline{\mathcal{Z}}'_{-n} \cup \overline{\mathcal{C}}'$ (if (G,ι) is rigid).

We have

(f) $X = \pi'(\tilde{X})$.

We show that

(g) $\tilde{X}$ generates the $\mathbf{Z}[v]$-module $\mathcal{M}$.

If (G, ι) is not rigid, this is clear. ♠ Assume now that (G, ι) is rigid. Let $\mathcal{M}'$ be the $\mathbf{Z}[v]$-submodule of $\mathcal{M}$ generated by $\tilde{X}$. If $u \in \overline{\mathcal{Z}}'_n$, then by the arguments in 2.13, we have $\bar{W}^u_n \in \sum_{u' \in \overline{\mathcal{Z}}'_n} \mathbf{Z}[v]u'$, hence $\bar{W}^u_n \in \mathcal{M}'$. If $u \in \overline{\mathcal{C}}_n$, then $\bar{W}^u_n = \bar{W}^{\dot{u}}_{-n}$ and this is in $\mathcal{M}'$ since $\dot{u} \in \overline{\mathcal{Z}}'_{-n}$ (we use the previous sentence with u, n replaced by $\dot{u}, -n$). If $u \in \overline{\mathcal{C}}'$, then $\bar{W}^u_n = u$ is again in $\mathcal{M}'$. Since $(\bar{W}^u_n)_{u \in \overline{\mathcal{Z}}_n}$ is a $\mathbf{Z}[v]$-basis of $\mathcal{M}$, we see that $\mathcal{M} = \mathcal{M}'$. ♠ This proves (g).

We show how the canonical basis and the PBW-bases are determined in terms of the subsets $\overline{\mathcal{Z}}'_n, \overline{\mathcal{Z}}'_{-n}$ (which are defined by the inductive construction in 1.11 (i)) and (in the rigid case) by the set $\overline{\mathcal{C}}'$ which is defined as in 2.14.

We first define $\tilde{X}$ as in (e). Note that $\mathcal{M}$ is defined in terms of $\tilde{X}$ as in (g), and then the basis X of $\mathcal{M}/v\mathcal{M}$ is defined in terms of $\tilde{X}$ as in (f).

Now the canonical basis can be reconstructed in terms of $\mathcal{M}$ and X: for any $x \in X$, there is a unique element $\hat{x} \in \mathcal{M}$ such that $\pi'(\hat{x}) = x$ and $\beta(\hat{x}) = \hat{x}$. The elements $\{\hat{x}; x \in X\}$ form the canonical basis. Now let $n \in \Delta$. We show how to reconstruct the PBW-basis $\overline{\mathcal{Z}}_n$. If (G, ι) is not rigid, then $\overline{\mathcal{Z}}_n = \overline{\mathcal{Z}}'_n$ is already known. ♠ Assume now that (G, ι) is rigid. Then the part $\overline{\mathcal{Z}}'_n \cup \overline{\mathcal{C}}'$ of $\overline{\mathcal{Z}}_n$ is already known. It remains to characterize the subset $\overline{\mathcal{C}}_n$ of $\overline{\mathcal{Z}}_n$. For $n \in \Delta$ let X_n be the set of all $x \in \pi'(\overline{\mathcal{Z}}'_{-n})$ such that $x \notin \pi'(\overline{\mathcal{Z}}'_n)$. For any $x \in X_n$, we can write uniquely $\hat{x} = x' + x''$ where x'' is in the subspace of $\bar{K}_G$ spanned by $\overline{\mathcal{Z}}'_n$ and x' is orthogonal under (:) to that subspace. Then $\overline{\mathcal{C}}_n$ consists of the elements x' for various $x \in X_n$. ♠

5.8 Let $n \in \Delta$. Assume that (G, ι) is rigid. Let Ξ_n be the set of all $\kappa \in \mathfrak{I}_{L_n G}$ such that $S_\kappa \neq \overset{\circ}{L}_n G$ and $S_{\dot{\kappa}} = \overset{\circ}{L}_{-n} G$. Here $\kappa \mapsto \dot{\kappa}$ is as in 3.12. It would be interesting to find a simple description of the set of local systems Ξ_n (without using Fourier–Deligne transform). In particular, we would like to know which G^ι-orbits in $L_n G$ are of the form S_κ for some $\kappa \in \Xi_n$. (Our results answer this question only in terms of an algorithm, not in closed form.) For example, if $G = GL_n(\mathbf{k})$, then Ξ_n has only one object: the local system $\bar{\mathbf{Q}}_l$ on the 0-dimensional orbit. In the case studied in 5.3, Ξ_n has two objects: the local system $\bar{\mathbf{Q}}_l$ on the 0 or 2-dimensional orbit.

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Crystal Base Elements of an Extremal Weight Module Fixed by a Diagram Automorphism II: Case of Affine Lie Algebras

Satoshi Naito and Daisuke Sagaki

Abstract We continue the study of the fixed point subset of a crystal under the action of a Dynkin diagram automorphism, by restricting ourselves to the case of the crystal base of an extremal weight module over a quantum affine algebra. In previous works, we introduced and studied a canonical injection from the fixed point subset of the crystal base above into the crystal base of a certain extremal weight module for the associated orbit Lie algebra, in the setting of general Kac–Moody algebras. The purpose of this paper is to prove that this (injective) map is also surjective, and hence bijective under the assumption that the Dynkin diagram automorphism fixes a distinguished vertex "0" of the (affine) Dynkin diagram.

Keywords Crystal bases · Extremal weight modules · Diagram automorphisms

Mathematics Subject Classifications (2000): Primary: 17B37, 17B10; Secondary: 81R50

1 Introduction

In previous works [NS1, NS3], we studied the relation between the crystal base elements of an extremal weight module fixed by a Dynkin diagram automorphism for a general Kac–Moody algebra and the crystal base elements of an extremal weight module for the associated orbit (Kac–Moody) Lie algebra. In this paper, we restrict our attention to the case of affine Lie algebras and continue the study of this relation.

We briefly recall the notation and main results of [NS1, NS3]. Let $\mathfrak{g} = \mathfrak{g}(A)$ be the Kac–Moody algebra over $\mathbb{Q}$ associated with a symmetrizable generalized Cartan matrix (GCM) $A = (a_{ij})_{i,j \in I}$ with Cartan subalgebra $\mathfrak{h}$, simple coroots $\{h_j\}_{j \in I} \subset \mathfrak{h}$, simple roots $\{\alpha_j\}_{j \in I} \subset \mathfrak{h}^*$, and fundamental weights $\{\Lambda_j\}_{j \in I} \subset \mathfrak{h}^*$.

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A diagram automorphism $\omega : I \to I$ of the Dynkin diagram of A naturally induces a Lie algebra automorphism $\omega : \mathfrak{g} \to \mathfrak{g}$ that preserves the triangular decomposition of $\mathfrak{g}$, and hence induces the contragredient map $\omega^* : \mathfrak{h}^* \to \mathfrak{h}^*$ of the restriction $\omega|_{\mathfrak{h}}$ of ω to $\mathfrak{h}$. To the data $(\mathfrak{g}(A), \omega)$, we can associate a certain symmetrizable Kac–Moody algebra $\widehat{\mathfrak{g}}$ with Cartan subalgebra $\widehat{\mathfrak{h}}$, called the orbit Lie algebra. It is known that there exists a natural $\mathbb{Q}$-linear isomorphism $P_\omega^* : \widehat{\mathfrak{h}}^* \xrightarrow{\sim} (\mathfrak{h}^0)^* \cong (\mathfrak{h}^*)^0$, where $\mathfrak{h}^0 := \{h \in \mathfrak{h} \mid \omega(h) = h\}$ and $(\mathfrak{h}^*)^0 := \{\lambda \in \mathfrak{h}^* \mid \omega^*(\lambda) = \lambda\}$.

For an integral weight $\lambda \in P \subset \mathfrak{h}^*$, we denote by $V(\lambda)$ the extremal weight module of extremal weight λ over the quantized universal enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$ over $\mathbb{Q}(q)$, and by $\mathcal{B}(\lambda)$ its crystal base. If $\lambda \in P \cap (\mathfrak{h}^*)^0$, then the crystal base $\mathcal{B}(\lambda)$ comes equipped with a natural action of the diagram automorphism ω, and the fixed point subset of $\mathcal{B}(\lambda)$ under this action of ω is denoted by $\mathcal{B}^0(\lambda)$. In [NS1], for $\lambda \in P \cap (\mathfrak{h}^*)^0$, we obtained a canonical injection $R_\lambda : \mathcal{B}^0(\lambda) \hookrightarrow \widehat{\mathcal{B}}(\widehat{\lambda})$, where $\widehat{\mathcal{B}}(\widehat{\lambda})$ denotes the crystal base of the extremal weight module of extremal weight $\widehat{\lambda} := (P_\omega^*)^{-1}(\lambda) \in \widehat{\mathfrak{h}}^*$ for the orbit Lie algebra $\widehat{\mathfrak{g}}$. Furthermore, in [NS3, Theorem 4.2.1], we proved that if the crystal graph of $\widehat{\mathcal{B}}(\widehat{\lambda})$ is connected, then the (injective) map R_λ above is surjective, and hence bijective. However, since this restriction on the weight $\widehat{\lambda} \in \widehat{\mathfrak{h}}^*$ is rather strong, there are many cases to which this result cannot be applied.

Now, let us explain the main result of this paper. Assume that $\mathfrak{g} = \mathfrak{g}(A)$ is an affine Lie algebra, with canonical central element $c = \sum_{j \in I} a_j^\vee h_j$, and that $\omega : I \to I$ fixes a distinguished index $0 \in I$. Note that in this case, the orbit Lie algebra $\widehat{\mathfrak{g}}$ is an affine Lie algebra $\mathfrak{g}(\widehat{A})$ associated with a certain GCM $\widehat{A} = (\widehat{a}_{ij})_{i,j \in \widehat{I}}$, where $\widehat{I}$ is a complete list of representatives of the ω-orbits in I (see Sect. 4.1 for the explicit Dynkin diagram of $\widehat{A}$). In addition, as is easily seen, there is no loss of generality in assuming that $\lambda \in P \cap (\mathfrak{h}^*)^0$ is "level-zero dominant," i.e. that $\lambda \in P \cap (\mathfrak{h}^*)^0$ is of the form $\lambda = \sum_{i \in I_0} m_i \varpi_i$, with $m_i \in \mathbb{Z}_{\geq 0}$ for $i \in I_0 := I \setminus \{0\}$, where $\varpi_i := \Lambda_i - a_0^\vee \Lambda_0$, $i \in I_0$, are the level-zero fundamental weights for $\mathfrak{g}$. Then, by making use of some deep results in [BN], we can prove the following (main) theorem.

Theorem. *Let the notation and assumptions be as above. Then the* (*injective*) *map* $R_\lambda : \mathcal{B}^0(\lambda) \hookrightarrow \widehat{\mathcal{B}}(\widehat{\lambda})$ *is also surjective, and hence bijective.*

Remark. If $\lambda = \sum_{i \in I_0} m_i \varpi_i \in P \cap (\mathfrak{h}^*)^0$, then $\widehat{\lambda} = (P_\omega^*)^{-1}(\lambda) \in \widehat{\mathfrak{h}}^*$ is of the form $\sum_{i \in \widehat{I}_0} m_i \widehat{\varpi}_i$, where $\widehat{I}_0 := \widehat{I} \setminus \{0\}$, and $\widehat{\varpi}_i$, $i \in \widehat{I}_0$, are the level-zero fundamental weights for the orbit (affine) Lie algebra $\widehat{\mathfrak{g}}$. We know from [BN, Theorem 4.16] that the crystal graph of $\widehat{\mathcal{B}}(\widehat{\lambda})$ is not connected if and only if there exists some $i \in \widehat{I}_0$ for which $m_i \geq 2$, in which case we cannot apply [NS3, Theorem 4.2.1] (cf. [NS3, Example 4.2.2]).

This paper is organized as follows. In Sect. 2, we recall some of the standard facts on the crystal bases of extremal weight modules, and also some deep results of [BN] needed in Sect. 4. In Sect. 3, we briefly review the relevant results from our previous works about the fixed point subset under a diagram automorphism for various crystal bases, including the crystal base of an extremal weight module. Also, we show a few technical lemmas, which will be used in the proof of our main theorem. In Sect. 4, we first fix our notation for diagram automorphisms of affine Lie algebras and associated orbit (affine) Lie algebras, and then state our main theorem (Theorem above). The remainder of Sect. 4 is devoted to the proof of this theorem, for which some deep results in [BN] are needed.

2 Crystal Bases of Extremal Weight Modules

In this section, we first recall some of the standard facts on the crystal bases of extremal weight modules from [Kas3] and [Kas4]. Next, in Sect. 2.6, we review some deep results obtained in [BN] on the crystal bases of extremal weight modules over quantum affine algebras, which are needed for the proof of our main theorem.

2.1 Kac–Moody Algebras and Quantized Universal Enveloping Algebras

Let $\mathfrak{g} := \mathfrak{g}(A)$ be the Kac–Moody algebra over $\mathbb{Q}$ associated with a symmetrizable generalized Cartan matrix (GCM) $A = (a_{ij})_{i,j\in I}$ with Cartan subalgebra $\mathfrak{h}$, simple coroots $\Pi^\vee := \{h_j\}_{j\in I} \subset \mathfrak{h}$, simple roots $\Pi := \{\alpha_j\}_{j\in I} \subset \mathfrak{h}^* := \mathrm{Hom}_{\mathbb{Q}}(\mathfrak{h}, \mathbb{Q})$, and Chevalley generators $\{E_j, F_j \mid j \in I\}$, where $\mathfrak{g}_{\alpha_j} = \mathbb{Q}E_j$ and $\mathfrak{g}_{-\alpha_j} = \mathbb{Q}F_j$. Denote by $W := \langle r_j \mid j \in I\rangle \subset \mathrm{GL}(\mathfrak{h}^*)$ the Weyl group of $\mathfrak{g}$, where $r_j \in \mathrm{GL}(\mathfrak{h}^*)$ is the simple reflection in $\alpha_j \in \mathfrak{h}^*$, and denote by $(\cdot\,,\cdot)$ the standard invariant bilinear form on $\mathfrak{h}^*$.

Take an integral weight lattice $P \subset \mathfrak{h}^*$ such that $\alpha_j \in P$ for all $j \in I$, and set $P^* := \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z}) \subset \mathfrak{h}$. Let q be an indeterminate, and set $q_s := q^{1/D}$, where D is the least positive integer such that $D(\alpha_j, \alpha_j)/2 \in \mathbb{Z}_{>0}$ for all $j \in I$. Following [Kas4, Definition 2.1] (and also [BN, Sect. 2.2]), we denote by $U_q(\mathfrak{g}) = \langle E_j, F_j, q^h \mid j \in I, h \in D^{-1}P^*\rangle$ the quantized universal enveloping algebra of $\mathfrak{g}$ over the field $\mathbb{Q}(q_s)$ of rational functions in q_s. Recall the weight space decomposition: $U_q(\mathfrak{g}) = \bigoplus_{\beta\in Q} U_q(\mathfrak{g})_\beta$, where for $\beta \in Q := \bigoplus_{j\in I} \mathbb{Z}\alpha_j$,

$$U_q(\mathfrak{g})_\beta := \{x \in U_q(\mathfrak{g}) \mid q^h x q^{-h} = q^{\beta(h)}x \text{ for all } h \in D^{-1}P^*\}.$$

Also, we have a $\mathbb{Q}(q_s)$-algebra antiautomorphism $* : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ defined by:

$$\begin{cases} (q^h)^* = q^{-h} & \text{for } h \in D^{-1}P^*, \\ E_j^* = E_j,\ F_j^* = F_j & \text{for } j \in I. \end{cases} \tag{2.1.1}$$

Let $\widetilde{U}_q(\mathfrak{g}) := \bigoplus_{\lambda \in P} U_q(\mathfrak{g})a_\lambda$ denote the modified quantized universal enveloping algebra of $\mathfrak{g}$, where $a_\lambda \in \widetilde{U}_q(\mathfrak{g})_\lambda$ for $\lambda \in P$ is a formal element of weight λ (for details, see [Kas3, Sect. 1.2]). We also have a $\mathbb{Q}(q_s)$-algebra antiautomorphism $* : \widetilde{U}_q(\mathfrak{g}) \to \widetilde{U}_q(\mathfrak{g})$ defined by:

$$\begin{cases} (q^h)^* = q^{-h} & \text{for } h \in D^{-1}P^*, \\ E_j^* = E_j,\ F_j^* = F_j & \text{for } i \in I, \\ a_\lambda^* = a_{-\lambda} & \text{for } \lambda \in P. \end{cases} \tag{2.1.2}$$

2.2 Crystal Bases of $U_q^{\mp}(\mathfrak{g})$ and $\widetilde{U}_q(\mathfrak{g})$

Let $U_q^+(\mathfrak{g})$ (resp., $U_q^-(\mathfrak{g})$) be the positive (resp., negative) part of $U_q(\mathfrak{g})$; that is, the $\mathbb{Q}(q_s)$-subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_j \mid j \in I\}$ (resp., $\{F_j \mid j \in I\}$). We denote by $\mathcal{B}(\pm\infty)$ the crystal base of $U_q^{\mp}(\mathfrak{g})$, equipped with the Kashiwara operators $e_j, f_j, j \in I$, and denote by $u_{\pm\infty}$ the element of $\mathcal{B}(\pm\infty)$ corresponding to the identity element $1 \in U_q^{\mp}(\mathfrak{g})$.

Recall from [Kas1, Proposition 5.2.4] that the crystal lattice $\mathcal{L}(\pm\infty)$ of $U_q^{\mp}(\mathfrak{g})$ is stable under the antiautomorphism $* : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$. Furthermore, we know from [Kas2, Theorem 2.1.1] that the $\mathbb{Q}$-linear automorphism (also denoted by $*$) on $\mathcal{L}(\pm\infty)/q\mathcal{L}(\pm\infty)$ induced by $* : \mathcal{L}(\pm\infty) \to \mathcal{L}(\pm\infty)$ stabilizes the crystal base $\mathcal{B}(\pm\infty)$, that is, $\mathcal{B}(\pm\infty)^* = \mathcal{B}(\pm\infty)$. We call this bijection $* : \mathcal{B}(\pm\infty) \to \mathcal{B}(\pm\infty)$ the $*$-operation on $\mathcal{B}(\pm\infty)$.

It is known from [Kas3, Theorem 2.1.2] that the modified quantized universal enveloping algebra $\widetilde{U}_q(\mathfrak{g})$ (considered as a left $U_q(\mathfrak{g})$-module by left multiplication) has the crystal base $\mathcal{B}(\widetilde{U}_q(\mathfrak{g})) = \bigsqcup_{\lambda \in P} \mathcal{B}(U_q(\mathfrak{g})a_\lambda)$, where $\mathcal{B}(U_q(\mathfrak{g})a_\lambda)$ is the crystal base of the left $U_q(\mathfrak{g})$-module $U_q(\mathfrak{g})a_\lambda$ for $\lambda \in P$. We denote by e_j, f_j, $j \in I$, the Kashiwara operators on $\mathcal{B}(\widetilde{U}_q(\mathfrak{g}))$. By [Kas3, Theorem 3.1.1], for each $\lambda \in P$, there exists an isomorphism

$$\Xi_\lambda : \mathcal{B}(U_q(\mathfrak{g})a_\lambda) \xrightarrow{\sim} \mathcal{B}(\infty) \otimes \mathcal{T}_\lambda \otimes \mathcal{B}(-\infty) \tag{2.2.1}$$

of crystals, where $\mathcal{T}_\lambda := \{t_\lambda\}$ is a crystal for $U_q(\mathfrak{g})$ consisting of a single element t_λ of weight $\lambda \in P$ (see [Kas2, Example 1.2.4]). We define an isomorphism

$$\Xi : \mathcal{B}(\widetilde{U}_q(\mathfrak{g})) \xrightarrow{\sim} \bigsqcup_{\lambda \in P} \mathcal{B}(\infty) \otimes \mathcal{T}_\lambda \otimes \mathcal{B}(-\infty) \tag{2.2.2}$$

of crystals by taking the direct sum of the isomorphisms Ξ_λ, $\lambda \in P$, in (2.2.1).

As in the case of $\mathcal{B}(\pm\infty)$, we have a bijection $* : \mathcal{B}(\widetilde{U}_q(\mathfrak{g})) \to \mathcal{B}(\widetilde{U}_q(\mathfrak{g}))$ induced by the $\mathbb{Q}(q_s)$-algebra antiautomorphism $* : \widetilde{U}_q(\mathfrak{g}) \to \widetilde{U}_q(\mathfrak{g})$ (see [Kas3, Theorem 4.3.2]), which we call the $*$-operation on $\mathcal{B}(\widetilde{U}_q(\mathfrak{g}))$. Furthermore, we define the $*$-operation

$$* : \bigsqcup_{\lambda \in P} \mathcal{B}(\infty) \otimes \mathcal{T}_\lambda \otimes \mathcal{B}(-\infty) \to \bigsqcup_{\lambda \in P} \mathcal{B}(\infty) \otimes \mathcal{T}_\lambda \otimes \mathcal{B}(-\infty) \tag{2.2.3}$$

so that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{B}(\widetilde{U}_q(\mathfrak{g})) & \xrightarrow{\Xi} & \bigsqcup_{\lambda \in P} \mathcal{B}(\infty) \otimes \mathcal{T}_\lambda \otimes \mathcal{B}(-\infty) \\ {\scriptstyle *}\big\downarrow & & \big\downarrow{\scriptstyle *} \\ \mathcal{B}(\widetilde{U}_q(\mathfrak{g})) & \xrightarrow{\Xi} & \bigsqcup_{\lambda \in P} \mathcal{B}(\infty) \otimes \mathcal{T}_\lambda \otimes \mathcal{B}(-\infty). \end{array} \tag{2.2.4}$$

Proposition 2.2.1 ([Kas3, Corollary 4.3.3]). *Let* $b_1 \in \mathcal{B}(\infty)$, $b_2 \in \mathcal{B}(-\infty)$, *and* $\lambda \in P$. *Then, we have* $(b_1 \otimes t_\lambda \otimes b_2)^* = b_1^* \otimes t_{\lambda'} \otimes b_2^*$, *where* $\lambda' := -\lambda - \mathrm{wt}\, b_1 - \mathrm{wt}\, b_2$.

2.3 Extremal Elements of Crystals

Let $\mathcal{B}$ be a normal crystal for $U_q(\mathfrak{g})$ in the sense of [Kas3, Sect. 1.5]; recall that the crystal base $\mathcal{B}(\widetilde{U}_q(\mathfrak{g})) \cong \bigsqcup_{\lambda \in P} \mathcal{B}(\infty) \otimes \mathcal{T}_\lambda \otimes \mathcal{B}(-\infty)$ is a normal crystal (see the comment at the end of [Kas3, Sect. 2.1]). We can define an action of the Weyl group W on the crystal $\mathcal{B}$ as follows (for more details, see [Kas3, Sect. 7]). For each $j \in I$, define $S_j : \mathcal{B} \to \mathcal{B}$ by:

$$S_j b = \begin{cases} f_j^n b & \text{if } n := (\mathrm{wt}\, b)(h_j) \geq 0, \\ e_j^{-n} b & \text{if } n := (\mathrm{wt}\, b)(h_j) < 0, \end{cases} \qquad \text{for } b \in \mathcal{B}. \tag{2.3.1}$$

Then we obtain a unique action $S : W \to \mathrm{Bij}(\mathcal{B})$, $w \mapsto S_w$, of the Weyl group W on the set $\mathcal{B}$ such that $S_{r_j} = S_j$ for all $j \in I$, where for a set X, $\mathrm{Bij}(X)$ denotes the group of all bijections from the set X to itself.

Definition 2.3.1. An element $b \in \mathcal{B}$ is said to be extremal (or more precisely, W-extremal) if for every $w \in W$, either $e_j S_w b = 0$ or $f_j S_w b = 0$ holds for each $j \in I$.

Remark 2.3.2. (1) Let $\mathcal{B}$ be a normal crystal for $U_q(\mathfrak{g})$. If $b \in \mathcal{B}$ is an extremal element, then $S_w b$ is an extremal element of $\mathcal{B}$ whose weight is $w(\mathrm{wt}\, b)$ for each $w \in W$.

(2) Let $\mathcal{B}_1$, $\mathcal{B}_2$ be normal crystals for $U_q(\mathfrak{g})$ and assume that $\Phi : \mathcal{B}_1 \xrightarrow{\sim} \mathcal{B}_2$ is an isomorphism of crystals. Then the following diagram commutes for all $w \in W$:

$$\begin{array}{ccc} \mathcal{B}_1 & \xrightarrow{\Phi} & \mathcal{B}_2 \\ {\scriptstyle S_w}\downarrow & & \downarrow{\scriptstyle S_w} \\ \mathcal{B}_1 & \xrightarrow{\Phi} & \mathcal{B}_2. \end{array} \tag{2.3.2}$$

Also, an element $b \in \mathcal{B}_1$ is extremal if and only if $\Phi(b) \in \mathcal{B}_2$ is extremal.

2.4 Extremal Weight Modules

Definition 2.4.1 (see [Kas3, Sect. 8]). Let M be an integrable $U_q(\mathfrak{g})$-module. A weight vector $v \in M$ of weight $\lambda \in P$ is said to be extremal if there exists a family $\{v_w\}_{w \in W}$ of weight vectors in M satisfying the following conditions:

(1) If w is the identity element e of W, then $v_w = v_e = v$;
(2) If $n := (w\lambda)(h_j) \geq 0$, then $E_j v_w = 0$ and $F_j^{(n)} v_w = v_{r_j w}$;
(3) If $n := (w\lambda)(h_j) \leq 0$, then $F_j v_w = 0$ and $E_j^{(-n)} v_w = v_{r_j w}$.

Here, for $j \in I$ and $k \in \mathbb{Z}_{\geq 0}$, $E_j^{(k)}$ and $F_j^{(k)}$ denote the kth q-divided powers of E_j and F_j, respectively.

For $\lambda \in P$, we set

$$I_\lambda := \bigoplus_{b \in \mathcal{B}(U_q(\mathfrak{g})a_\lambda) \setminus \mathcal{B}(\lambda)} \mathbb{Q}(q) G(b) \subset U_q(\mathfrak{g}) a_\lambda,$$

where $G(b)$ denotes the global basis element corresponding to $b \in \mathcal{B}(U_q(\mathfrak{g})a_\lambda)$, and

$$\mathcal{B}(\lambda) := \big\{ b \in \mathcal{B}(U_q(\mathfrak{g})a_\lambda) \mid b^* \in \mathcal{B}(\widetilde{U}_q(\mathfrak{g})) \text{ is extremal} \big\}.$$

Theorem 2.4.2 ([Kas3, Theorem 8.2.2]). *Let $\lambda \in P$ be an integral weight.*

(1) *The subspace I_λ is a $U_q(\mathfrak{g})$-submodule of $U_q(\mathfrak{g})a_\lambda$.*
(2) *The quotient $U_q(\mathfrak{g})$-module $V(\lambda) := U_q(\mathfrak{g})a_\lambda / I_\lambda$ is an integrable $U_q(\mathfrak{g})$-module, which is isomorphic to the $U_q(\mathfrak{g})$-module generated by a single element u_λ subject to the defining relation that the u_λ is an extremal weight vector of weight λ.*
(3) *The subset $\mathcal{B}(\lambda)$ of $\mathcal{B}(U_q(\mathfrak{g})a_\lambda)$ is precisely a crystal base of the integrable $U_q(\mathfrak{g})$-module $V(\lambda)$.*
(4) *For each $w \in W$, there exists an isomorphism $V(\lambda) \cong V(w\lambda)$ of $U_q(\mathfrak{g})$-modules between $V(\lambda)$ and $V(w\lambda)$.*
(5) *If we set $S_w^* := * \circ S_w \circ *$ for each $w \in W$, then we have $S_w^*(\mathcal{B}(\lambda)) \subset \mathcal{B}(w\lambda)$ for all $w \in W$. Moreover, the map $S_w^* : \mathcal{B}(\lambda) \to \mathcal{B}(w\lambda)$ is an isomorphism of crystals from $\mathcal{B}(\lambda)$ onto $\mathcal{B}(w\lambda)$.*

Definition 2.4.3. The $U_q(\mathfrak{g})$-module $V(\lambda)$ in the theorem above is called the extremal weight module of extremal weight λ.

Remark 2.4.4 (*see* [Kas3, Sects. 8.2 and 8.3]). If λ is a dominant (resp., antidominant) integral weight, then the extremal weight module $V(\lambda)$ is isomorphic, as a $U_q(\mathfrak{g})$-module, to the integrable highest (resp., lowest) weight module of highest (resp., lowest) weight λ. Thus, the crystal base $\mathcal{B}(\lambda)$ is isomorphic, as a crystal, to the crystal base of the integrable highest (resp., lowest) weight module of highest (resp., lowest) weight λ.

2.5 Extremal Weight Modules Over Quantum Affine Algebras

In this and the next subsection, we assume that $\mathfrak{g} = \mathfrak{g}(A)$ is an affine Lie algebra, i.e. that the GCM $A = (a_{ij})_{i,j \in I}$ of $\mathfrak{g}$ is of affine type; the vertices of the Dynkin diagram of $A = (a_{ij})_{i,j \in I}$ are numbered as in [Kac, Sect. 4.8, Tables Aff 1 – Aff 3], with “0” being a distinguished element of the index set I. Recall from [Kac, Sect. 6.2] that the Cartan subalgebra of $\mathfrak{g}$ is given by: $\mathfrak{h} = \left(\bigoplus_{j \in I} \mathbb{Q} h_j\right) \oplus \mathbb{Q} d$, where $d \in \mathfrak{h}$ is the scaling element, and that $\alpha_j(d) = \delta_{j,0}$, $\Lambda_j(d) = 0$ for all $j \in I$. Note that the standard invariant bilinear form $(\cdot\,,\,\cdot)$ on $\mathfrak{h}^*$ is normalized as in [Kac, Sect. 6.2], so that $\lambda(c) = (\lambda, \delta)$ for all $\lambda \in \mathfrak{h}^*$, where $\delta = \sum_{j \in I} a_j \alpha_j \in \mathfrak{h}^*$ and $c = \sum_{j \in I} a_j^{\vee} h_j \in \mathfrak{h}$ denote the null root and the canonical central element of $\mathfrak{g}$, respectively. We take an integral weight lattice P as follows:

$$P = \left(\bigoplus_{j \in I} \mathbb{Z} \Lambda_j\right) \oplus \mathbb{Z}\,(a_0^{-1}\delta) \subset \mathfrak{h}^*. \tag{2.5.1}$$

Here we should note that $a_0 = 2$ if $\mathfrak{g}$ is of type $A_{2n}^{(2)}$, $n \geq 2$, and $a_0 = 1$ otherwise. Then we have

$$P^* = \left(\bigoplus_{j \in I} \mathbb{Z} h_j\right) \oplus \mathbb{Z} d \ \subset \mathfrak{h}.$$

Let $\lambda \in P$ be an integral weight. If $\lambda(c) > 0$ (resp., $\lambda(c) < 0$), then λ is conjugate, under the Weyl group W of $\mathfrak{g}$, to a dominant (resp., antidominant) integral weight $\Lambda \in P$. Hence it follows from Theorem 2.4.2 (4), (5) and Remark 2.4.4 that the extremal weight module $V(\lambda)$ is isomorphic, as a $U_q(\mathfrak{g})$-module, to the integrable highest (resp., lowest) weight module of highest (reps., lowest) weight Λ, and that the crystal base $\mathcal{B}(\lambda)$ is isomorphic, as a crystal, to the crystal base of the integrable highest (resp., lowest) weight module of highest (reps., lowest) weight Λ. Accordingly, we are reduced to studying the extremal weight module $V(\lambda)$ and its crystal base $\mathcal{B}(\lambda)$ in the case $\lambda(c) = 0$.

Definition 2.5.1. An integral weight $\lambda \in P$ is said to be level-zero if $\lambda(c) = 0$.

We set $I_0 := I \setminus \{0\}$, and for each $i \in I_0$, define a level-zero fundamental weight $\varpi_i \in P$ for the affine Lie algebra $\mathfrak{g}$ by:

$$\varpi_i = \Lambda_i - a_i^\vee \Lambda_0; \tag{2.5.2}$$

it is easy to check that $\varpi_i(c) = 0$.

Let $\lambda \in P$ be a level-zero integral weight. We see from the definitions of an extremal weight module and its crystal base that for $k \in \mathbb{Z}$,

$$V(\lambda + k\delta) \cong V(\lambda) \otimes \mathbb{Q}(q_s)v_{k\delta} \quad \text{as } U_q(\mathfrak{g})\text{-modules}, \tag{2.5.3}$$

$$\mathcal{B}(\lambda + k\delta) \cong \mathcal{B}(\lambda) \otimes \mathcal{T}_{k\delta} \quad \text{as crystals}, \tag{2.5.4}$$

where $\mathbb{Q}(q_s)v_{k\delta}$ is the one-dimensional $U_q(\mathfrak{g})$-module defined by: $E_j v_{k\delta} = F_j v_{k\delta} = 0$ for all $j \in I$, and $q^h v_{k\delta} = q^{k\delta(h)} v_{k\delta}$ for all $h \in D^{-1}P^*$. Therefore, we may assume that $\lambda(d) = 0$, and hence $\lambda \in \sum_{i \in I_0} \mathbb{Z}\varpi_i$. Furthermore, the element $\lambda \in \sum_{i \in I_0} \mathbb{Z}\varpi_i$ is conjugate, under the Weyl group W of $\mathfrak{g}$, to an element in $\sum_{i \in I_0} \mathbb{Z}_{\geq 0}\varpi_i$. Thus, in view of Theorem 2.4.2 (4) and (5), we may assume from the beginning that λ is contained in the set $\sum_{i \in I_0} \mathbb{Z}_{\geq 0}\varpi_i$.

2.6 Some Results of Beck and Nakajima

In this subsection, we review some deep results in [BN], which are used in the proof of our main theorem. In order to state these results, we need (much) more notation.

First, we consider the case in which $\mathfrak{g}$ is not of type $A_{2n}^{(2)}$, $n \geq 2$; in this case, our numbering of the vertices of the Dynkin diagram is the same as the one in [BN] (see [BN, Sect. 2.1]). Let $\varpi_i^\vee$, $i \in I_0$, be the element of $\bigoplus_{j \in I_0} \mathbb{Q}\alpha_j \subset \mathfrak{h}^*$ such that $(\alpha_j, \varpi_i^\vee) = \delta_{ij}$ for $i, j \in I_0$, and set $\widetilde{\varpi}_i := d_i \varpi_i^\vee$ with $d_i := \max\{1, (\alpha_i, \alpha_i)/2\}$. For each $i \in I_0$, define an endomorphism $t_{\widetilde{\varpi}_i}$ of the vector space $\mathfrak{h}^*$ by the same formula as [Kac, (6.5.2)]:

$$t_{\widetilde{\varpi}_i}(\lambda) = \lambda + \lambda(c)\,\widetilde{\varpi}_i - \left\{(\lambda, \widetilde{\varpi}_i) + \frac{\lambda(c)}{2}(\widetilde{\varpi}_i, \widetilde{\varpi}_i)\right\}\delta \quad \text{for } \lambda \in \mathfrak{h}^*.$$

It is known (see, for example, [FoSS, Sect. 2] and also [W, Sect. 1.2]) that there exist an element $w_i \in W$ and a diagram automorphism $\tau_i : I \to I$ of the Dynkin diagram of $\mathfrak{g}$ such that $w_i^{-1} t_{\widetilde{\varpi}_i}(\alpha_j) = \alpha_{\tau_i(j)}$ for all $j \in I$. Let $U_q'(\mathfrak{g})$ denote the $\mathbb{Q}(q_s)$-subalgebra of $U_q(\mathfrak{g})$ generated by E_j, F_j, $t_j := q^{(\alpha_j, \alpha_j)h_j/2}$, $j \in I$, and define a $\mathbb{Q}(q_s)$-algebra automorphism $T_{\widetilde{\varpi}_i} \in \mathrm{Aut}(U_q'(\mathfrak{g}))$ by: $T_{\widetilde{\varpi}_i} = T''_{w_i,1} \circ \tau_i$, where, for $w \in W$, $T''_{w,1}$ is the $\mathbb{Q}(q_s)$-algebra automorphism of $U_q'(\mathfrak{g})$ defined in

[L, Chap. 39], and $\tau_i \in \mathrm{Aut}(U_q'(\mathfrak{g}))$ is given by: $\tau_i(E_j) = E_{\tau_i(j)}$, $\tau_i(F_j) = F_{\tau_i(j)}$, and $\tau_i(t_j) = t_{\tau_i(j)}$ for $j \in I$. Following [B, Sect. 4.6] (see also [BN, Remark 3.6]), we set

$$E_{kd_i\delta-\alpha_i} := T^k_{\widetilde{\varpi}_i}(T''_{r_i,1})^{-1}(E_i) \quad \text{for } i \in I_0 \text{ and } k \in \mathbb{Z}_{>0},$$

which is an element of $U_q^+(\mathfrak{g}) \cap U_q(\mathfrak{g})_{kd_i\delta-\alpha_i}$ by [L, Proposition 40.1.3]. We define

$$\widetilde{\psi}_{i,kd_i} := E_{kd_i\delta-\alpha_i}E_i - q_i^2 E_i E_{kd_i\delta-\alpha_i} \in U_q^+(\mathfrak{g}) \cap U_q(\mathfrak{g})_{kd_i\delta}$$

for $i \in I_0$ and $k \in \mathbb{Z}_{>0}$, where $q_i := q^{(\alpha_i,\alpha_i)/2}$ for $i \in I$, and then define $E_{i,kd_i\delta} \in U_q^+(\mathfrak{g}) \cap U_q(\mathfrak{g})_{kd_i\delta}$ for $i \in I_0$ and $k \in \mathbb{Z}_{>0}$ by the following generating function in a variable x:

$$(q_i - q_i^{-1})\sum_{k=1}^{\infty} E_{i,kd_i\delta}x^k = \log\left(1 + \sum_{k=1}^{\infty}(q_i - q_i^{-1})\widetilde{\psi}_{i,kd_i}x^k\right).$$

Then we know (see, for example, [BN, p. 351]) that

$$\left[E_{i,kd_i\delta},\ E_{j,ld_j\delta}\right] = 0 \quad \text{for all } i,\ j \in I_0 \text{ and } k,\ l \in \mathbb{Z}_{>0}. \tag{2.6.1}$$

Furthermore, for each $i \in I_0$ and $k \in \mathbb{Z}_{\geq 0}$, we define the "integral" imaginary root vectors $\widetilde{P}_{i,kd_i} \in U_q^+(\mathfrak{g})$ by the following generating function in a variable x as in [BN, (3.7)]:

$$\sum_{k\geq 0}\widetilde{P}_{i,kd_i}x^k = \exp\left(\sum_{k\geq 0}\frac{E_{i,kd_i\delta}}{[k]_i}x^k\right),$$

where $[k]_i := (q_i^k - q_i^{-k})/(q_i - q_i^{-1})$ for $i \in I$ and $k \in \mathbb{Z}_{\geq 0}$. Note that $\left[\widetilde{P}_{i,kd_i},\ \widetilde{P}_{j,ld_j}\right] = 0$ for all $i,\ j \in I_0$ and $k,\ l \in \mathbb{Z}_{\geq 0}$ by (2.6.1). Also, it is easily seen that the weight of $\widetilde{P}_{i,kd_i}$ is equal to $kd_i\delta$ for each $i \in I_0$ and $k \in \mathbb{Z}_{\geq 0}$. By convention, we set $\widetilde{P}_{i,kd_i} := 0 \in U_q^+(\mathfrak{g})$ for $i \in I_0$ and $k \in \mathbb{Z}_{<0}$.

For a partition ρ, i.e. for a weakly decreasing sequence $\rho = (\rho_1 \geq \rho_2 \geq \cdots)$ of nonnegative integers with finitely many nonzero parts (i.e. $\rho_k = 0$ for k sufficiently large), we denote by $\ell(\rho)$ its length, i.e. the number of nonzero parts of ρ, and set $|\rho| := \sum_{k\geq 1}\rho_k$. Now, assume that $\lambda \in P$ is of the form $\lambda = \sum_{i\in I_0} m_i\varpi_i$, with $m_i \in \mathbb{Z}_{\geq 0}$ for $i \in I_0$. We denote by $\mathbf{N}(\lambda)$ the set of I_0-tuples $(\rho^{(i)})_{i\in I_0}$ of partitions $\rho^{(i)}$, $i \in I_0$, such that $\ell(\rho^{(i)}) \leq m_i$ for all $i \in I_0$; this set is denoted by $\mathbf{N}^{\mathcal{R}}(\lambda)$ in [BN, Definition 4.2]. Let $\mathbf{c}_0 = (\rho^{(i)})_{i\in I_0} \in \mathbf{N}(\lambda)$. For each $i \in I_0$, we write the conjugate ${}^t\rho^{(i)}$ of the partition $\rho^{(i)}$ in the form ${}^t\rho^{(i)} = ({}^t\rho_1^{(i)} \geq {}^t\rho_2^{(i)} \geq \cdots)$. Set

$$S_{\mathbf{c}_0} := \prod_{i\in I_0} S_{\rho^{(i)}}, \quad \text{where } S_{\rho^{(i)}} := \det(\widetilde{P}_{i,({}^t\rho_k^{(i)}-k+m)d_i})_{1\leq k,m\leq \ell}. \tag{2.6.2}$$

Here we take (an arbitrary) $\ell \in \mathbb{Z}_{\geq 1}$ such that $\ell \geq \ell({}^t\rho^{(i)})$ for all $i \in I_0$; note that the element $S_{\mathbf{c}_0} \in U_q^+(\mathfrak{g})$ does not depend on the choice of such an ℓ. Since $\widetilde{P}_{i,kd_i} \in U_q^+(\mathfrak{g}) \cap U_q(\mathfrak{g})_{kd_i\delta}$ for each $i \in I_0$ and $k \in \mathbb{Z}_{\geq 0}$, it follows that $S_{\mathbf{c}_0}$ is an element of $U_q^+(\mathfrak{g})$ whose weight is equal to $|\mathbf{c}_0|\delta$, where $|\mathbf{c}_0| := \sum_{i \in I_0} |\rho^{(i)}| d_i$. Also, we set

$$S_{\mathbf{c}_0}^- := \overline{S_{\mathbf{c}_0}^\vee}, \tag{2.6.3}$$

where $\vee : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ denotes the $\mathbb{Q}(q_s)$-algebra automorphism of $U_q(\mathfrak{g})$ defined by:

$$\begin{cases} (q^h)^\vee = q^{-h} & \text{for } h \in D^{-1}P^*, \\ E_j^\vee = F_j, \ F_j^\vee = E_j & \text{for } j \in I, \end{cases} \tag{2.6.4}$$

and ${}^-: U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ denotes the $\mathbb{Q}$-algebra automorphism of $U_q(\mathfrak{g})$ defined by:

$$\begin{cases} \overline{q} = q^{-1}, \ \overline{q^h} = q^{-h} & \text{for } h \in D^{-1}P^*, \\ \overline{E_j} = E_j, \ \overline{F_j} = F_j & \text{for } j \in I. \end{cases} \tag{2.6.5}$$

Since $S_{\mathbf{c}_0} \in U_q^+(\mathfrak{g}) \cap U_q(\mathfrak{g})_{|\mathbf{c}_0|\delta}$, it follows from the definitions of the maps $\vee$ and ${}^-$ that $S_{\mathbf{c}_0}^- = \overline{S_{\mathbf{c}_0}^\vee}$ is an element of $U_q^-(\mathfrak{g})$ whose weight is equal to $-|\mathbf{c}_0|\delta$.

Next, we consider the case in which $\mathfrak{g}$ is of type $A_{2n}^{(2)}$, $n \geq 2$, with $I = \{0, 1, 2, \dots, n\}$ as the index set. In this case, we should remark that our numbering of the vertices of the Dynkin diagram is in an order reverse to the one in [BN] (see [BN, Sect. 2.1]). Namely, for each $1 \leq i \leq n$, the vertex i of the Dynkin diagram under our numbering is identical to the vertex $n - i$ of the Dynkin diagram under the numbering in [BN].

We set $I_n := I \setminus \{n\} = \{0, 1, \dots, n-1\}$. Note that $d_i = \max\{1, (\alpha_i, \alpha_i)/2\} = 1$ for all $i \in I_n$. For each $i \in I_n$ and $k \in \mathbb{Z}_{\geq 0}$, we define $E_{i,kd_i\delta} = E_{i,k\delta} \in U_q^+(\mathfrak{g}) \cap U_q(\mathfrak{g})_{k\delta}$ in exactly the same way as above, with I_0 replaced by I_n, and then define the integral imaginary root vectors $\widetilde{P}_{i,kd_i} = \widetilde{P}_{i,k} \in U_q^+(\mathfrak{g}) \cap U_q(\mathfrak{g})_{k\delta}$ by the following generating function in a variable x:

$$\sum_{k \geq 0} \widetilde{P}_{i,k} x^k = \begin{cases} \exp\left(\displaystyle\sum_{k \geq 0} \frac{E_{i,k\delta}}{[k]_i} x^k \right) & \text{if } i \neq 0, \\ \exp\left(\displaystyle\sum_{k \geq 0} \frac{E_{0,k\delta}}{[2k]_0} x^k \right) & \text{if } i = 0. \end{cases}$$

By convention, we set $\widetilde{P}_{i,kd_i} = \widetilde{P}_{i,k} := 0 \in U_q^+(\mathfrak{g})$ for $i \in I_n$ and $k \in \mathbb{Z}_{<0}$. Remark that for each $i \in I_0 = \{1, 2, \dots, n\}$ and $k \in \mathbb{Z}$, the integral imaginary root vector "$\widetilde{P}_{i,kd_i} = \widetilde{P}_{i,k}$" in [BN, (3.7)] (in their notation) is identical to the element $\widetilde{P}_{n-i,kd_{n-i}} = \widetilde{P}_{n-i,k}$ above.

Now, assume that $\lambda \in P$ is of the form $\lambda = \sum_{i \in I_0} m_i \varpi_i$, with $m_i \in \mathbb{Z}_{\geq 0}$ for $i \in I_0 = \{1, 2, \ldots, n\}$. We denote by $\mathbf{N}(\lambda)$ the set of I_0-tuples $(\rho^{(i)})_{i \in I_0}$ of partitions $\rho^{(i)}$, $i \in I_0$, such that $\ell(\rho^{(i)}) \leq m_i$ for all $i \in I_0$. Here we make the following remark.

Remark 2.6.1. Set

$$\eta_0 := 2\Lambda_0 - \Lambda_n, \qquad \eta_i := \Lambda_i - \Lambda_n \quad \text{for } 1 \leq i \leq n-1. \tag{2.6.6}$$

Then, for each $i \in I_0$, the ith level-zero fundamental weight under the numbering of the vertices of the Dynkin diagram in [BN] (see [BN, p. 348]) is identical to the η_{n-i} above. But, we see from [KNT, Lemma 5.11] that there exists $w \in W_0 := \langle r_j \mid j \in I_0 \rangle \subset W$ such that $w(\varpi_i) = \eta_{n-i}$ for all $i \in I_0 = \{1, 2, \ldots, n\}$. If $\lambda = \sum_{i \in I_0} m_i \varpi_i$ with $m_i \in \mathbb{Z}_{\geq 0}$ for $i \in I_0$, then $w\lambda = \sum_{i \in I_0} m_i \eta_{n-i} = \sum_{i \in I_n} m_{n-i} \eta_i$. Thus, the set $\mathbf{N}(\lambda)$ is identical to the set $\mathbf{N}^{\mathcal{R}}(w\lambda)$ in the notation of [BN, Definition 4.2].

For each $\mathbf{c}_0 = (\rho^{(i)})_{i \in I_0} \in \mathbf{N}(\lambda)$, we define $S_{\mathbf{c}_0} \in U_q^+(\mathfrak{g})$ by:

$$S_{\mathbf{c}_0} := \prod_{i \in I_0} S_{\rho^{(i)}}, \quad \text{where } S_{\rho^{(i)}} := \det(\widetilde{P}_{n-i, {}^t\rho_k^{(i)} - k + m})_{1 \leq k, m \leq \ell}, \tag{2.6.7}$$

with $\ell \in \mathbb{Z}_{\geq 1}$ such that $\ell \geq \ell({}^t\rho^{(i)})$ for all $i \in I_0$, and then set

$$S_{\mathbf{c}_0}^- := \overline{S_{\mathbf{c}_0}^\vee}, \tag{2.6.8}$$

where the automorphisms $\vee : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ and $^- : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ are defined by the same formulas as (2.6.4) and (2.6.5), respectively. Then, $S_{\mathbf{c}_0}^- = \overline{S_{\mathbf{c}_0}^\vee}$ is an element of $U_q^-(\mathfrak{g}) \cap U_q(\mathfrak{g})_{-|\mathbf{c}_0|\delta}$, where $|\mathbf{c}_0| := \sum_{i \in I_0} |\rho^{(i)}| d_{n-i} = \sum_{i \in I_0} |\rho^{(i)}|$.

We now summarize some results in [BN] in the following

Proposition 2.6.2. *Assume that $\lambda \in P$ is of the form $\lambda = \sum_{i \in I_0} m_i \varpi_i$, with $m_i \in \mathbb{Z}_{\geq 0}$ for $i \in I_0$, and set*

$$\Lambda := \begin{cases} \displaystyle\sum_{i \in I_0} m_i \eta_{n-i} & \text{if } \mathfrak{g} \text{ is of type } A_{2n}^{(2)}, n \geq 2, \\ \lambda & \text{otherwise.} \end{cases} \tag{2.6.9}$$

(1) *Let $\mathbf{c}_0 \in \mathbf{N}(\lambda)$. Then, the element $S_{\mathbf{c}_0}^- \in U_q^-(\mathfrak{g})$ is contained in the crystal lattice $\mathcal{L}(\infty)$ of $U_q^-(\mathfrak{g})$. Moreover, the element $S_{\mathbf{c}_0}^- \bmod q_s\mathcal{L}(\infty)$ of $\mathcal{L}(\infty)/q_s\mathcal{L}(\infty)$ is contained in the crystal base $\mathcal{B}(\infty)$ of $U_q^-(\mathfrak{g})$.*

(2) *If the elements* $\mathbf{c}_0, \mathbf{c}_0' \in \mathbf{N}(\lambda)$ *are not equal, then* $S_{\mathbf{c}_0}^- \bmod q_s\mathcal{L}(\infty) \neq S_{\mathbf{c}_0'}^- \bmod q_s\mathcal{L}(\infty)$*, and hence* $b_{\mathbf{c}_0} \neq b_{\mathbf{c}_0'}$*. Here, for* $\mathbf{c}_0 \in \mathbf{N}(\lambda)$*, we set*

$$b_{\mathbf{c}_0} := \Xi_\Lambda^{-1}\big((S_{\mathbf{c}_0}^- \bmod q_s\mathcal{L}(\infty)) \otimes t_\Lambda \otimes u_{-\infty}\big) \in \mathcal{B}(U_q(\mathfrak{g})a_\Lambda), \qquad (2.6.10)$$

and similarly for $\mathbf{c}_0' \in \mathbf{N}(\lambda)$*.*

(3) *Let* $\mathbf{c}_0 \in \mathbf{N}(\lambda)$*. Then, the element* $b_{\mathbf{c}_0} \in \mathcal{B}(U_q(\mathfrak{g})a_\Lambda)$ *is contained in the crystal base* $\mathcal{B}(\Lambda)$ *of the extremal weight module* $V(\Lambda)$ *and is an extremal element of weight* $\Lambda - |\mathbf{c}_0|\delta$*.*

(4) *If* $b \in \mathcal{B}(\Lambda)$ *is an extremal element and* $\Xi_\Lambda(b) = b_1 \otimes t_\Lambda \otimes u_{-\infty}$ *for some* $b_1 \in \mathcal{B}(\infty)$ *such that* $\mathrm{wt}(b_1) \in \mathbb{Z}_{\le 0}\,\delta$*, then* $b = b_{\mathbf{c}_0}$ *for some* $\mathbf{c}_0 \in \mathbf{N}(\lambda)$*.*

(5) *Each element of* $\mathcal{B}(\Lambda)$ *is connected to* $b_{\mathbf{c}_0}$ *for some* $\mathbf{c}_0 \in \mathbf{N}(\lambda)$*.*

Proof. The first assertion of part (1) follows immediately from [BN, Corollary 3.26] and the comment after it; remark that the element $S_{\mathbf{c}_0}$ is equal to "$L(\mathbf{c}, 0)$ with $\mathbf{c} = (0, \mathbf{c}_0, 0)$" in the notation therein (see [BN, (3.11)]). The second assertion of part (1) follows from [BN, Proposition 3.27] and the equality "$\mathrm{sgn}(\mathbf{c}, 0) = 1$" (in the notation therein) proved in [BN, Sect. 5]. Part (2) is obvious by [BN, Proposition 3.23]. Parts (3)–(5) are precisely [BN, Proposition 4.3] along with the equality $\mathrm{sgn}(\mathbf{c}, 0) = 1$. □

Definition 2.6.3 (see the comment after [BN, (3.11)]). Let $\mathbf{c}_0 \in \mathbf{N}(\lambda)$. The extremal element $b_{\mathbf{c}_0} \in \mathcal{B}(\Lambda)$ is called the purely imaginary extremal (PIE for short) element corresponding to $\mathbf{c}_0$.

Remark 2.6.4. Suppose that $\lambda \in P$ is of the form $\lambda = \sum_{i \in I_0} m_i \varpi_i$, with $m_i \in \mathbb{Z}_{\ge 0}$ for $i \in I_0$, and let Λ be as defined in (2.6.9). Recall from Remark 2.6.1 that λ is W-conjugate to Λ, and hence $\mathcal{B}(\Lambda) \cong \mathcal{B}(\lambda)$ as crystals by Theorem 2.4.2 (5). Also, we know from [BN, Theorem 4.16] that the connected components of the crystal base $\mathcal{B}(\Lambda)$, and hence those of $\mathcal{B}(\lambda)$, are parametrized by the subset $\mathbf{N}(\lambda)'$ of $\mathbf{N}(\lambda)$ consisting of all I_0-tuples $(\rho^{(i)})_{i \in I_0}$ of partitions $\rho^{(i)}$, $i \in I_0$, such that $\ell(\rho^{(i)}) < m_i$ for all $i \in I_0$. Therefore, the crystal graph of $\mathcal{B}(\lambda)$ is connected if and only if $m_i \in \{0, 1\}$ for all i (see also [Kas4, Proposition 5.4 (ii)]).

3 Fixed Point Subsets of Crystal Bases

In this section, we recall some basic material on diagram automorphisms and orbit Lie algebras, and then briefly review the relevant results from our previous works about the fixed point subset under a diagram automorphism for various crystal bases, including the crystal base of an extremal weight module. Also, we show a few technical lemmas, which will be used in the proof of our main theorem. Throughout this section, we assume that $\mathfrak{g} = \mathfrak{g}(A)$ is a Kac–Moody algebra over $\mathbb{Q}$ associated with a symmetrizable GCM $A = (a_{ij})_{i,j \in I}$, and use the notation of Sects. 2.1–2.4.

3.1 Diagram Automorphisms and Orbit Lie Algebras

Let $\omega : I \to I$ be a bijection of the index set I of the GCM $A = (a_{ij})_{i,j\in I}$ such that $a_{\omega(i),\omega(j)} = a_{ij}$ for all $i, j \in I$, called a (Dynkin) diagram automorphism; we denote its order by N. The diagram automorphism ω naturally induces a Lie algebra automorphism $\omega \in \mathrm{Aut}(\mathfrak{g})$ of order N such that $\omega(\mathfrak{h}) = \mathfrak{h}$, and $\omega(E_j) = E_{\omega(j)}$, $\omega(F_j) = F_{\omega(j)}$, $\omega(h_j) = h_{\omega(j)}$ for all $j \in I$. We then define a $\mathbb{Q}$-linear automorphism $\omega^* : \mathfrak{h}^* \to \mathfrak{h}^*$ by: $(\omega^*(\lambda))(h) = \lambda(\omega^{-1}(h))$ for $\lambda \in \mathfrak{h}^*$, $h \in \mathfrak{h}$, and set

$$\mathfrak{h}^0 := \{h \in \mathfrak{h} \mid \omega(h) = h\}, \qquad (\mathfrak{h}^*)^0 := \{\lambda \in \mathfrak{h}^* \mid \omega^*(\lambda) = \lambda\}. \tag{3.1.1}$$

Also, we define a subgroup $\widetilde{W}$ of W by:

$$\widetilde{W} = \{w \in W \mid \omega^* w = w\omega^*\}. \tag{3.1.2}$$

Note that $(\mathfrak{h}^*)^0$ is stable under the action of $\widetilde{W} \subset W$.

Remark 3.1.1. It is easy to verify that

$$\omega^*(\alpha_j) = \alpha_{\omega(j)}, \quad \omega^* r_j (\omega^*)^{-1} = r_{\omega(j)} \qquad \text{for all } j \in I. \tag{3.1.3}$$

Let $P \subset \mathfrak{h}^*$ be an ω^*-stable integral weight lattice such that $\alpha_j \in P$ for all $j \in I$. We know the following lemma from [NS3, Lemma 1.2.1].

Lemma 3.1.2. (1) *There exists a $\mathbb{Q}(q_s)$-algebra automorphism $\omega \in \mathrm{Aut}(U_q(\mathfrak{g}))$ of order N such that $\omega(E_j) = E_{\omega(j)}$, $\omega(F_j) = F_{\omega(j)}$, and $\omega(q^h) = q^{\omega(h)}$ for all $j \in I$ and $h \in D^{-1}P^*$.*

(2) *There exists a $\mathbb{Q}(q_s)$-algebra automorphism $\omega \in \mathrm{Aut}(\widetilde{U}_q(\mathfrak{g}))$ of order N such that $\omega(xa_\lambda) = \omega(x)a_{\omega^*(\lambda)}$ for all $x \in U_q(\mathfrak{g})$ and $\lambda \in P$.*

We set $c_{ij} := \sum_{k=0}^{N_j-1} a_{i,\omega^k(j)}$ for $i, j \in I$, where $N_j := \#\{\omega^k(j) \mid k \geq 0\}$. For our purposes, we may (and do) assume that the diagram automorphism $\omega : I \to I$ satisfies the following condition, called the linking condition:

$$c_{jj} = 1 \text{ or } 2 \quad \text{for each } j \in I. \tag{3.1.4}$$

Remark 3.1.3 (*see* [FuSS, Sect. 2.2]). If $c_{jj} = 1$, then $a_{j,\omega^{N_j/2}(j)} = -1$ and $a_{j,\omega^k(j)} = 0$ for any other $1 \leq k \leq N_j - 1$, $k \neq N_j/2$, where N_j is an even integer. Hence the Dynkin subdiagram corresponding to the ω-orbit of this j is of type $A_2 \times \cdots \times A_2$ ($N_j/2$ times). If $c_{jj} = 2$, then $a_{j,\omega^k(j)} = 0$ for all $1 \leq k \leq N_j - 1$. Hence the Dynkin subdiagram corresponding to the ω-orbit of this j is of type $A_1 \times \cdots \times A_1$ (N_j times).

We choose a complete list $\widehat{I}$ of representatives of the ω-orbits in I, and set $\widehat{a}_{ij} := 2c_{ij}/c_{jj}$ for $i, j \in \widehat{I}$. We know from [FuRS, Lemma 2.1] that a matrix $\widehat{A} = (\widehat{a}_{ij})_{i,j\in\widehat{I}}$ is a symmetrizable GCM. Let $\widehat{\mathfrak{g}} := \mathfrak{g}(\widehat{A})$ be the Kac–Moody algebra over $\mathbb{Q}$ associated with $\widehat{A}$ with Cartan subalgebra $\widehat{\mathfrak{h}}$, simple coroots $\widehat{\Pi}^{\vee} = \{\widehat{h}_j\}_{j\in\widehat{I}} \subset \widehat{\mathfrak{h}}$, and simple roots $\widehat{\Pi} = \{\widehat{\alpha}_j\}_{j\in\widehat{I}} \subset \widehat{\mathfrak{h}}^* := \mathrm{Hom}_{\mathbb{Q}}(\widehat{\mathfrak{h}}, \mathbb{Q})$. We call $\widehat{\mathfrak{g}} = \mathfrak{g}(\widehat{A})$ the orbit Lie algebra (associated with the diagram automorphism $\omega : I \to I$). Denote by $\widehat{W} := \langle \widehat{r}_j \mid j \in \widehat{I} \rangle \subset \mathrm{GL}(\widehat{\mathfrak{h}}^*)$ the Weyl group of the orbit Lie algebra $\widehat{\mathfrak{g}}$, where $\widehat{r}_j \in \mathrm{GL}(\widehat{\mathfrak{h}}^*)$ is the simple reflection in $\widehat{\alpha}_j \in \widehat{\mathfrak{h}}^*$. Let $U_q(\widehat{\mathfrak{g}})$ denote the quantized universal enveloping algebra of the orbit Lie algebra $\widehat{\mathfrak{g}}$ over $\mathbb{Q}(q_s)$, with weight lattice $\widehat{P} := (P_{\omega}^*)^{-1}(P \cap (\mathfrak{h}^*)^0)$, where q_s is defined as in Sect. 2.1. Also, let $\widetilde{U}_q(\widehat{\mathfrak{g}}) := \bigoplus_{\widehat{\lambda}\in\widehat{P}} U_q(\widehat{\mathfrak{g}})\widehat{a}_{\widehat{\lambda}}$ denote the modified quantized universal enveloping algebra of the orbit Lie algebra $\widehat{\mathfrak{g}}$, where $\widehat{a}_{\widehat{\lambda}}$ is a formal element of weight $\widehat{\lambda}$ for $\widehat{\lambda} \in \widehat{P}$.

From [FuRS, Sect. 2], we know that there exist $\mathbb{Q}$-linear isomorphisms $P_{\omega} : \mathfrak{h}^0 \xrightarrow{\sim} \widehat{\mathfrak{h}}$ and $P_{\omega}^* : \widehat{\mathfrak{h}}^* \xrightarrow{\sim} (\mathfrak{h}^0)^* \cong (\mathfrak{h}^*)^0$ such that

$$P_{\omega}(\widetilde{h}_j) = \widehat{h}_j, \quad P_{\omega}^*(\widehat{\alpha}_j) = \widetilde{\alpha}_j \qquad \text{for all } j \in \widehat{I},$$
$$\big(P_{\omega}^*(\widehat{\lambda})\big)(h) = \widehat{\lambda}\big(P_{\omega}(h)\big) \text{ for all } \widehat{\lambda} \in \widehat{\mathfrak{h}}^* \text{ and } h \in \mathfrak{h}^0, \tag{3.1.5}$$

where we set for $j \in \widehat{I}$,

$$\widetilde{h}_j := \frac{1}{N_j}\sum_{k=0}^{N_j-1} h_{\omega^k(j)} \in \mathfrak{h}^0, \qquad \widetilde{\alpha}_j := \frac{2}{c_{jj}}\sum_{k=0}^{N_j-1} \alpha_{\omega^k(j)} \in (\mathfrak{h}^*)^0. \tag{3.1.6}$$

Also, we define $w_j \in W$ by:

$$w_j = \begin{cases} \displaystyle\prod_{k=0}^{N_j/2-1} (r_{\omega^k(j)} r_{\omega^{k+N_j/2}(j)} r_{\omega^k(j)}) & \text{if } c_{jj} = 1, \\ \displaystyle\prod_{k=0}^{N_j-1} r_{\omega^k(j)} & \text{if } c_{jj} = 2, \end{cases} \tag{3.1.7}$$

for each $j \in \widehat{I}$ (see Remark 3.1.3). It is easily seen that $w_j \in \widetilde{W}$ for all $j \in \widehat{I}$. Furthermore, we know from [FuRS, Sect. 3] that there exists a group isomorphism $\Theta : \widehat{W} \xrightarrow{\sim} \widetilde{W}$ such that $\Theta(\widehat{w})|_{(\mathfrak{h}^*)^0} = P_{\omega}^* \circ \widehat{w} \circ (P_{\omega}^*)^{-1}$ for all $\widehat{w} \in \widehat{W}$, and $\Theta(\widehat{r}_j) = w_j$ for all $j \in \widehat{I}$.

3.2 Fixed Point Subset of the Crystal Base $\mathcal{B}(U_q(\mathfrak{g})a_\lambda)$

Let $\lambda \in P \cap (\mathfrak{h}^*)^0$. We see from [NS1, Lemma 2.4.2] that there exists a natural action of ω on the crystal base $\mathcal{B}(U_q(\mathfrak{g})a_\lambda)$ of $U_q(\mathfrak{g})a_\lambda$, induced by the restriction of $\omega \in \mathrm{Aut}(\widetilde{U}_q(\mathfrak{g}))$ to $U_q(\mathfrak{g})a_\lambda$, satisfying

$$\begin{aligned} &\omega \circ e_j = e_{\omega(j)} \circ \omega \quad \text{and} \quad \omega \circ f_j = f_{\omega(j)} \circ \omega \quad \text{for all } j \in I, \\ &\mathrm{wt}(\omega(b)) = \omega^*(\mathrm{wt}\, b) \quad \text{for all } b \in \mathcal{B}(U_q(\mathfrak{g})a_\lambda), \end{aligned} \tag{3.2.1}$$

where it is understood that $\omega(0) = 0$. We set

$$\mathcal{B}^0(U_q(\mathfrak{g})a_\lambda) := \{b \in \mathcal{B}(U_q(\mathfrak{g})a_\lambda) \mid \omega(b) = b\}, \tag{3.2.2}$$

and define ω-Kashiwara operators $\widetilde{e}_j$ and $\widetilde{f}_j$, $j \in \widehat{I}$, on $\mathcal{B}(U_q(\mathfrak{g})a_\lambda)$ by:

$$\widetilde{x}_j = \begin{cases} \displaystyle\prod_{k=0}^{N_j/2-1} (x_{\omega^k(i)} x^2_{\omega^{k+N_j/2}(j)} x_{\omega^k(j)}) & \text{if } c_{jj} = 1, \\ \displaystyle\prod_{k=0}^{N_j-1} x_{\omega^k(j)} & \text{if } c_{jj} = 2, \end{cases} \tag{3.2.3}$$

where x is either e or f.

Let us denote by $\widehat{\mathcal{B}}(\widetilde{U}_q(\widehat{\mathfrak{g}})) = \bigsqcup_{\widehat{\lambda} \in \widehat{P}} \widehat{\mathcal{B}}(U_q(\widehat{\mathfrak{g}})\widehat{a}_{\widehat{\lambda}})$ the crystal base of the modified quantized universal enveloping algebra $\widetilde{U}_q(\widehat{\mathfrak{g}})$ of the orbit Lie algebra $\widehat{\mathfrak{g}}$, and by $\widehat{e}_j$, $\widehat{f}_j$, $j \in \widehat{I}$, the Kashiwara operators on $\widehat{\mathcal{B}}(\widetilde{U}_q(\widehat{\mathfrak{g}}))$.

Proposition 3.2.1 ([NS1, Proposition 2.4.4]). *Let $\lambda \in P \cap (\mathfrak{h}^*)^0$, and set $\widehat{\lambda} := (P_\omega^*)^{-1}(\lambda)$. Then, the set $\mathcal{B}^0(U_q(\mathfrak{g})a_\lambda) \cup \{0\}$ is stable under ω-Kashiwara operators $\widetilde{e}_j$ and $\widetilde{f}_j$ for all $j \in \widehat{I}$. Moreover, there exists a canonical bijection $Q_\lambda : \mathcal{B}^0(U_q(\mathfrak{g})a_\lambda) \xrightarrow{\sim} \widehat{\mathcal{B}}(U_q(\widehat{\mathfrak{g}})\widehat{a}_{\widehat{\lambda}})$ such that*

$$(P_\omega^*)^{-1}(\mathrm{wt}(b)) = \mathrm{wt}(Q_\lambda(b)) \quad \text{for all } b \in \mathcal{B}^0(U_q(\mathfrak{g})a_\lambda), \tag{3.2.4}$$

$$Q_\lambda \circ \widetilde{e}_j = \widehat{e}_j \circ Q_\lambda \quad \text{and} \quad Q_\lambda \circ \widetilde{f}_j = \widehat{f}_j \circ Q_\lambda \quad \text{for all } j \in \widehat{I}, \tag{3.2.5}$$

where it is understood that $Q_\lambda(0) = 0$.

Also, we have a natural action of ω on the crystal base $\mathcal{B}(\pm\infty)$ of $U_q^\mp(\mathfrak{g})$, induced by the restriction of $\omega \in \mathrm{Aut}(U_q(\mathfrak{g}))$ to $U_q^\mp(\mathfrak{g})$, such that $\omega \circ e_j = e_{\omega(j)} \circ \omega$ and $\omega \circ f_j = f_{\omega(j)} \circ \omega$ for all $j \in I$, and $\mathrm{wt}(\omega(b)) = \omega^*(\mathrm{wt}\, b)$ for all $b \in \mathcal{B}(\pm\infty)$ (see [NS2, Sect. 3.4]); here it is understood that $\omega(0) = 0$. We set

$$\mathcal{B}^0(\pm\infty) := \{b \in \mathcal{B}(\pm\infty) \mid \omega(b) = b\}. \tag{3.2.6}$$

Let $U_q^+(\widehat{\mathfrak{g}})$ (resp., $U_q^-(\widehat{\mathfrak{g}})$) denote the positive (resp., negative) part of $U_q(\widehat{\mathfrak{g}})$, and $\widehat{\mathcal{B}}(\pm\infty)$ the crystal base of $U_q^{\mp}(\widehat{\mathfrak{g}})$, equipped with Kashiwara operators $\widehat{e}_j$, $\widehat{f}_j$, $j \in \widehat{I}$.

Proposition 3.2.2 ([NS2, Theorem 3.4.1]). *The set* $\mathcal{B}^0(\pm\infty) \cup \{0\}$ *is stable under the* ω*-Kashiwara operators* $\widetilde{e}_j$ *and* $\widetilde{f}_j$, $j \in \widehat{I}$, *which are defined by the same formula as* (3.2.3)*. Moreover, there exist canonical bijections* $P_{\pm\infty} : \mathcal{B}^0(\pm\infty) \xrightarrow{\sim} \widehat{\mathcal{B}}(\pm\infty)$ *such that*

$$(P_\omega^*)^{-1}(\mathrm{wt}(b)) = \mathrm{wt}(P_{\pm\infty}(b)) \quad \text{for all } b \in \mathcal{B}^0(\pm\infty), \tag{3.2.7}$$

$$P_{\pm\infty} \circ \widetilde{e}_j = \widehat{e}_j \circ P_{\pm\infty} \quad \text{and} \quad P_{\pm\infty} \circ \widetilde{f}_j = \widehat{f}_j \circ P_{\pm\infty} \quad \text{for all } j \in \widehat{I}, \tag{3.2.8}$$

where it is understood that $P_{\pm\infty}(0) = 0$.

Remark 3.2.3. We see that $\omega(u_{\pm\infty}) = u_{\pm\infty}$, and hence $u_{\pm\infty} \in \mathcal{B}^0(\pm\infty)$. Let $\widehat{u}_{\pm\infty}$ denote the element of $\widehat{\mathcal{B}}(\pm\infty)$ corresponding to $1 \in U_q^{\mp}(\widehat{\mathfrak{g}})$. Since $u_{\pm\infty}$ (resp., $\widehat{u}_{\pm\infty}$) is a unique element of weight $0 \in P$ (resp., $0 \in \widehat{P}$) in $\mathcal{B}(\pm\infty)$ (resp., in $\widehat{\mathcal{B}}(\pm\infty)$), we deduce from (3.2.7) that $P_{\pm\infty}(u_{\pm\infty}) = \widehat{u}_{\pm\infty}$.

We also know from [NS3, Remark 2.4.3] that

$$\Xi_\lambda\big(\mathcal{B}^0(U_q(\mathfrak{g})a_\lambda)\big) = \mathcal{B}^0(\infty) \otimes \mathcal{T}_\lambda \otimes \mathcal{B}^0(-\infty) \tag{3.2.9}$$

for $\lambda \in P \cap (\mathfrak{h}^*)^0$. Let $\widehat{\Xi}_{\widehat{\lambda}} : \widehat{\mathcal{B}}(U_q(\widehat{\mathfrak{g}})\widehat{a}_{\widehat{\lambda}}) \xrightarrow{\sim} \widehat{\mathcal{B}}(\infty) \otimes \widehat{\mathcal{T}}_{\widehat{\lambda}} \otimes \widehat{\mathcal{B}}(-\infty)$ denote the corresponding isomorphism of crystals for the orbit Lie algebra $\widehat{\mathfrak{g}}$, where $\widehat{\mathcal{T}}_{\widehat{\lambda}} := \{\widehat{t}_{\widehat{\lambda}}\}$ is a crystal for $U_q(\widehat{\mathfrak{g}})$ consisting of a single element $\widehat{t}_{\widehat{\lambda}}$ of weight $\widehat{\lambda} \in \widehat{P}$.

Proposition 3.2.4. *Let* $\lambda \in P \cap (\mathfrak{h}^*)^0$*, and set* $\widehat{\lambda} := (P_\omega^*)^{-1}(\lambda)$*. Then, the following diagram commutes*:

$$\begin{tikzcd}
\mathcal{B}^0(U_q(\mathfrak{g})a_\lambda) \arrow[r, "\Xi_\lambda"] \arrow[d, "Q_\lambda"'] & \mathcal{B}^0(\infty) \otimes \mathcal{T}_\lambda \otimes \mathcal{B}^0(-\infty) \arrow[d, "P_\infty \otimes Q^\lambda \otimes P_{-\infty}"] \\
\widehat{\mathcal{B}}(U_q(\widehat{\mathfrak{g}})\widehat{a}_{\widehat{\lambda}}) \arrow[r, "\widehat{\Xi}_{\widehat{\lambda}}"] & \widehat{\mathcal{B}}(\infty) \otimes \widehat{\mathcal{T}}_{\widehat{\lambda}} \otimes \widehat{\mathcal{B}}(-\infty),
\end{tikzcd} \tag{3.2.10}$$

where $Q^\lambda : \mathcal{T}_\lambda \to \widehat{\mathcal{T}}_{\widehat{\lambda}}$ *is defined by*: $Q^\lambda(t_\lambda) = \widehat{t}_{\widehat{\lambda}}$.

3.3 Fixed Point Subset of the Crystal Base $\mathcal{B}(\lambda)$

If $\lambda \in P \cap (\mathfrak{h}^*)^0$, then we know from [NS1, Lemma 4.3.1] that the crystal base $\mathcal{B}(\lambda) \subset \mathcal{B}(U_q(\mathfrak{g})a_\lambda)$ of the extremal weight $U_q(\mathfrak{g})$-module $V(\lambda)$ of extremal weight λ is stable under the action of ω on $\mathcal{B}(U_q(\mathfrak{g})a_\lambda)$. We set

$$\mathcal{B}^0(\lambda) := \{b \in \mathcal{B}(\lambda) \mid \omega(b) = b\} \subset \mathcal{B}^0(U_q(\mathfrak{g})a_\lambda) \quad \text{for } \lambda \in P \cap (\mathfrak{h}^*)^0. \quad (3.3.1)$$

Now, let $\widehat{\mathcal{B}}$ be a normal crystal for $U_q(\widehat{\mathfrak{g}})$. Denote by $\widehat{S} : \widehat{W} \to \mathrm{Bij}(\widehat{\mathcal{B}})$, $\widehat{w} \mapsto \widehat{S}_{\widehat{w}}$, the unique action of the Weyl group $\widehat{W}$ on the set $\widehat{\mathcal{B}}$ such that $\widehat{S}_{\widehat{r}_j} = \widehat{S}_j$ for all $j \in \widehat{I}$, where $\widehat{S}_j$ for $j \in \widehat{I}$ is defined by the same formula as (2.3.1), with the hat $\widehat{}$ on all symbols involved. An element $\widehat{b} \in \widehat{\mathcal{B}}$ is said to be extremal (or more precisely, $\widehat{W}$-extremal) if either $\widehat{e}_j \widehat{S}_{\widehat{w}} \widehat{b} = 0$ or $\widehat{f}_j \widehat{S}_{\widehat{w}} \widehat{b} = 0$ holds for each $\widehat{w} \in \widehat{W}$ and $j \in \widehat{I}$.

Let $* : \widehat{\mathcal{B}}(\widetilde{U}_q(\widehat{\mathfrak{g}})) \to \widehat{\mathcal{B}}(\widetilde{U}_q(\widehat{\mathfrak{g}}))$ be the $*$-operation on the crystal base $\widehat{\mathcal{B}}(\widetilde{U}_q(\widehat{\mathfrak{g}}))$ for the orbit Lie algebra $\widehat{\mathfrak{g}}$. For each $\widehat{\lambda} \in \widehat{P}$, we set

$$\widehat{\mathcal{B}}(\widehat{\lambda}) := \{\widehat{b} \in \widehat{\mathcal{B}}(U_q(\widehat{\mathfrak{g}})\widehat{a}_{\widehat{\lambda}}) \mid \widehat{b}^* \in \widehat{\mathcal{B}}(\widetilde{U}_q(\widehat{\mathfrak{g}})) \text{ is extremal}\} \subset \widehat{\mathcal{B}}(U_q(\widehat{\mathfrak{g}})\widehat{a}_{\widehat{\lambda}}), \quad (3.3.2)$$

which is precisely a crystal base of the extremal weight $U_q(\widehat{\mathfrak{g}})$-module of extremal weight $\widehat{\lambda}$.

Now, combining [NS1, Theorem 4.3.2] and [NS3, Theorem 3.2.1] (see also [NS3, Remark 4.1.2]), we obtain the following theorem.

Theorem 3.3.1. *Let $\lambda \in P \cap (\mathfrak{h}^*)^0$, and set $\widehat{\lambda} := (P_\omega^*)^{-1}(\lambda)$. We define a map R_λ to be the restriction $Q_\lambda|_{\mathcal{B}^0(\lambda)}$ to $\mathcal{B}^0(\lambda) \subset \mathcal{B}^0(U_q(\mathfrak{g})a_\lambda)$ of the bijection $Q_\lambda : \mathcal{B}^0(U_q(\mathfrak{g})a_\lambda) \to \widehat{\mathcal{B}}(U_q(\widehat{\mathfrak{g}})\widehat{a}_{\widehat{\lambda}})$ of Proposition 3.2.1. Then, the map R_λ is an injection from $\mathcal{B}^0(\lambda)$ into $\widehat{\mathcal{B}}(\widehat{\lambda})$.*

The main result ([NS3, Theorem 4.2.1]) of our previous paper [NS3] asserts that the (injective) map $R_\lambda : \mathcal{B}^0(\lambda) \hookrightarrow \widehat{\mathcal{B}}(\widehat{\lambda})$ above is also surjective, and hence bijective if the crystal graph of $\widehat{\mathcal{B}}(\widehat{\lambda})$ is connected. However, this restriction on the weight $\widehat{\lambda} \in \widehat{\mathfrak{h}}^*$ is so strong that there are many cases to which the result above cannot be applied (see Remark 2.6.4).

3.4 Technical Lemmas

In this subsection, we collect a few technical lemmas, which will be used to deal with Case (e) of Sect. 4.1. We define an isomorphism

$$\widehat{\Xi} : \widehat{\mathcal{B}}(\widetilde{U}_q(\widehat{\mathfrak{g}})) \xrightarrow{\sim} \bigsqcup_{\widehat{\lambda} \in \widehat{P}} \widehat{\mathcal{B}}(\infty) \otimes \widehat{\mathcal{T}}_{\widehat{\lambda}} \otimes \widehat{\mathcal{B}}(-\infty)$$

of crystals for $U_q(\widehat{\mathfrak{g}})$, and the $*$-operation

$$* : \bigsqcup_{\widehat{\lambda} \in \widehat{P}} \widehat{\mathcal{B}}(\infty) \otimes \widehat{\mathcal{T}}_{\widehat{\lambda}} \otimes \widehat{\mathcal{B}}(-\infty) \to \bigsqcup_{\widehat{\lambda} \in \widehat{P}} \widehat{\mathcal{B}}(\infty) \otimes \widehat{\mathcal{T}}_{\widehat{\lambda}} \otimes \widehat{\mathcal{B}}(-\infty)$$

in the same manner as in Sect. 2.2. The next lemma follows immediately from Proposition 2.2.1, the tensor product rule for crystals, and the definition of $\widehat{S}_j$ for $j \in \widehat{I}$ (see also [Kas4, pp. 173–174]).

Lemma 3.4.1. *Let $\widehat{b}_1 \otimes \widehat{t}_{\widehat{\lambda}} \otimes \widehat{b}_2 \in \widehat{\mathcal{B}}(\infty) \otimes \widehat{\mathcal{T}}_{\widehat{\lambda}} \otimes \widehat{\mathcal{B}}(-\infty)$, and assume that $\widehat{\lambda}(\widehat{h}_j) > 0$ for some $j \in \widehat{I}$. Then, the equation*

$$\widehat{S}_j^*(\widehat{b}_1 \otimes \widehat{t}_{\widehat{\lambda}} \otimes \widehat{b}_2) = \widehat{e}_j^{*p_1} \widehat{b}_1 \otimes \widehat{t}_{\widehat{r}_j \widehat{\lambda}} \otimes \widehat{e}_j^{*p_2} \widehat{b}_2 \tag{3.4.1}$$

holds for some $p_1, p_2 \in \mathbb{Z}_{\geq 0}$ such that $p_1 + p_2 = \widehat{\lambda}(\widehat{h}_j)$, where we set $\widehat{S}_{\widehat{w}}^* := * \circ \widehat{S}_{\widehat{w}} \circ *$ for $\widehat{w} \in \widehat{W}$, and $\widehat{e}_j^* := * \circ \widehat{e}_j \circ *$ for $j \in I$.

For an element $\widehat{b}$ of $\widehat{\mathcal{B}}(\infty) \otimes \widehat{\mathcal{T}}_{\widehat{\lambda}} \otimes \widehat{\mathcal{B}}(-\infty)$ or $\widehat{\mathcal{B}}(-\infty)$, we set

$$\widehat{\varphi}_j(\widehat{b}) := \max\bigl\{k \geq 0 \mid \widehat{f}_j^k \widehat{b} \neq 0\bigr\}, \quad \text{and} \quad \widehat{f}_j^{\max} \widehat{b} := \widehat{f}_j^{\widehat{\varphi}_j(\widehat{b})} \widehat{b}.$$

Lemma 3.4.2. *Let $\widehat{b} = \widehat{b}_1 \otimes \widehat{t}_{\widehat{\lambda}} \otimes \widehat{b}_2 \in \widehat{\mathcal{B}}(\infty) \otimes \widehat{\mathcal{T}}_{\widehat{\lambda}} \otimes \widehat{\mathcal{B}}(-\infty)$, and $j \in \widehat{I}$. Assume that $\widehat{b}$ is an extremal element such that $(\mathrm{wt}(\widehat{b}))(\widehat{h}_j) > 0$. Then, the equation*

$$\widehat{S}_j(\widehat{b}_1 \otimes \widehat{t}_{\widehat{\lambda}} \otimes \widehat{b}_2) = \widehat{f}_j^{p} \widehat{b}_1 \otimes \widehat{t}_{\widehat{\lambda}} \otimes \widehat{f}_j^{\max} \widehat{b}_2 \tag{3.4.2}$$

holds for some $p \in \mathbb{Z}_{\geq 0}$.

Proof. Since $\widehat{b}$ is an extremal element of the normal crystal $\widehat{\mathcal{B}}(\infty) \otimes \widehat{\mathcal{T}}_{\widehat{\lambda}} \otimes \widehat{\mathcal{B}}(-\infty)$ and $(\mathrm{wt}(\widehat{b}))(\widehat{h}_j) > 0$, it follows from the definitions of $\widehat{S}_j$ and $\widehat{f}_j^{\max}$ that $\widehat{S}_j \widehat{b} = \widehat{f}_j^{\max} \widehat{b}$. The assertion of the lemma now follows using a formula for "$\widetilde{f}_i^{\max} b$" (in the notation therein) on page 173 of [Kas4]. This proves the lemma. □

Let $\widehat{w} \in \widehat{W}$, and let $\widehat{w} = \widehat{r}_{j_p} \widehat{r}_{j_{p-1}} \cdots \widehat{r}_{j_1}$ be a reduced expression. We define a subset $\widehat{\mathcal{B}}_{\widehat{w}}(-\infty)$ of $\widehat{\mathcal{B}}(-\infty)$ by:

$$\widehat{\mathcal{B}}_{\widehat{w}}(-\infty) = \bigl\{\widehat{e}_{j_p}^{s_p} \widehat{e}_{j_{p-1}}^{s_{p-1}} \cdots \widehat{e}_{j_1}^{s_1} \widehat{u}_{-\infty} \mid s_p, s_{p-1}, \ldots, s_1 \in \mathbb{Z}_{\geq 0}\bigr\}.$$

We know from [Kas2, Proposition 3.2.5] (see also [Kas4, Sect. 2.3]) that the subset $\widehat{\mathcal{B}}_{\widehat{w}}(-\infty)$ does not depend on the choice of a reduced expression of $\widehat{w}$. Furthermore, we have the following lemma.

Lemma 3.4.3. (1) *There holds $\bigl(\widehat{\mathcal{B}}_{\widehat{w}}(-\infty)\bigr)^* = \widehat{\mathcal{B}}_{\widehat{w}^{-1}}(-\infty)$ in $\widehat{\mathcal{B}}(-\infty)$.*

(2) *Let $\widehat{w} \in \widehat{W}$, and let $\widehat{w} = \widehat{r}_{j_p} \widehat{r}_{j_{p-1}} \cdots \widehat{r}_{j_1}$ be a reduced expression. Then, we have*

$$\widehat{f}_{j_1}^{\max} \widehat{f}_{j_2}^{\max} \cdots \widehat{f}_{j_p}^{\max} \widehat{b} = \widehat{u}_{-\infty} \quad \text{for all } \widehat{b} \in \widehat{\mathcal{B}}_{\widehat{w}}(-\infty). \tag{3.4.3}$$

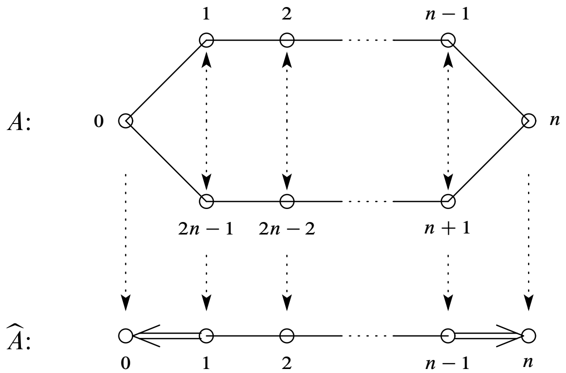
Proof. We know part (1) from [Kas2, Proposition 3.3.1]. Part (2) follows from [Kas2, Proposition 3.2.5 (iii) and Lemma 3.3.3] (see also [Kas4, Sect. 2.3, statements (iv) and (2.9)]). □

4 Surjectivity of the Map R_λ: Case of Affine Lie Algebras

4.1 Affine Lie Algebras, Diagram Automorphisms, and Orbit Lie Algebras

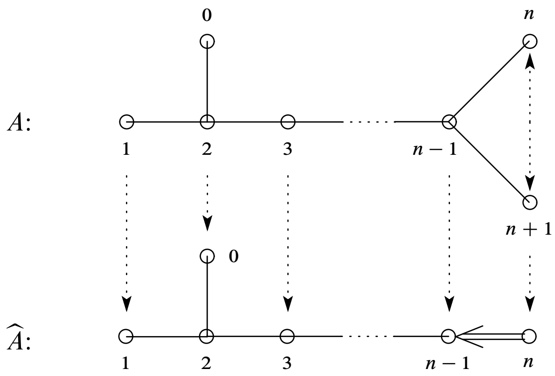
In the remainder of this paper, we assume that $\mathfrak{g} = \mathfrak{g}(A)$ is an affine Lie algebra of type $A_n^{(1)}$ ($n \geq 2$), $D_n^{(1)}$ ($n \geq 4$), or $E_6^{(1)}$. We further assume that the diagram automorphism $\omega : I \to I$ satisfies the (additional) condition that $\omega(0) = 0$, where $0 \in I$ is a distinguished index (specified as below). In this subsection, we give all pairs $(\mathfrak{g}, \omega)$ of an affine Lie algebra $\mathfrak{g}$ and a nontrivial diagram automorphism $\omega : I \to I$ satisfying the condition above, and also give the associated orbit Lie algebras $\widehat{\mathfrak{g}}$.

Case (a). The Cartan matrix $A = (a_{ij})_{i,j \in I}$ of $\mathfrak{g}$ is the affine Cartan matrix of type $A_{2n-1}^{(1)}$ ($n \geq 2$), and the diagram automorphism $\omega : I \to I$ is given by: $\omega(0) = 0$ and $\omega(j) = 2n - j$ for $j \in I \setminus \{0\}$ (note that the order N of ω is equal to 2). Then the Cartan matrix $\widehat{A} = (\widehat{a}_{ij})_{i,j \in \widehat{I}}$ of the orbit Lie algebra $\widehat{\mathfrak{g}}$ is the affine Cartan matrix of type $D_{n+1}^{(2)}$:

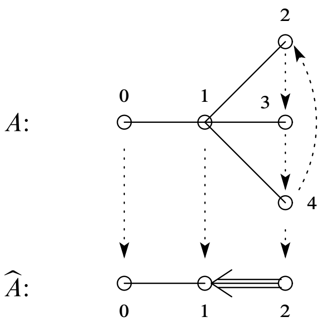


Case (b). The Cartan matrix $A = (a_{ij})_{i,j \in I}$ of $\mathfrak{g}$ is the affine Cartan matrix of type $D_{n+1}^{(1)}$ ($n \geq 3$), and the diagram automorphism $\omega : I \to I$ is given by: $\omega(j) = j$ for $j \in I \setminus \{n, n+1\}$, and $\omega(n) = n + 1$, $\omega(n + 1) = n$ (note that the order N of

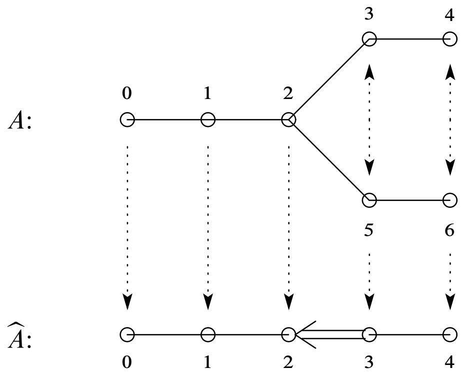
ω is equal to 2). Then the Cartan matrix $\widehat{A} = (\widehat{a}_{ij})_{i,j \in \widehat{I}}$ of the orbit Lie algebra $\widehat{\mathfrak{g}}$ is the affine Cartan matrix of type $A^{(2)}_{2n-1}$:



Case (c). The Cartan matrix $A = (a_{ij})_{i,j \in I}$ of $\mathfrak{g}$ is the affine Cartan matrix of type $D^{(1)}_4$, and the diagram automorphism $\omega : I \to I$ is given by: $\omega(0) = 0$, $\omega(1) = 1$, $\omega(2) = 3$, $\omega(3) = 4$, and $\omega(4) = 2$ (note that the order N of ω is equal to 3). Then the Cartan matrix $\widehat{A} = (\widehat{a}_{ij})_{i,j \in \widehat{I}}$ of the orbit Lie algebra $\widehat{\mathfrak{g}}$ is the affine Cartan matrix of type $D^{(3)}_4$:

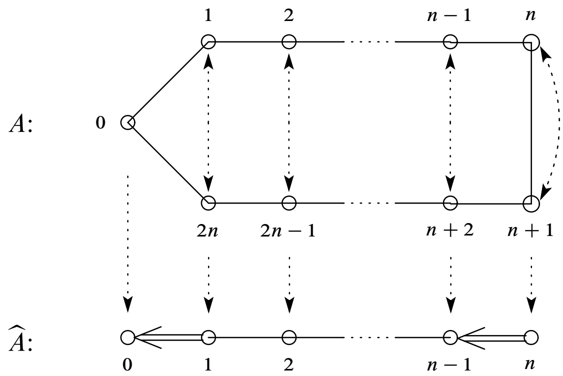


Case (d). The Cartan matrix $A = (a_{ij})_{i,j \in I}$ of $\mathfrak{g}$ is the affine Cartan matrix of type $E^{(1)}_6$, and the diagram automorphism $\omega : I \to I$ is given by: $\omega(0) = 0$, $\omega(1) = 1$, $\omega(2) = 2$, $\omega(3) = 5$, $\omega(4) = 6$, $\omega(5) = 3$, and $\omega(6) = 4$ (note that the order N of ω is equal to 2). Then the Cartan matrix $\widehat{A} = (\widehat{a}_{ij})_{i,j \in \widehat{I}}$ of the orbit Lie algebra $\widehat{\mathfrak{g}}$ is the affine Cartan matrix of type $E^{(2)}_6$:

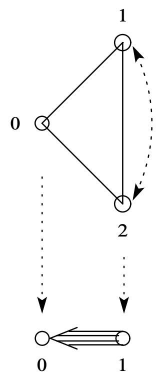


Case (e). The Cartan matrix $A = (a_{ij})_{i,j \in I}$ of $\mathfrak{g}$ is the affine Cartan matrix of type $A_{2n}^{(1)}$ ($n \geq 1$), and the diagram automorphism $\omega : I \to I$ is given by: $\omega(0) = 0$ and $\omega(j) = 2n + 1 - j$ for $j \in I \setminus \{0\}$ (note that the order of ω is equal to 2). Then the Cartan matrix $\widehat{A} = (\widehat{a}_{ij})_{i,j \in \widehat{I}}$ of the orbit Lie algebra $\widehat{\mathfrak{g}}$ is the affine Cartan matrix of type $A_{2n}^{(2)}$ (it should be mentioned again that the vertices of the Dynkin diagram of type $A_{2n}^{(2)}$ are numbered in an order reverse to the one in [BN]; see [BN, Sect. 2.1]):

If $n \geq 2$, then



If $n = 1$, then



Recall from Sect. 2.5 that the Cartan subalgebra $\mathfrak{h}$ of the affine Lie algebra $\mathfrak{g}$ is given by: $\mathfrak{h} = \left(\bigoplus_{j \in I} \mathbb{Q} h_j \right) \oplus \mathbb{Q} d$, where d is the scaling element, and that $\alpha_j(d) = \delta_{j,0}$, $\Lambda_j(d) = 0$ for $j \in I$. Here we remark that since $\omega(0) = 0$, the automorphism $\omega : \mathfrak{h} \to \mathfrak{h}$ is given by: $\omega(h_j) = h_{\omega(j)}$ for $j \in I$, and $\omega(d) = d$. Note that $\omega(c) = c$ and $\omega^*(\delta) = \delta$ (see Remark 3.1.1). Also, since $\omega(d) = d$, it follows that $\omega^*(\Lambda_j) = \Lambda_{\omega(j)}$ for all $j \in I$. In particular, the integral weight lattice (see (2.5.1); note that $a_0 = 1$)

$$P = \left(\bigoplus_{j \in I} \mathbb{Z}\Lambda_j\right) \oplus \mathbb{Z}\delta \subset \mathfrak{h}^*$$

is ω^*-stable.

Similarly, the Cartan subalgebra $\widehat{\mathfrak{h}}$ of the orbit Lie algebra $\widehat{\mathfrak{g}}$ is given by: $\widehat{\mathfrak{h}} = \left(\bigoplus_{j \in \widehat{I}} \mathbb{Q}\widehat{h}_j\right) \oplus \mathbb{Q}\widehat{d}$, where $\widehat{d}$ is the scaling element. Also, the simple roots $\widehat{\alpha}_j$, $j \in \widehat{I}$, and fundamental weights $\widehat{\Lambda}_j \in \widehat{\mathfrak{h}}^*$, $j \in \widehat{I}$, for the orbit Lie algebra $\widehat{\mathfrak{g}}$ satisfy the following: $\widehat{\alpha}_j(\widehat{d}) = \delta_{j,0}$ and $\widehat{\Lambda}_j(\widehat{d}) = 0$ for $j \in \widehat{I}$. Let $\widehat{\delta} := \sum_{j \in \widehat{I}} \widehat{a}_j \widehat{\alpha}_j \in \widehat{\mathfrak{h}}^*$ and $\widehat{c} := \sum_{j \in \widehat{I}} \widehat{a}_j^\vee \widehat{h}_j \in \widehat{\mathfrak{h}}$ denote the null root and the canonical central element of the orbit Lie algebra $\widehat{\mathfrak{g}}$, respectively. Note that $\widehat{a}_0 = 1$ in Cases (a)–(d), and $\widehat{a}_0 = 2$ in Case (e). Then the integral weight lattice $\widehat{P} \subset \widehat{\mathfrak{h}}^*$ is given by (see (2.5.1)):

$$\widehat{P} = \left(\bigoplus_{j \in \widehat{I}} \mathbb{Z}\widehat{\Lambda}_j\right) \oplus \mathbb{Z}\left(\widehat{a}_0^{-1}\widehat{\delta}\right) \subset \widehat{\mathfrak{h}}^*.$$

Since $\omega(0) = 0$, the $\mathbb{Q}$-linear isomorphism $P_\omega : \mathfrak{h}^0 \xrightarrow{\sim} \widehat{\mathfrak{h}}$ is given by: $P_\omega(\widetilde{h}_j) = \widehat{h}_j$ for $j \in I$, and $P_\omega(d) = \widehat{d}$. It is easy to verify that

$$P_\omega(c) = \widehat{c}, \quad \text{and} \quad P_\omega^*(\widehat{\delta}) = \widehat{a}_0\delta. \tag{4.1.1}$$

Also, since

$$P_\omega^*(\widehat{\Lambda}_j) = \sum_{k=0}^{N_j-1} \Lambda_{\omega^k(j)} \quad \text{for all } j \in \widehat{I}, \tag{4.1.2}$$

we have $\widehat{P} = (P_\omega^*)^{-1}(P \cap (\mathfrak{h}^*)^0)$.

4.2 Main Result

The purpose of this paper is to prove that in Cases (a)–(e) of Sect. 4.1, the (injective) map $R_\lambda : \mathcal{B}^0(\lambda) \hookrightarrow \widehat{\mathcal{B}}(\widehat{\lambda})$ of Theorem 3.3.1 is also surjective, and hence bijective for every $\lambda \in P \cap (\mathfrak{h}^*)^0$. We see from (4.1.1) and the definition of P_ω^* (see (3.1.5)) that $\lambda \in P \cap (\mathfrak{h}^*)^0$ satisfies $\lambda(c) > 0$ (resp., $\lambda(c) < 0$) if and only if $\widehat{\lambda} := (P_\omega^*)^{-1}(\lambda)$ satisfies $\widehat{\lambda}(\widehat{c}) > 0$ (resp., $\widehat{\lambda}(\widehat{c}) < 0$). Hence the argument in Sect. 2.5 shows that if $\lambda \in P \cap (\mathfrak{h}^*)^0$ satisfies $\lambda(c) > 0$ or $\lambda(c) < 0$, then the crystal graph of $\widehat{\mathcal{B}}(\widehat{\lambda})$ is connected. Thus, in these cases, [NS3, Theorem 4.2.1] shows that the (injective) map $R_\lambda : \mathcal{B}^0(\lambda) \hookrightarrow \widehat{\mathcal{B}}(\widehat{\lambda})$ is surjective, and hence bijective. Therefore, it remains to consider the case in which $\lambda \in P \cap (\mathfrak{h}^*)^0$ is level-zero. Note that if $\lambda \in P \cap (\mathfrak{h}^*)^0$ is level-zero, then $\widehat{\lambda} = (P_\omega^*)^{-1}(\lambda) \in \widehat{P}$ is also level-zero, i.e. $\widehat{\lambda}(\widehat{c}) = 0$.

We set $\widehat{I}_0 := \widehat{I} \setminus \{0\}$, and for each $i \in \widehat{I}_0$, define a level-zero fundamental weight $\widehat{\varpi}_i \in \widehat{P}$ for the orbit Lie algebra $\widehat{\mathfrak{g}}$ by: $\widehat{\varpi}_i = \widehat{\Lambda}_i - \widehat{a}_i^\vee \widehat{\Lambda}_0$. A simple calculation yields the following equations:

$$\omega(\varpi_i) = \varpi_{\omega(i)} \quad \text{for all } i \in I_0, \tag{4.2.1}$$

$$P_\omega^*(\widehat{\varpi}_i) = \sum_{k=0}^{N_i-1} \varpi_{\omega^k(i)} \quad \text{for all } i \in \widehat{I}_0. \tag{4.2.2}$$

Let $\lambda \in P \cap (\mathfrak{h}^*)^0$ be a level-zero integral weight. Because the crystal bases $\mathcal{B}(\lambda)$ and $\mathcal{B}(\lambda + k\delta)$ are "isomorphic" up to shift of weights by $k\delta$ for $k \in \mathbb{Z}$ by (2.5.4), we may (and do) assume that $\lambda(d) = 0$, and hence $\lambda \in \left(\sum_{i \in I_0} \mathbb{Z}\varpi_i\right) \cap (\mathfrak{h}^*)^0$. Then, since $m_{\omega(i)} = m_i$ for all $i \in I_0$ by (4.2.1), it follows from (4.2.2) that $\widehat{\lambda} = (P_\omega^*)^{-1}(\lambda) \in \sum_{i \in \widehat{I}_0} \mathbb{Z}\widehat{\varpi}_i$. Take $\widehat{w} \in \widehat{W}_0 := \langle \widehat{r}_j \mid j \in \widehat{I}_0 \rangle \subset \widehat{W}$ such that $\widehat{w}\widehat{\lambda} \in \sum_{i \in \widehat{I}_0} \mathbb{Z}_{\geq 0}\widehat{\varpi}_i$, and set $w := \Theta^{-1}(\widehat{w}) \in \langle w_j \mid j \in \widehat{I}_0 \rangle \subset \widetilde{W}$. Because $w\lambda = \left(P_\omega^* \circ \widehat{w} \circ (P_\omega^*)^{-1}\right)(\lambda) = P_\omega^*(\widehat{w}\widehat{\lambda})$ (see Sect. 3.1), we have $w\lambda \in \left(\sum_{i \in I_0} \mathbb{Z}_{\geq 0}\varpi_i\right) \cap (\mathfrak{h}^*)^0$ by (4.2.2). Here we recall from Theorem 2.4.2 (5) that $S_w^* = * \circ S_w \circ *$ (resp., $\widehat{S}_{\widehat{w}}^* = * \circ \widehat{S}_{\widehat{w}} \circ *$) gives an isomorphism of crystals from $\mathcal{B}(\lambda) \subset \mathcal{B}(U_q(\mathfrak{g})a_\lambda)$ onto $\mathcal{B}(w\lambda) \subset \mathcal{B}(U_q(\mathfrak{g})a_{w\lambda})$ (resp., from $\widehat{\mathcal{B}}(\widehat{\lambda}) \subset \widehat{\mathcal{B}}(U_q(\widehat{\mathfrak{g}})\widehat{a}_{\widehat{\lambda}})$ onto $\widehat{\mathcal{B}}(\widehat{w}\widehat{\lambda}) \subset \widehat{\mathcal{B}}(U_q(\widehat{\mathfrak{g}})\widehat{a}_{\widehat{w}\widehat{\lambda}})$). Since $w \in \widetilde{W}$, and $\omega \circ * = * \circ \omega$ from the definitions, we can easily deduce that $\omega \circ S_w^* = S_{\omega^* w (\omega^*)^{-1}}^* \circ \omega = S_w^* \circ \omega$ using (3.1.3). Therefore, under the isomorphism $S_w^* : \mathcal{B}(\lambda) \xrightarrow{\sim} \mathcal{B}(w\lambda)$, the fixed point subset $\mathcal{B}^0(\lambda) \subset \mathcal{B}(\lambda)$ is mapped onto the fixed point subset $\mathcal{B}^0(w\lambda) \subset \mathcal{B}(w\lambda)$. Furthermore, we can show, using [NS1, Proposition 2.4.4(4)] (see also the proof of [NS1, Theorem 3.3.3]), that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{B}^0(\lambda) & \xrightarrow{R_\lambda} & \widehat{\mathcal{B}}(\widehat{\lambda}) \\
{\scriptstyle S_w^*}\downarrow & & \downarrow{\scriptstyle \widehat{S}_{\widehat{w}}^*} \\
\mathcal{B}^0(w\lambda) & \xrightarrow{R_{w\lambda}} & \widehat{\mathcal{B}}(\widehat{w}\widehat{\lambda}).
\end{array}$$

Hence the map $R_\lambda : \mathcal{B}^0(\lambda) \to \widehat{\mathcal{B}}(\widehat{\lambda})$ is surjective if and only if the map $R_{w\lambda} : \mathcal{B}^0(w\lambda) \to \widehat{\mathcal{B}}(\widehat{w}\widehat{\lambda})$ is surjective. Thus, we may assume from the beginning that λ is contained in the set $\left(\sum_{i \in I_0} \mathbb{Z}_{\geq 0}\varpi_i\right) \cap (\mathfrak{h}^*)^0$.

Now we are ready to state the main result of this paper.

Theorem 4.2.1. *Assume that* $\mathfrak{g} = \mathfrak{g}(A)$ *is an affine Lie algebra of type* $A_n^{(1)}$ $(n \ge 2)$, $D_n^{(1)}$ $(n \ge 4)$, *or* $E_6^{(1)}$, *and that the diagram automorphism* $\omega : I \to I$ *fixes the index* $0 \in I$ *specified as in Sect. 4.1. Let* $\lambda \in \left(\sum_{i \in I_0} \mathbb{Z}_{\ge 0} \varpi_i\right) \cap (\mathfrak{h}^*)^0$, *and set* $\widehat{\lambda} := (P_\omega^*)^{-1}(\lambda)$. *Then, the (injective) map* $R_\lambda : \mathcal{B}^0(\lambda) \hookrightarrow \widehat{\mathcal{B}}(\widehat{\lambda})$ *of Theorem 3.3.1 is also surjective, and hence bijective.*

4.3 PIE Elements Fixed by a Diagram Automorphism

We take (and fix) a level-zero integral weight $\lambda \in P$ of the form $\lambda = \sum_{i \in I_0} m_i \varpi_i$, with $m_i \in \mathbb{Z}_{\ge 0}$ for $i \in I_0$, such that $\omega^*(\lambda) = \lambda$; note that $m_{\omega(i)} = m_i$ for all $i \in I_0$ by (4.2.1). Also, note that $d_i = \max\{1, (\alpha_i, \alpha_i)/2\}$ is equal to 1 for all $i \in I_0$, since the Cartan matrix $A = (a_{ij})_{i,j \in I}$ of $\mathfrak{g}$ is symmetric.

Recall from Sect. 2.6 the set $\mathbf{N}(\lambda)$ of all I_0-tuples $(\rho^{(i)})_{i \in I_0}$ of partitions $\rho^{(i)}$, $i \in I_0$, such that $\ell(\rho^{(i)}) \le m_i$ for all $i \in I_0$. We define an action of the diagram automorphism ω on $\mathbf{N}(\lambda)$ as follows. Let $\mathbf{c}_0 = (\rho^{(i)})_{i \in I_0} \in \mathbf{N}(\lambda)$. Then we define $\omega(\mathbf{c}_0) := (\chi^{(i)})_{i \in I_0}$, where $\chi^{(i)} := \rho^{(\omega(i))}$ for each $i \in I_0$. Since $m_i = m_{\omega(i)}$ for all $i \in I_0$, it follows that $\omega(\mathbf{c}_0) \in \mathbf{N}(\lambda)$. We set

$$\mathbf{N}^0(\lambda) := \left\{\mathbf{c}_0 \in \mathbf{N}(\lambda) \mid \omega(\mathbf{c}_0) = \mathbf{c}_0\right\}. \tag{4.3.1}$$

Note that $\mathbf{c}_0 = (\rho^{(i)})_{i \in I_0} \in \mathbf{N}(\lambda)$ is fixed by the diagram automorphism ω if and only if $\rho^{(i)} = \rho^{(\omega(i))}$ for all $i \in I_0$. This, in particular, implies that in Case (e), the number $|\mathbf{c}_0| = \sum_{i \in I_0} |\rho^{(i)}| d_i = \sum_{i \in I_0} |\rho^{(i)}|$ is contained in $2\mathbb{Z}_{\ge 0} = \widehat{a}_0 \mathbb{Z}_{\ge 0}$ for all $\mathbf{c}_0 \in \mathbf{N}^0(\lambda)$. For each $k \in \mathbb{Z}_{\ge 0}$, we set

$$\mathbf{N}^0(\lambda)_k := \left\{\mathbf{c}_0 \in \mathbf{N}^0(\lambda) \mid |\mathbf{c}_0| = \widehat{a}_0 k\right\}. \tag{4.3.2}$$

Lemma 4.3.1. *Let* $\mathbf{c}_0 \in \mathbf{N}(\lambda)$. *Then the element* $S_{\mathbf{c}_0}^- \in U_q^-(\mathfrak{g})$ *is mapped to* $S_{\omega(\mathbf{c}_0)}^-$ *under* $\omega \in \mathrm{Aut}(U_q(\mathfrak{g}))$. *Moreover, the element* $S_{\mathbf{c}_0}^- \in U_q^-(\mathfrak{g})$ *is fixed by* $\omega \in \mathrm{Aut}(U_q(\mathfrak{g}))$ *if and only if* $\mathbf{c}_0 \in \mathbf{N}^0(\lambda)$.

Proof. We see from the definition (see Sect. 2.6) that the integral imaginary root vector $\widetilde{P}_{i,k}$ $(= \widetilde{P}_{i,kd_i})$ is mapped to $\widetilde{P}_{\omega(i),k}$ $(= \widetilde{P}_{\omega(i),kd_{\omega(i)}})$ under $\omega \in \mathrm{Aut}(U_q(\mathfrak{g}))$ for each $i \in I_0$ and $k \in \mathbb{Z}_{\ge 0}$. Therefore, we deduce from the definitions (2.6.2) and (2.6.7) that $\omega(S_{\mathbf{c}_0}) = S_{\omega(\mathbf{c}_0)}$. Also, it follows immediately from (2.6.4) and (2.6.5) that $\omega \circ \vee = \vee \circ \omega$ and $\omega \circ {}^- = {}^- \circ \omega$. Hence we deduce that $\omega(S_{\mathbf{c}_0}^-) = S_{\omega(\mathbf{c}_0)}^-$, which is the first assertion. The second assertion follows from the first one and Proposition 2.6.2 (2). This proves the lemma. □

Because the action of the diagram automorphism ω on $\mathcal{B}(\infty)$ is induced by $\omega \in \mathrm{Aut}(U_q(\mathfrak{g}))$, it follows from Lemma 4.3.1 and (3.2.9) that

$$\begin{aligned}
&\mathbf{c}_0 \in \mathbf{N}^0(\lambda) \\
&\iff \quad (S_{\mathbf{c}_0}^- \bmod q_s\mathcal{L}(\infty)) \otimes t_\lambda \otimes u_{-\infty} \in \mathcal{B}^0(\infty) \otimes \mathcal{T}_\lambda \otimes \mathcal{B}^0(-\infty) \\
&\iff \quad b_{\mathbf{c}_0} = \Xi_\lambda^{-1}\big((S_{\mathbf{c}_0}^- \bmod q_s\mathcal{L}(\infty)) \otimes t_\lambda \otimes u_{-\infty}\big) \in \mathcal{B}^0(U_q(\mathfrak{g})a_\lambda).
\end{aligned}$$

Therefore, we conclude from Proposition 2.6.2 (3) that

$$\{b_{\mathbf{c}_0} \mid \mathbf{c}_0 \in \mathbf{N}^0(\lambda)\} \subset \mathcal{B}^0(U_q(\mathfrak{g})a_\lambda) \cap \mathcal{B}(\lambda) = \mathcal{B}^0(\lambda). \tag{4.3.3}$$

Now, for each $k \in \mathbb{Z}_{\geq 0}$, we set

$$\mathcal{B}^0(\lambda)_{\mathrm{PIE},k} := \left\{ b \in \mathcal{B}^0(\lambda) \;\middle|\; \begin{array}{l} b \text{ is extremal, and } \Xi_\lambda(b) = b_1 \otimes t_\lambda \otimes u_{-\infty} \\ \text{for some } b_1 \in \mathcal{B}^0(\infty) \text{ such that } \mathrm{wt}(b_1) = -\widehat{a}_0 k\delta \end{array} \right\}.$$

Then it follows from Proposition 2.6.2 (3) and (4.3.3) that the set $\{b_{\mathbf{c}_0} \mid \mathbf{c}_0 \in \mathbf{N}^0(\lambda)_k\}$ is contained in $\mathcal{B}^0(\lambda)_{\mathrm{PIE},k}$. Therefore, in view of Proposition 2.6.2 (2), we obtain

$$\#\mathcal{B}^0(\lambda)_{\mathrm{PIE},k} \geq \#\mathbf{N}^0(\lambda)_k \qquad \text{for all } k \in \mathbb{Z}_{\geq 0}. \tag{4.3.4}$$

4.4 PIE Elements for Orbit Lie Algebras

As in Sect. 4.3, we assume that $\lambda \in P$ is a level-zero integral weight of the form $\lambda = \sum_{i \in I_0} m_i \varpi_i$, with $m_i \in \mathbb{Z}_{\geq 0}$ for $i \in I_0$, such that $\omega^*(\lambda) = \lambda$. We set $\widehat{\lambda} = (P_\omega^*)^{-1}(\lambda)$, which is equal to $\sum_{i \in \widehat{I}_0} m_i \widehat{\varpi}_i$ by (4.2.2).

Recall that in Case (e), the vertices of the Dynkin diagram of type $A_{2n}^{(2)}$ are numbered in an order reverse to the one in [BN] (see [BN, Sect. 2.1]). We define $\widehat{\eta}_i \in \widehat{P}$, $i \in \widehat{I}_n := \widehat{I} \setminus \{n\}$, as in (2.6.6). Then, $\widehat{\lambda} = \sum_{i \in \widehat{I}_0} m_i \widehat{\varpi}_i$ is conjugate, under the (finite) Weyl group $\widehat{W}_0 := \langle \widehat{r}_j \mid j \in \widehat{I}_0 \rangle \subset \widehat{W}$, to $\widehat{\Lambda} := \sum_{i \in \widehat{I}_0} m_i \widehat{\eta}_{n-i}$ (see Remark 2.6.1).

Let $\widehat{\mathbf{N}}(\widehat{\lambda})$ denote the set of $\widehat{I}_0$-tuples $(\widehat{\rho}^{(i)})_{i \in \widehat{I}_0}$ of partitions $\widehat{\rho}^{(i)}$, $i \in \widehat{I}_0$, such that $\ell(\widehat{\rho}^{(i)}) \leq m_i$ for all $i \in \widehat{I}_0$. Also, for each $k \in \mathbb{Z}_{\geq 0}$, we set

$$\widehat{\mathbf{N}}(\widehat{\lambda})_k := \{\widehat{\mathbf{c}}_0 \in \widehat{\mathbf{N}}(\widehat{\lambda}) \mid |\widehat{\mathbf{c}}_0| = k\},$$

where $|\widehat{\mathbf{c}}_0| := \sum_{i \in \widehat{I}_0} |\widehat{\rho}^{(i)}| \widehat{d}_i$, with

$$\widehat{d}_i := \begin{cases} \max\{1, (\widehat{\alpha}_i, \widehat{\alpha}_i)/2\} & \text{in Cases (a)–(d),} \\ \max\{1, (\widehat{\alpha}_{n-i}, \widehat{\alpha}_{n-i})/2\} = 1 & \text{in Case (e),} \end{cases} \qquad \text{for } i \in \widehat{I}_0. \tag{4.4.1}$$

Remark 4.4.1. Recall that $N_i \in \mathbb{Z}_{\geq 1}$ is defined to be the number $\#\{\omega^k(i) \mid k \geq 0\}$ for $i \in \widehat{I}$. We can check that $\widehat{d}_i = N_i \widehat{a}_0^{-1}$ for all $i \in \widehat{I}_0$ in Cases (a)–(e).

For each $k \in \mathbb{Z}_{\geq 0}$, we set

$$\widehat{\mathcal{B}}(\widehat{\mu})_{\mathrm{PIE},k} := \left\{ \widehat{b} \in \widehat{\mathcal{B}}(\widehat{\mu}) \;\middle|\; \begin{array}{l} \widehat{b} \text{ is extremal, and } \widehat{\Xi}_{\widehat{\mu}}(\widehat{b}) = \widehat{b}_1 \otimes \widehat{t}_{\widehat{\mu}} \otimes \widehat{u}_{-\infty} \\ \text{for some } \widehat{b}_1 \in \widehat{\mathcal{B}}(\infty) \text{ such that } \mathrm{wt}(\widehat{b}_1) = -k\widehat{\delta} \end{array} \right\}, \tag{4.4.2}$$

where $\widehat{\mu} = \widehat{\lambda}$ in Cases (a)–(e), and $\widehat{\mu} = \widehat{\lambda}$ or $\widehat{\mu} = \widehat{\Lambda}$ in Case (e).

Lemma 4.4.2. *Let $\widehat{\lambda} = (P_\omega^*)^{-1}(\lambda) = \sum_{i \in \widehat{I}_0} m_i \widehat{\varpi}_i$ be the level-zero integral weight above for the orbit Lie algebra $\widehat{\mathfrak{g}}$. Then, the inequality $\#\widehat{\mathcal{B}}(\widehat{\lambda})_{\mathrm{PIE},k} \leq \#\widehat{\mathbf{N}}(\widehat{\lambda})_k$ holds for every $k \in \mathbb{Z}_{\geq 0}$.*

Proof. Let us fix $k \in \mathbb{Z}_{\geq 0}$ arbitrarily. In Cases (a)–(d), we see from Proposition 2.6.2 (2)–(4) that the equality $\#\widehat{\mathcal{B}}(\widehat{\lambda})_{\mathrm{PIE},k} = \#\widehat{\mathbf{N}}(\widehat{\lambda})_k$ holds. Hence it remains to consider Case (e). Let $\widehat{w} \in \widehat{W}_0$ be the unique shortest element such that $\widehat{w}\widehat{\lambda} = \widehat{\Lambda}$, and let $\widehat{w} = \widehat{r}_{j_p}\widehat{r}_{j_{p-1}} \cdots \widehat{r}_{j_1}$ be a reduced expression. Then it follows (for example, from [Kac, Lemma 3.11 b)]) that

$$\big(\widehat{r}_{j_q}\widehat{r}_{j_{q-1}} \cdots \widehat{r}_{j_1}(\widehat{\lambda})\big)(\widehat{h}_{j_{q+1}}) > 0 \quad \text{for } q = 0, 1, \ldots, p-1. \tag{4.4.3}$$

Claim. *If $\widehat{b} \in \widehat{\mathcal{B}}(\widehat{\lambda})_{\mathrm{PIE},k}$, then the element $\widehat{S}_{\widehat{w}}\widehat{S}^*_{\widehat{w}}(\widehat{b})$ is contained in $\widehat{\mathcal{B}}(\widehat{\Lambda})_{\mathrm{PIE},k}$. Namely, we have $\widehat{S}_{\widehat{w}}\widehat{S}^*_{\widehat{w}}\big(\widehat{\mathcal{B}}(\widehat{\lambda})_{\mathrm{PIE},k}\big) \subset \widehat{\mathcal{B}}(\widehat{\Lambda})_{\mathrm{PIE},k}$.*

Proof of the claim. We see from (4.4.2) that the weight of an element of $\widehat{\mathcal{B}}(\widehat{\lambda})_{\mathrm{PIE},k}$ is $\widehat{\lambda} - k\widehat{\delta}$. Because $\widehat{S}^*_{\widehat{w}}$ gives an isomorphism of crystals from $\widehat{\mathcal{B}}(\widehat{\lambda})$ onto $\widehat{\mathcal{B}}(\widehat{w}\widehat{\lambda}) = \widehat{\mathcal{B}}(\widehat{\Lambda})$, $\widehat{S}^*_{\widehat{w}}(\widehat{b})$ is an extremal element of $\widehat{\mathcal{B}}(\widehat{\Lambda})$ whose weight is $\widehat{\lambda} - k\widehat{\delta}$ by Remark 2.3.2 (2). Also, it follows from Remark 2.3.2 (1) that $\widehat{S}_{\widehat{w}}\widehat{S}^*_{\widehat{w}}(\widehat{b})$ is an extremal element of $\widehat{\mathcal{B}}(\widehat{\Lambda})$ whose weight is $\widehat{w}(\widehat{\lambda} - k\widehat{\delta}) = \widehat{\Lambda} - k\widehat{\delta}$.

Now, using lemmas in Sect. 3.4, we compute the element $\widehat{\Xi}_{\widehat{\Lambda}}(\widehat{S}_{\widehat{w}}\widehat{S}^*_{\widehat{w}}(\widehat{b}))$. Write $\widehat{\Xi}_{\widehat{\lambda}}(\widehat{b})$ as $\widehat{b}_1 \otimes \widehat{t}_{\widehat{\lambda}} \otimes \widehat{u}_{-\infty}$ for some $\widehat{b}_1 \in \widehat{\mathcal{B}}(\infty)$ such that $\mathrm{wt}(\widehat{b}_1) = -k\widehat{\delta}$. Then we have

$$\begin{aligned} \widehat{\Xi}_{\widehat{\Lambda}}(\widehat{S}_{\widehat{w}}\widehat{S}^*_{\widehat{w}}(\widehat{b})) = \widehat{\Xi}(\widehat{S}_{\widehat{w}}\widehat{S}^*_{\widehat{w}}(\widehat{b})) &= \widehat{S}_{\widehat{w}}\widehat{\Xi}\big(\widehat{S}^*_{\widehat{w}}(\widehat{b})\big) \quad \text{by Remark 2.3.2 (2)} \\ &= \widehat{S}_{\widehat{w}}\widehat{S}^*_{\widehat{w}}\widehat{\Xi}(\widehat{b}) \quad \text{by Remark 2.3.2 (2), along with (2.2.4)} \\ &= \widehat{S}_{\widehat{w}}\widehat{S}^*_{\widehat{w}}\widehat{\Xi}_{\widehat{\lambda}}(\widehat{b}) = \widehat{S}_{\widehat{w}}\widehat{S}^*_{\widehat{w}}(\widehat{b}_1 \otimes \widehat{t}_{\widehat{\lambda}} \otimes \widehat{u}_{-\infty}). \end{aligned}$$

Noting (4.4.3), we obtain, by repeated application of Lemma 3.4.1,

$$\begin{aligned}\widehat{S}^*_{\widehat{w}}(\widehat{b}_1 \otimes \widehat{t}_{\widehat{\lambda}} \otimes \widehat{u}_{-\infty}) &= \widehat{S}^*_{j_p}\widehat{S}^*_{j_{p-1}} \cdots \widehat{S}^*_{j_1}(\widehat{b}_1 \otimes \widehat{t}_{\widehat{\lambda}} \otimes \widehat{u}_{-\infty}) \\ &= \widehat{b}'_1 \otimes \widehat{t}_{\widehat{\Lambda}} \otimes \big(\widehat{e}^{*s_p}_{j_p}\widehat{e}^{*s_{p-1}}_{j_{p-1}} \cdots \widehat{e}^{*s_1}_{j_1}\widehat{u}_{-\infty}\big)\end{aligned}$$

for some $s_p, s_{p-1}, \dots, s_1 \in \mathbb{Z}_{\geq 0}$ and $\widehat{b}'_1 \in \widehat{\mathcal{B}}(\infty)$. Furthermore, again noting (4.4.3), we obtain, by repeated application of Lemma 3.4.2,

$$\begin{aligned}\widehat{S}_w\widehat{S}^*_{\widehat{w}}(\widehat{b}_1 \otimes \widehat{t}_{\widehat{\lambda}} \otimes \widehat{u}_{-\infty}) &= \widehat{S}_{j_p}\widehat{S}_{j_{p-1}} \cdots \widehat{S}_{j_1}\Big(\widehat{b}'_1 \otimes \widehat{t}_{\widehat{\Lambda}} \otimes \big(\widehat{e}^{*s_p}_{j_p}\widehat{e}^{*s_{p-1}}_{j_{p-1}} \cdots \widehat{e}^{*s_1}_{j_1}\widehat{u}_{-\infty}\big)\Big) \\ &= \widehat{b}''_1 \otimes \widehat{t}_{\widehat{\Lambda}} \otimes \big(\widehat{f}^{\max}_{j_p}\widehat{f}^{\max}_{j_{p-1}} \cdots \widehat{f}^{\max}_{j_1}\widehat{e}^{*s_p}_{j_p}\widehat{e}^{*s_{p-1}}_{j_{p-1}} \cdots \widehat{e}^{*s_1}_{j_1}\widehat{u}_{-\infty}\big)\end{aligned}$$

for some $\widehat{b}''_1 \in \widehat{\mathcal{B}}(\infty)$. Here, since $\widehat{e}^{*s_p}_{j_p}\widehat{e}^{*s_{p-1}}_{j_{p-1}} \cdots \widehat{e}^{*s_1}_{j_1}\widehat{u}_{-\infty} = \big(\widehat{e}^{s_p}_{j_p}\widehat{e}^{s_{p-1}}_{j_{p-1}} \cdots \widehat{e}^{s_1}_{j_1}\widehat{u}_{-\infty}\big)^*$, it is contained in $\big(\widehat{\mathcal{B}}_{\widehat{w}}(-\infty)\big)^*$, and hence in $\widehat{\mathcal{B}}_{\widehat{w}^{-1}}(-\infty)$ by Lemma 3.4.3 (1). It follows from Lemma 3.4.3 (2) that

$$\widehat{f}^{\max}_{j_p}\widehat{f}^{\max}_{j_{p-1}} \cdots \widehat{f}^{\max}_{j_1}\underbrace{\widehat{e}^{*s_p}_{j_p}\widehat{e}^{*s_{p-1}}_{j_{p-1}} \cdots \widehat{e}^{*s_1}_{j_1}\widehat{u}_{-\infty}}_{\in \widehat{\mathcal{B}}_{\widehat{w}^{-1}}(-\infty)} = \widehat{u}_{-\infty},$$

since $\widehat{w}^{-1} = \widehat{r}_{j_1}\widehat{r}_{j_2} \cdots \widehat{r}_{j_p}$ is a reduced expression of $\widehat{w}^{-1}$. Therefore, from the above, we conclude that $\widehat{\Xi}_{\widehat{\Lambda}}(\widehat{S}_{\widehat{w}}\widehat{S}^*_{\widehat{w}}(\widehat{b})) = \widehat{b}''_1 \otimes \widehat{t}_{\widehat{\Lambda}} \otimes \widehat{u}_{-\infty}$. Also, since $\mathrm{wt}\big(\widehat{\Xi}_{\widehat{\Lambda}}(\widehat{S}_{\widehat{w}}\widehat{S}^*_{\widehat{w}}(\widehat{b}))\big) = \mathrm{wt}\big(\widehat{S}_{\widehat{w}}\widehat{S}^*_{\widehat{w}}(\widehat{b})\big) = \widehat{\Lambda} - k\widehat{\delta}$, it follows that $\mathrm{wt}(\widehat{b}''_1) = -k\widehat{\delta}$. This proves the claim. ▮

Now, we continue the proof of the lemma. We see again from Proposition 2.6.2 (2)–(4) that the equality $\#\widehat{\mathcal{B}}(\widehat{\Lambda})_{\mathrm{PIE},k} = \#\widehat{\mathbf{N}}(\widehat{\lambda})_k$ holds. Since $\widehat{S}_{\widehat{w}}\widehat{S}^*_{\widehat{w}}$ is bijective (in particular, injective), it follows from the claim above that the inequality $\#\widehat{\mathcal{B}}(\widehat{\lambda})_{\mathrm{PIE},k} \leq \#\widehat{\mathcal{B}}(\widehat{\Lambda})_{\mathrm{PIE},k} = \#\widehat{\mathbf{N}}(\widehat{\lambda})_k$ holds. Thus we have proved the lemma. □

4.5 *Relation Between* $\mathbf{N}^0(\lambda)$ *and* $\widehat{\mathbf{N}}(\widehat{\lambda})$

As in Sects. 4.3 and 4.4, we assume that $\lambda \in P$ is a level-zero integral weight of the form $\lambda = \sum_{i \in I_0} m_i\varpi_i$, with $m_i \in \mathbb{Z}_{\geq 0}$ for $i \in I_0$, such that $\omega^*(\lambda) = \lambda$, and set $\widehat{\lambda} = (P^*_\omega)^{-1}(\lambda) = \sum_{i \in \widehat{I}_0} m_i\widehat{\varpi}_i$.

Lemma 4.5.1. *Let $k \in \mathbb{Z}_{\geq 0}$. Then, there exists a bijection $\Psi_k : \mathbf{N}^0(\lambda)_k \xrightarrow{\sim} \widehat{\mathbf{N}}(\widehat{\lambda})_k$. In particular, the equation $\#\mathbf{N}^0(\lambda)_k = \#\widehat{\mathbf{N}}(\widehat{\lambda})_k$ holds.*

Proof. Let $(\rho^{(i)})_{i \in I_0} \in \mathbf{N}^0(\lambda)_k$. Recall from the definition (4.3.2) that $\ell(\rho^{(i)}) \leq m_i$ and $\rho^{(\omega(i))} = \rho^{(i)}$ for all $i \in I_0$, and $|(\rho^{(i)})_{i \in I_0}| = \sum_{i \in I_0} |\rho^{(i)}| = \widehat{a}_0 k$. Now we define $\Psi_k((\rho^{(i)})_{i\in I_0})$ by: $\Psi_k((\rho^{(i)})_{i\in I_0}) = (\rho^{(i)})_{i\in \widehat{I}_0}$; this definition makes sense since $\rho^{(\omega(i))} = \rho^{(i)}$ for all $i \in I_0$. Since $\ell(\rho^{(i)}) \le m_i$ for all $i \in I_0$, it follows that $\Psi_k((\rho^{(i)})_{i\in I_0}) = (\rho^{(i)})_{i\in \widehat{I}_0}$ is contained in $\widehat{\mathbf{N}}(\widehat{\lambda})$. Furthermore, we have

$$\begin{aligned}
|\Psi_k((\rho^{(i)})_{i\in I_0})| = |(\rho^{(i)})_{i\in \widehat{I}_0}| &= \sum_{i\in \widehat{I}_0} |\rho^{(i)}|\widehat{d}_i \quad \text{by the definitions} \\
&= \sum_{i\in \widehat{I}_0} |\rho^{(i)}| N_i \widehat{a}_0^{-1} \quad \text{by Remark 4.4.1} \\
&= \widehat{a}_0^{-1} \sum_{i\in \widehat{I}_0} \sum_{k=0}^{N_i-1} |\rho^{(\omega^k(i))}| \quad \text{since } \rho^{(\omega(i))} = \rho^{(i)} \text{ for every } i \in I_0 \\
&= \widehat{a}_0^{-1} \sum_{i\in I_0} |\rho^{(i)}| = k \quad \text{since } (\rho^{(i)})_{i\in I_0} \in \mathbf{N}^0(\lambda)_k,
\end{aligned}$$

which implies that $\Psi_k((\rho^{(i)})_{i\in I_0}) \in \widehat{\mathbf{N}}(\widehat{\lambda})_k$. Thus, we have obtained a map $\Psi_k : \mathbf{N}^0(\lambda)_k \to \widehat{\mathbf{N}}(\widehat{\lambda})_k$. The injectivity of the map Ψ_k is clear. Hence it remains to show the surjectivity of Ψ_k. Let $(\widehat{\rho}^{(i)})_{i\in \widehat{I}_0} \in \widehat{\mathbf{N}}(\widehat{\lambda})_k$. Define an I_0-tuple $(\rho^{(i)})_{i\in I_0}$ by: $\rho^{(\omega^k(i))} = \widehat{\rho}^{(i)}$ for $i \in \widehat{I}_0$ and $0 \le k \le N_i - 1$. Since $\ell(\rho^{(\omega^k(i))}) = \ell(\widehat{\rho}^{(i)}) \le m_i = m_{\omega^k(i)}$ for all $i \in \widehat{I}_0$ and $0 \le k \le N_i - 1$, it follows that $(\rho^{(i)})_{i\in I_0} \in \mathbf{N}(\lambda)$. It is obvious that $(\rho^{(i)})_{i\in I_0}$ is fixed by the diagram automorphism ω, i.e. $(\rho^{(i)})_{i\in I_0} \in \mathbf{N}^0(\lambda)$. Also, a computation similar to the one above yields that $|(\rho^{(i)})_{i\in I_0}| = \widehat{a}_0|(\widehat{\rho}^{(i)})_{i\in \widehat{I}_0}| = \widehat{a}_0 k$, i.e. $(\rho^{(i)})_{i\in I_0} \in \mathbf{N}^0(\lambda)_k$. The surjectivity of the map Ψ_k now follows since $\Psi_k((\rho^{(i)})_{i\in I_0}) = (\widehat{\rho}^{(i)})_{i\in \widehat{I}_0}$ by definition. This proves the lemma. □

4.6 Proof of the Main Result

Let the notation and assumptions be as in Sect. 4.2. In order to give a proof of Theorem 4.2.1, we need the following lemma.

Lemma 4.6.1. *Let $k \in \mathbb{Z}_{\ge 0}$. Then, we have $R_\lambda(\mathcal{B}^0(\lambda)_{\mathrm{PIE},k}) = \widehat{\mathcal{B}}(\widehat{\lambda})_{\mathrm{PIE},k}$.*

Proof. Let $b \in \mathcal{B}^0(\lambda)_{\mathrm{PIE},k}$. We see from [NS1, Theorem 3.3.3] that $R_\lambda(b)$ is a $\widehat{W}$-extremal element of $\widehat{\mathcal{B}}(\widehat{\lambda})$. Write $\Xi_\lambda(b)$ as $b_1 \otimes t_\lambda \otimes u_{-\infty} \in \mathcal{B}^0(\infty) \otimes \mathcal{T}_\lambda \otimes \mathcal{B}^0(-\infty)$ for some $b_1 \in \mathcal{B}^0(\infty)$ such that $\mathrm{wt}(b_1) = -\widehat{a}_0 k\delta$. Then we see that

$$\begin{aligned}
\widehat{\Xi}_{\widehat{\lambda}}(R_\lambda(b)) = \widehat{\Xi}_{\widehat{\lambda}}(Q_\lambda(b)) &= (P_\infty \otimes Q^\lambda \otimes P_{-\infty})(\Xi_\lambda(b)) \quad \text{by Proposition 3.2.4} \\
&= P_\infty(b_1) \otimes \widehat{t}_{\widehat{\lambda}} \otimes \widehat{u}_{-\infty} \quad \text{by Remark 3.2.3.}
\end{aligned}$$

Also, since $\mathrm{wt}(b_1) = -\widehat{a}_0 k\delta$, we have

$$\begin{aligned}\mathrm{wt}(P_\infty(b_1)) &= (P_\omega^*)^{-1}(\mathrm{wt}(b_1)) \qquad \text{by (3.2.7)}\\ &= -k\widehat{\delta} \qquad \text{by (4.1.2).}\end{aligned}$$

Thus we obtain $R_\lambda(b) \in \widehat{\mathcal{B}}(\widehat{\lambda})_{\mathrm{PIE},k}$, which implies that

$$R_\lambda(\mathcal{B}^0(\lambda)_{\mathrm{PIE},k}) \subset \widehat{\mathcal{B}}(\widehat{\lambda})_{\mathrm{PIE},k}. \tag{4.6.1}$$

Here we note that $\#R_\lambda(\mathcal{B}^0(\lambda)_{\mathrm{PIE},k}) = \#\mathcal{B}^0(\lambda)_{\mathrm{PIE},k}$ since R_λ is an injection. Therefore, we have

$$\begin{aligned}\#R_\lambda(\mathcal{B}^0(\lambda)_{\mathrm{PIE},k}) &= \#\mathcal{B}^0(\lambda)_{\mathrm{PIE},k}\\ &\geq \#\mathbf{N}^0(\lambda)_k \quad \text{by (4.3.4)}\\ &= \#\widehat{\mathbf{N}}(\widehat{\lambda})_k \quad \text{by Lemma 4.5.1}\\ &\geq \#\widehat{\mathcal{B}}(\widehat{\lambda})_{\mathrm{PIE},k} \quad \text{by Lemma 4.4.2.}\end{aligned} \tag{4.6.2}$$

Hence, by comparing the cardinalities, we conclude from the inclusion (4.6.1) that

$$R_\lambda(\mathcal{B}^0(\lambda)_{\mathrm{PIE},k}) = \widehat{\mathcal{B}}(\widehat{\lambda})_{\mathrm{PIE},k}.$$

This proves the lemma. □

Finally, we are in a position to complete the proof of Theorem 4.2.1. Take $\widehat{b} \in \widehat{\mathcal{B}}(\widehat{\lambda})$ arbitrarily, and define $b \in \mathcal{B}^0(U_q(\mathfrak{g})a_\lambda)$ to be the inverse image $Q_\lambda^{-1}(\widehat{b})$ of $\widehat{b}$ under the bijection $Q_\lambda : \mathcal{B}^0(U_q(\mathfrak{g})a_\lambda) \xrightarrow{\sim} \widehat{\mathcal{B}}(U_q(\widehat{\mathfrak{g}})\widehat{a}_{\widehat{\lambda}})$ of Proposition 3.2.1 (recall that $\widehat{\mathcal{B}}(\widehat{\lambda})$ is a subcrystal of $\widehat{\mathcal{B}}(U_q(\widehat{\mathfrak{g}})\widehat{a}_{\widehat{\lambda}})$). To prove that the map $R_\lambda : \mathcal{B}^0(\lambda) \hookrightarrow \widehat{\mathcal{B}}(\widehat{\lambda})$ is surjective, it suffices to show that this $b \in \mathcal{B}^0(U_q(\mathfrak{g})a_\lambda)$ is contained in $\mathcal{B}^0(\lambda)$; indeed, if $b \in \mathcal{B}^0(\lambda)$, then it follows from the definition of the map R_λ that $R_\lambda(b) = R_\lambda(Q_\lambda^{-1}(\widehat{b})) = Q_\lambda(Q_\lambda^{-1}(\widehat{b})) = \widehat{b}$. Also, since $\mathcal{B}^0(\lambda) = \mathcal{B}(\lambda) \cap \mathcal{B}^0(U_q(\mathfrak{g})a_\lambda)$, and $b \in \mathcal{B}^0(U_q(\mathfrak{g})a_\lambda)$ by definition, it is obvious that $b \in \mathcal{B}^0(\lambda)$ if and only if $b \in \mathcal{B}(\lambda)$.

Claim. *There exist a monomial $\widehat{X}_1$ in the Kashiwara operators $\widehat{e}_j, \widehat{f}_j$ for $j \in \widehat{I}$ and an element $\widehat{b}_1 \in \widehat{\mathcal{B}}(\widehat{\lambda})_{\mathrm{PIE},k}$ for some $k \in \mathbb{Z}_{\geq 0}$ such that $\widehat{b} = \widehat{X}_1\widehat{b}_1$.*

Proof of the claim. In Cases (a)–(d), the assertion follows from Proposition 2.6.2 (3) and (5). Hence it remains to consider Case (e). As in Sect. 4.4, let $\widehat{\lambda} = (P_\omega^*)^{-1}(\lambda)$ be of the form $\widehat{\lambda} = \sum_{i\in\widehat{I}_0} m_i\widehat{\varpi}_i$, and $\widehat{\Lambda} = \sum_{i\in\widehat{I}_0} m_i\widehat{\eta}_{n-i}$. We take the unique shortest element $\widehat{w} \in \widehat{W}_0$ such that $\widehat{w}\widehat{\lambda} = \widehat{\Lambda}$. Since $\widehat{S}^*_{\widehat{w}}(\widehat{b}) \in \widehat{\mathcal{B}}(\widehat{\Lambda})$ by Theorem 2.4.2 (5), it follows from Proposition 2.6.2 (3) and (5) that there exist a monomial $\widehat{X}_2$ in the

Kashiwara operators $\widehat{e}_j$, $\widehat{f}_j$ for $j \in \widehat{I}$ and an element $\widehat{b}_2 \in \widehat{\mathcal{B}}(\widehat{\Lambda})_{\mathrm{PIE},k}$ for some $k \in \mathbb{Z}_{\geq 0}$ such that $\widehat{S}^*_{\widehat{w}}(\widehat{b}) = \widehat{X}_2\widehat{b}_2$. Also, we have

$$(\widehat{S}_{\widehat{w}}\widehat{S}^*_{\widehat{w}})^{-1}\widehat{S}^*_{\widehat{w}}(\widehat{b}) = \widehat{S}^*_{\widehat{w}^{-1}}\widehat{S}_{\widehat{w}^{-1}}\widehat{S}^*_{\widehat{w}}(\widehat{b}) = \widehat{S}_{\widehat{w}^{-1}}\widehat{S}^*_{\widehat{w}^{-1}}\widehat{S}^*_{\widehat{w}}(\widehat{b}),$$

where the second equality follows from Remark 2.3.2 (2) since $\widehat{S}^*_{\widehat{w}^{-1}} : \widehat{\mathcal{B}}(\widehat{\Lambda}) \to \widehat{\mathcal{B}}(\widehat{\lambda})$ is an isomorphism of crystals for $U_q(\widehat{\mathfrak{g}})$ (see Theorem 2.4.2 (5)). Thus we obtain

$$(\widehat{S}_{\widehat{w}}\widehat{S}^*_{\widehat{w}})^{-1}\widehat{S}^*_{\widehat{w}}(\widehat{b}) = \widehat{S}_{\widehat{w}^{-1}}\widehat{S}^*_{\widehat{w}^{-1}}\widehat{S}^*_{\widehat{w}}(\widehat{b}) = \widehat{S}_{\widehat{w}^{-1}}(\widehat{b}).$$

Similarly, we have

$$\begin{aligned}(\widehat{S}_{\widehat{w}}\widehat{S}^*_{\widehat{w}})^{-1}(\widehat{X}_2\widehat{b}_2) &= \widehat{S}^*_{\widehat{w}^{-1}}\widehat{S}_{\widehat{w}^{-1}}(\widehat{X}_2\widehat{b}_2) \\ &= \widehat{S}_{\widehat{w}^{-1}}\big(\widehat{X}_2\widehat{S}^*_{\widehat{w}^{-1}}(\widehat{b}_2)\big) \quad \text{by Theorem 2.4.2 (5) and Remark 2.3.2 (2).}\end{aligned}$$

Therefore, we deduce that $\widehat{S}_{\widehat{w}^{-1}}(\widehat{b}) = \widehat{S}_{\widehat{w}^{-1}}\big(\widehat{X}_2\widehat{S}^*_{\widehat{w}^{-1}}(\widehat{b}_2)\big)$, and hence $\widehat{b} = \widehat{X}_2\widehat{S}^*_{\widehat{w}^{-1}}(\widehat{b}_2)$. Furthermore, again by Theorem 2.4.2 (5) and Remark 2.3.2 (2), we see that

$$\widehat{S}^*_{\widehat{w}^{-1}}(\widehat{b}_2) = \widehat{S}^*_{\widehat{w}^{-1}}\widehat{S}_{\widehat{w}}\widehat{S}_{\widehat{w}^{-1}}(\widehat{b}_2) = \widehat{S}_{\widehat{w}}\widehat{S}^*_{\widehat{w}^{-1}}\widehat{S}_{\widehat{w}^{-1}}(\widehat{b}_2) = \widehat{S}_{\widehat{w}}(\widehat{S}_{\widehat{w}}\widehat{S}^*_{\widehat{w}})^{-1}(\widehat{b}_2).$$

Hence, if we set $\widehat{b}_1 := (\widehat{S}_{\widehat{w}}\widehat{S}^*_{\widehat{w}})^{-1}(\widehat{b}_2)$, then we get $\widehat{b} = \widehat{X}_2\widehat{S}_{\widehat{w}}(\widehat{b}_1)$. Since $\widehat{S}_{\widehat{w}}(\widehat{b}_1) = \widehat{X}_3\widehat{b}_1$ for some monomial $\widehat{X}_3$ in the Kashiwara operators $\widehat{e}_j$, $\widehat{f}_j$ for $j \in \widehat{I}$ by the definition of $\widehat{S}_{\widehat{w}}$, it suffices to show that $\widehat{b}_1 \in \widehat{\mathcal{B}}(\widehat{\lambda})_{\mathrm{PIE},k}$. Recall from the proof of Lemma 4.4.2 that

$$\widehat{S}_{\widehat{w}}\widehat{S}^*_{\widehat{w}}\big(\widehat{\mathcal{B}}(\widehat{\lambda})_{\mathrm{PIE},k}\big) \subset \widehat{\mathcal{B}}(\widehat{\Lambda})_{\mathrm{PIE},k} \quad \text{and} \quad \#\widehat{\mathcal{B}}(\widehat{\lambda})_{\mathrm{PIE},k} \leq \#\widehat{\mathcal{B}}(\widehat{\Lambda})_{\mathrm{PIE},k} = \#\widehat{\mathbf{N}}(\widehat{\lambda})_k.$$

Also, we see from (4.6.1) and (4.6.2) that $\#\widehat{\mathcal{B}}(\widehat{\lambda})_{\mathrm{PIE},k} = \#\widehat{\mathbf{N}}(\widehat{\lambda})_k$. Consequently, we deduce that $\widehat{S}_{\widehat{w}}\widehat{S}^*_{\widehat{w}}\big(\widehat{\mathcal{B}}(\widehat{\lambda})_{\mathrm{PIE},k}\big) = \widehat{\mathcal{B}}(\widehat{\Lambda})_{\mathrm{PIE},k}$, since $\widehat{S}_{\widehat{w}}\widehat{S}^*_{\widehat{w}}$ is bijective (in particular, injective). Since $\widehat{b}_2 \in \widehat{\mathcal{B}}(\widehat{\Lambda})_{\mathrm{PIE},k}$, this implies that $\widehat{b}_1 \in \widehat{\mathcal{B}}(\widehat{\lambda})_{\mathrm{PIE},k}$, as desired. Thus we have proved the claim. ∎

Now, we continue the proof of Theorem 4.2.1. Write the monomial $\widehat{X}_1$ in the claim above as $\widehat{X}_1 = \widehat{x}_{j_1}\widehat{x}_{j_2}\cdots\widehat{x}_{j_l}$ for some $l \in \mathbb{Z}_{\geq 0}$, where $\widehat{x}_j$ is either $\widehat{e}_j$ or $\widehat{f}_j$ for each $j \in \widehat{I}$. We know from Lemma 4.6.1 that there exists $b_1 \in \mathcal{B}^0(\lambda)_{\mathrm{PIE},k}$ such that $R_\lambda(b_1) = \widehat{b}_1$; note that $b_1 = Q_\lambda^{-1}(\widehat{b}_1)$ by the definition of R_λ. Therefore, it follows from Proposition 3.2.1 that

$$\begin{aligned}b = Q_\lambda^{-1}(\widehat{b}) = Q_\lambda^{-1}(\widehat{x}_{j_1}\widehat{x}_{j_2}\cdots\widehat{x}_{j_l}\widehat{b}_1) &= \widetilde{x}_{j_1}\widetilde{x}_{j_2}\cdots\widetilde{x}_{j_l}Q_\lambda^{-1}(\widehat{b}_1) \\ &= \widetilde{x}_{j_1}\widetilde{x}_{j_2}\cdots\widetilde{x}_{j_l}b_1.\end{aligned} \tag{4.6.3}$$

Because $b_1 \in \mathcal{B}^0(\lambda)_{\mathrm{PIE},k} \subset \mathcal{B}(\lambda)$, and $\mathcal{B}(\lambda)$ is a subcrystal of $\mathcal{B}(U_q(\mathfrak{g})a_\lambda)$, we conclude from (4.6.3) that $b \in \mathcal{B}(\lambda)$, as desired. This finishes the proof of Theorem 4.2.1.

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t-Analogs of q-Characters of Quantum Affine Algebras of Type E_6, E_7, E_8

Hiraku Nakajima

Abstract We compute t-analogs of q-characters of all l-fundamental representations of the quantum affine algebras of type $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$ by a supercomputer. (Here l- stands for the loop.) In particular, we prove the fermionic formula for Kirillov–Reshetikhin modules conjectured by Hatayama et al. [Remarks on fermionic formula (1999)] for these classes of representations. We also give explicitly the monomial realization of the crystal of the corresponding fundamental representations of the quantum enveloping algebras associated with finite dimensional Lie algebras of types E_6, E_7, E_8. These are computations of Betti numbers of graded quiver varieties, quiver varieties and determination of all irreducible components of the lagrangian subvarieties of quiver varieties of types E_6, E_7, E_8, respectively.

Keywords Quantum affine algebras · Fundamental representations · q-character · Supercomputer

Mathematics Subject Classifications (2000): Primary 17B37; Secondary 14D21, 14L30, 16G20

1 Introduction

Let $\mathfrak{g}$ be a simple Lie algebra of type *ADE* over $\mathbb{C}$ with the index set I of simple roots, $\mathbf{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ be its loop algebra, and $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ be its quantum universal enveloping algebra, or the quantum loop algebra for short. It is a subquotient of the quantum affine algebra $\mathbf{U}_q(\widehat{\mathfrak{g}})$, i.e., without central extension and degree operator. It contains the quantum enveloping algebra $\mathbf{U}_q(\mathfrak{g})$ associated with $\mathfrak{g}$ as a subalgebra. In this chapter, we understand q as a nonzero complex number, which we assume not to be a root of unity, for simplicity. And the algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ should be understood as the specialization of an integral form of the quantum loop algebra. In [17],

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the specialization was denoted by $\mathbf{U}_\varepsilon(\mathbf{L}\mathfrak{g})$, but we use $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ for a brevity in this chapter. (See [17] for the definition of the specialization.)

By Drinfeld [3] and Chari-Pressley [2], simple $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$-modules are parametrized by I-tuples of polynomials $P = (P_i(u))_{i\in I}$ with normalization $P_i(0) = 1$. They are called *Drinfeld polynomials*. Let us denote by $L(P)$ the simple module with Drinfeld polynomial P. When P is given by $P_i(u) = (1-au)^{\delta_{iN}}$ for a given $N \in I$, we call corresponding module an Nth *l-fundamental representation*. (It has been called a *level* 0 *fundamental module* or simply *fundamental representation* in some literature.) We can assume $a = 1$ without the loss of generality as the general module is a pullback of the module with $a = 1$ by an algebra automorphism of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$.

Let $\chi_{q,t}(L(P))$ be the t-analog of q-character of a simple module $L(P)$ defined by the author [14, 17]. It is defined via the geometry of graded quiver varieties. It values in certain Laurent polynomial ring with infinitely many variables with integer coefficients. It is a t-analog of the q-character $\chi_q(L(P))$ introduced earlier [5, 12], which was a refinement of the ordinary character of the restriction of $L(P)$ to a $\mathbf{U}_q(\mathfrak{g})$-module. In [14, 17] we "computed" $\chi_{q,t}(L(P))$ for arbitrary given $L(P)$, in the sense that we gave a purely combinatorial algorithm to write down all monomials and coefficients in $\chi_{q,t}(L(P))$, where the final expression involves only $+$, $\times$, integers and variables.

In order to clarify in what sense our result is new compared with earlier results, we *define* what the word compute mean precisely. When we write the word compute in the quotation marks, it means that we give a combinatorial algorithm to compute something in the above sense. It does not necessarily mean that we actually compute it. We can write a computer program in principle, but the question whether we can actually compute it or not depends on the size of computer memory. (For example, it is clear that the rank n of $\mathfrak{g}$ cannot be larger than the size of the memory.) On the other hand, when we write the word compute without the quotation mark, we mean to compute something in a strict sense, i.e., we express something so that it contains only finitely many $\pm$, $\times$, integers and variables. For example, if we write $x = \sum_{i=1}^{2^{(2^{100})}} a_i$ for some explicit a_i, we "compute" x, but we do not compute x unless we actually compute the sum. On the other hand, we do not require that the final expression can be read by the human, as such a concept cannot make precise.

The algorithm is separated into three steps:

(1) "Computation" of $\chi_{q,t}$ for l-fundamental representations.
(2) "Computation" of $\chi_{q,t}$ for standard modules, i.e., tensor products of l-fundamental representations.
(3) "Computation" of the t-analog of the composition factors of simple modules in standard modules.

The third step is analogous to the definition of Kazhdan–Lusztig basis. If $M(P)$ denote the standard module, we have

$$\overline{\chi_{q,t}(L(P))} = \chi_{q,t}(L(P)), \quad \chi_{q,t}(L(P)) = \chi_{q,t}(M(P)) + \sum_{Q:Q<P} a_{PQ}(t)\chi_{q,t}(M(Q)) \tag{1.1}$$

for some $a_{PQ}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$, where "$<$" is a certain explicitly defined ordering. Thus $a_{PQ}(t)$ is analogous to Kazhdan–Lusztig polynomials. The above characterization allows us to "compute" $a_{PQ}(t)$, once $\chi_{q,t}(M(P))$ is "computed." (And it is known that the actual computation of Kazhdan–Lusztig polynomials is very hard.)

In the second step, we express $\chi_{q,t}(M(P))$ as a twisted multiplication of $\chi_{q,t}$ of l-fundamental representations. It is almost the same as usual multiplication on the polynomials, but a product of two monomials m, m' is twisted as $t^{2d(m,m')}mm'$. Therefore, this step is very simple. It is clear that $\chi_{q,t}(M(P))$ can be "computed" if $\chi_{q,t}$ of l-fundamental representations are "computed".

This chapter concerns the first step. Our "computation" in [14, 17] was t-analog of the "computation" by Frenkel–Mukhin [4]. It is based on the observation that (a) $\chi_{q,t}$ satisfies a certain analog of the Weyl group invariance of the ordinary characters, and (b) the l-fundamental representation satisfies a certain property analogous to that of minuscule representations of $\mathfrak{g}$. Recall that a simple finite dimensional representation of $\mathfrak{g}$ is called *minuscule* if all weights are conjugates of the highest weight under the Weyl group each occurring with multiplicity 1.

When $\mathfrak{g}$ is of classical type, i.e., of type A, D, the author gave a tableaux sum expression of $\chi_{q,t}$ of l-fundamental representations [16]. It means that we give another "computation" of $\chi_{q,t}$, which are more familiar to us than the above one. It does not mean we compute $\chi_{q,t}$ in our strict sense. In fact, the comparison of two methods does not make sense unless we define what we mean by "familiar." In practice, it just means that we have a faster algorithm for the actual computer calculation.

In this chapter we report the actual computer computation of $\chi_{q,t}$ of l-fundamental representations when $\mathfrak{g}$ is of type $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$. Our algorithm is implemented in the computer language **C**. The source code is available at `http://www.math.kyoto-u.ac.jp/~nakajima/Qchar/`. The author's personal computer (Dell Dimension 9100) can give the answer up to the sixth l-fundamental representation of E_8, where our numbering of I is the following:

$$\begin{array}{ccccccccccccc} 7 & - & 6 & - & 5 & - & 4 & - & 3 & - & 2 & - & 1 \\ & & & & | & & & & & & & & \\ & & & & 8 & & & & & & & & \end{array}.$$

We need about 120 Mbtyes of the memory for this calculation. For the fourth and fifth l-fundamental representations, the computation was done on a supercomputer FUJITSU HPC 2500 at Kyoto University. The calculation required about 2.6 Gbytes (for fourth) and 120 Gbytes (for fifth) of memory, and it took 6 and 350 h for the calculation, respectively. The final answers (stored in a compressed format as explained below) are 3.2 and 180 Gbytes, respectively. In fact, the calculation of the fourth one was done several years ago and was mentioned in some of the author's papers. However, we needed to wait for the Kyoto University to renovate the supercomputer so that we can use 120 Gbytes of memory in a single program, and then wait for the author to get an enough budget to use the supercomputer.

As far as the author knows, the computation (in our strict sense) for the fifth one was not known before. Frenkel–Mukhin, Hernandez–Schedler told the author that they wrote computer programs calculating $\chi_{q,t=1}$ and $\chi_{q,t}$, respectively. But both had a problem of computer memory.

In conclusion, we can now delete the quotation mark for computation in the first step of the algorithm for type E above.

As an application, we can compute t-analog of the ordinary characters of the restrictions of l-fundamental representations to $\mathbf{U}_q(\mathfrak{g})$-modules. The l-fundamental modules are examples of the so-called Kirillov–Reshetikhin modules. Kirillov–Reshetikhin gave conjectural formula for the ordinary character of the restriction of a Kirillov–Reshetikhin module [10]. Its graded version (i.e., t-analog) together with an interpretation in terms of the conjectural crystal base was given by Hatayama, Kuniba, Okado, Takagi and Yamada [6]. Then Lusztig conjectured that their conjectural grading is the same as the cohomological degree [13], in a certain class of Kirillov–Reshetikhin modules including l-fundamental representations. Therefore, the formula in [6], in the class, gives the generating function of Poincaré polynomials of quiver varieties.

In general, the conjectural formula is expressed as a summation over partitions and called a *fermionic formula*. The author gave an expression for $t = 1$ in [15, Corollary 1.3] (the result was extended to type *BCFG* in [7]). It is again given as a summation over partition, but the definition of the binomial coefficient appearing in the coefficients is different. The equivalence between two expressions is not known so far[1] therefore, the original fermionic formula is remained open.

For an l-fundamental representation, the original fermionic formula can be given by an explicit polynomial by the so-called Kleber's algorithm [11]. Here we do not make precise what we mean by "explicit." For types A, D, it was shown in [16] that this 'explicit' expression for an l-fundamental representation is equal to the "computation" in [14]. For type E, the algorithm can be used to compute the fermionic formula in our strict sense. Then the result can be checked in some special cases previously computed (at least for $t = 1$) (e.g., [1]), but most of all l-fundamental representations have remained open. Remark that Kleber's algorithm does not apply to the modified formula in [15]; so it is not known that the modified formula gives the computation in the strict sense.

Our computation of $\chi_{q,t}$ gives the explicit expression and we find that it is the same as one given in [6]. Therefore, we prove Lusztig's conjecture for all l-fundamental representations.

[1] The equivalence between two expressions is proved by P. Di Francesco and R. Kedem, *Proof of the combinatorial Kirillov–Reshetikhin conjecture*, Internat. Math. Res. Notices **7** (2008), Art. ID rnn006, 57 pp.

Also as another application, we determine all monomials appearing in the monomial realization of the crystal corresponding to fundamental representations of type E. For types A, D, they were determined in [16] as an application of the explicit description of $\chi_{q,t}$ of l-fundamental representations. For types B, C, they were determined in [9]. For types F, G, they can be easily determined (cf. [8]). In conclusion, we describe the monomial realization of the crystals of all fundamental representations explicitly.

2 t-Analogs of q-Characters

We shall not give the definition of quantum loop algebras, nor their finite dimensional representations in this chapter. (See [14] for a survey.) We just review properties of $\chi_{q,t}$, as axiomatized in [17].

Let $\mathscr{Y}_t \stackrel{\text{def.}}{=} \mathbb{Z}[t, t^{-1}, Y_{i,a}, Y_{i,a}^{-1}]_{i\in I, a\in\mathbb{C}^*}$ be a Laurent polynomial ring of uncountably many variables $Y_{i,a}$'s with coefficients in $\mathbb{Z}[t, t^{-1}]$. A *monomial* in $\mathscr{Y}_t$ means a monomial only in $Y_{i,a}^{\pm}$, containing no t's. Therefore, a polynomial is a sum of monomials multiplied by Laurent polynomials in t, called coefficients as usual. Let

$$A_{i,a} \stackrel{\text{def.}}{=} Y_{i,aq} Y_{i,aq^{-1}} \prod_{j:j\neq i} Y_{j,a}^{c_{ij}},$$

where c_{ij} is the (i, j)-entry of the Cartan matrix. Recall that q is a nonzero complex number, at which we make a specialization of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$. Let $\mathcal{M}$ be the set of monomials in $\mathscr{Y}_t$.

Definition 2.1. (1) For a monomial $m \in \mathcal{M}$, we define $u_{i,a}(m) \in \mathbb{Z}$ be the degree in $Y_{i,a}$, i.e.,

$$m = \prod_{i,a} Y_{i,a}^{u_{i,a}(m)}.$$

(2) A monomial $m \in \mathcal{M}$ is said i*-dominant* if $u_{i,a}(m) \geq 0$ for all a. It is said *l-dominant* if it is i-dominant for all i.

(3) Let m, m' be monomials in $\mathcal{M}$. We say $m \leq m'$ if m/m' is a monomial in $A_{i,a}^{-1}$ ($i \in I$, $a \in \mathbb{C}^*$). Here a monomial in $A_{i,a}^{-1}$ means a product of nonnegative powers of $A_{i,a}^{-1}$. It does not contain any factors $A_{i,a}$. In such a case, we define $v_{i,a}(m, m') \in \mathbb{Z}_{\geq 0}$ by

$$m = m' \prod_{i,a} A_{i,a}^{-v_{i,a}(m,m')}.$$

This is well-defined since the q-analog of the Cartan matrix is invertible. We say $m < m'$ if $m \leq m'$ and $m \neq m'$.

(4) For an i-dominant monomial $m \in \mathcal{M}$, we define

$$E_i(m) \stackrel{\text{def.}}{=} m \prod_a \sum_{r_a=0}^{u_{i,a}(m)} t^{r_a(u_{i,a}(m)-r_a)} \begin{bmatrix} u_{i,a}(m) \\ r_a \end{bmatrix}_t A_{i,aq}^{-r_a},$$

where $\left[{n \atop r}\right]_t$ is the t-binomial coefficient.

(5) We define a ring involution $\overline{}$ on $\mathscr{Y}_t$ by $\bar{t} = t^{-1}$, $\overline{Y_{i,a}^{\pm}} = Y_{i,a}^{\pm}$.

Suppose that l-dominant monomials m_{P^1}, m_{P^2} and monomials $m^1 \le m_{P^1}$, $m^2 \le m_{P^2}$ are given. We define an integer $d(m^1, m_{P^1}; m^2, m_{P^2})$ by

$$\begin{aligned} &d(m^1, m_{P^1}; m^2, m_{P^2}) \\ &\qquad \stackrel{\text{def.}}{=} \sum_{i,a} \left(v_{i,aq}(m^1, m_{P^1}) u_{i,a}(m^2) + u_{i,aq}(m_{P^1}) v_{i,a}(m^2, m_{P^2}) \right). \end{aligned} \tag{2.2}$$

For an I-tuple of rational functions $Q/R = (Q_i(u)/R_i(u))_{i \in I}$ with $Q_i(0) = R_i(0) = 1$, we set

$$m_{Q/R} \stackrel{\text{def.}}{=} \prod_{i \in I} \prod_{\alpha} \prod_{\beta} Y_{i,\alpha} Y_{i,\beta}^{-1},$$

where α (resp. β) runs roots of $Q_i(1/u) = 0$ (resp. $R_i(1/u) = 0$), i.e., $Q_i(u) = \prod_\alpha (1 - \alpha u)$ (resp. $R_i(u) = \prod_\beta (1 - \beta u)$). As a special case, an I-tuple of polynomials $P = (P_i(u))_{i \in I}$ defines $m_P = m_{P/1}$. The l-dominant monomial m_{P^α} appeared above is associated with an I-tuple of polynomials $P = (P_i(u))_{i \in I}$. In this way, the set $\mathcal{M}$ of monomials are identified with the set of I-tuple of rational functions, and the set of l-dominant monomials are identified with the set of I-tuple of polynomials.

The t-analog of the Grothendieck ring $\mathbf{R}_t$ is a free $\mathbb{Z}[t, t^{-1}]$-module with base $\{M(P)\}$ where $P = (P_i(u))_{i \in I}$ is the Drinfeld polynomial. (We do not recall the definition of standard modules $M(P)$ here, but the reader safely consider them as formal variables.)

The t-analog of the q-character homomorphism is a $\mathbb{Z}[t, t^{-1}]$-linear homomorphism $\chi_{q,t} \colon \mathbf{R}_t \to \mathscr{Y}_t$. It is defined as the generating function of Poincaré polynomials of graded quiver varieties, or the generating function of graded dimensions of l-weight spaces of a $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$-module [18], and will not be reviewed in this chapter.

We also need a slightly modified version:

$$\widetilde{\chi_{q,t}}(M(P)) = \sum_m t^{d(m, m_P; m, m_P)} a_m(t) m \qquad \text{if } \chi_{q,t}(M(P)) = \sum_m a_m(t) m.$$

If we know one of $\chi_{q,t}$ and $\widetilde{\chi_{q,t}}$, we know the remaining one.

The following was proved in [14, 17]:

Fact 2.3. (1) *The $\chi_{q,t}$ of a standard module $M(P)$ has a form*

$$\chi_{q,t}(M(P)) = m_P + \sum a_m(t) m,$$

where the summation runs over monomials $m < m_P$.

(2) *For each $i \in I$, $\widetilde{\chi_{q,t}}(M(P))$ can be expressed as a linear combination (over $\mathbb{Z}[t, t^{-1}]$) of $E_i(m)$ with i-dominant monomials m.*

(3) *Suppose that two I-tuples of polynomials $P^1 = (P_i^1)$, $P^2 = (P_i^2)$ satisfy the following condition:*

$$\begin{aligned}&a/b \notin \{q^n \mid n \in \mathbb{Z}, n \geq 2\} \text{ for any pair } a, b \text{ with } P_i^1(1/a) = 0,\\ &P_j^2(1/b) = 0\ (i, j \in I).\end{aligned} \tag{2.4}$$

Then we have

$$\widetilde{\chi_{q,t}}(M(P^1P^2)) = \sum_{m^1,m^2} t^{2d(m^1,m_{P^1};m^2,m_{P^2})} a_{m^1}(t) a_{m^2}(t) m^1 m^2,$$

where $\widetilde{\chi_{q,t}}(M(P^a)) = \sum_{m^a} a_{m^a}(t) m^a$ with $a = 1, 2$.
Moreover, properties (1),(2),(3) *uniquely determine $\chi_{q,t}(M(P))$.*

(4) *The $\chi_{q,t}$ of the simple module $L(P)$ is given by* (1.1).

Apart from the existence problem, one can consider the above properties (1)–(3) as the definition of $\chi_{q,t}$ (an axiomatic definition). We only use the above properties, and the reader can safely forget the original definition. Note that we will prove the existence of $\chi_{q,t}$ by our computer calculation.

By the property (1), we call the monomial m_P corresponding to the Drinfeld polynomial P *l-highest weight monomial.*

3 Algorithm

In this section, we shall explain our algorithm to determine $\widetilde{\chi_{q,t}}(L(P))$ recursively starting from the l-dominant weight monomial m_P. It is a slight modification of one in [4]. We shall also explain why we require large memory to compute $\chi_{q,t}$ of the fifth l-fundamental representation of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ with $\mathfrak{g} = E_8$. The problem does not exist for the other l-fundamental representations.

We take a Drinfeld polynomial $P = (P_i(u))$ $P_i(u) = (1-u)^{\delta_{iN}}$ corresponding to the Nth l-fundamental representation.

One of the key property of $\chi_{q,t}$ of an l-fundamental representation is that all monomials appearing in $\chi_{q,t}$ are not l-dominant except the l-highest one. This was proved in [4, Corollary 4.5] and [17, 4.13].

For each monomial m in $\widetilde{\chi_{q,t}}(L(P))$, we determine the coefficient $a_m(t) \in \mathbb{Z}[t]$ and the I-tuple of polynomial $(a_{m,i}(t))_{i\in I} \in \mathbb{Z}[t]$ (called *colouring*) recursively. Let us introduce several concepts. We say m is *admissible* if all $a_{m,i}(t)$ are the same for any i such that m is not i-dominant. We say *the algorithm fails at m* if m is not admissible. We say *the algorithm stops at m* if m is l-dominant.

Now we explain the algorithm. At the first stage, we set $a_{m_P}(t) = 1$ and $a_{m_P,i}(t) = 0$ for all $i \in I$ for the l-highest weight monomial m_P. Next, take a monomial m such that $a_m(t)$ and $a_{m,i}(t)$ are determined. If m is not i-dominant for any i (this will happen if m the l-lowest weight vector), we do nothing on m and go to the next monomial. If m is i-dominant, we compute $(a_m(t) - a_{m,i}(t))E_i(m)$.

We call this procedure the *i-expansion at m*. We add a monomial m' appearing there to the list. And for a monomial m' in the list, we set $a_{m',i}(t)$ be the sum of the contribution to m' in the i-expansion at m for various $m < m'$ which is i-dominant. As there is only finitely many $m < m'$, $a_{m',i}(t)$ will be eventually determined. After all $a_{m',i}(t)$ are determined in this way, we can ask m' is admissible or not. If m' is not admissible (i.e., the algorithm fails at m'), we stop. If m' is l-dominant (i.e., the algorithm stop at m'), we stop. If m' is admissible and not l-dominant, we set $a_{m'}(t) = a_{m',i}(t)$ for some (and any by admissibility) i such that m' is not i-dominant. We continue this procedure until all $a_m(t)$ and $a_{m,i}(t)$ are determined, and all $(a_m(t) - a_{m,i}(t))E_i(m)$ are expanded, or we stop at some m.

Now we apply the algorithm starting from the l-highest weight monomial m_P. As $\widetilde{\chi_{q,t}}(L(P))$ satisfies the properties (1) and (2) in Fact 2.3, the algorithm cannot fail. As $\widetilde{\chi_{q,t}}(L(P))$ does not contain l-dominant monomials other than l-highest one, the algorithm cannot stop. Finally as $L(P)$ is a finite dimensional, $\widetilde{\chi_{q,t}}(L(P))$ contains only finitely many monomials. Therefore, we eventually determine all $a_m(t)$ and $a_{m,i}(t)$.

Remark 3.1. If we apply the same algorithm in case $\mathfrak{g}$ is a Kac–Moody Lie algebra (say an affine Lie algebra), the algorithm does not fail, does not stop, but we always get a new monomial in the expansion. Therefore, the procedure never end.

Now we consider the fifth l-fundamental representation of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ with $\mathfrak{g} = E_8$, and we will explain the reason why we need various tricks to save the size of data. Because of these tricks, we had not known how big the total size is in advance; so we used the following guess: We know that the dimension of the fourth fundamental representation of $\mathfrak{g}$ is 146325270, while fifth one is 6899079264. Therefore, we expect that the corresponding $\chi_{q,t}$'s have a similar ratio. We first compute the fourth l-fundamental representations and expect that the total size of the fifth one is about 50 times as much. This turned out to be approximately correct as we can see from the data in Introduction.

By [16, Proposition 3.4], the set of monomials appearing in the q-character of an l-fundamental representation has a $\mathbf{U}_q(\mathfrak{g})$-crystal structure, which is isomorphic to the corresponding fundamental representation of $\mathbf{U}_q(\mathfrak{g})$. In particular, the number of the monomials appearing in the fifth l-fundamental representation is equal to the dimension of the fifth fundamental representation of $\mathfrak{g} = E_8$, i.e., $6899079264 \approx 6.4 \times 2^{30} = 6.4$ Giga. For each monomial m, we must remember (a) the expression of the monomial and (b) the colouring, i.e., an I-tuple of polynomials in t.

Let us first consider how we can express the monomial. It is known that l-lowest weight monomial, i.e., the unique monomial with (ordinary) weight $-\varpi_5$, is $Y_{5,q^{30}}^{-1}$ (see e.g., [4, 6.8]). We have

$$\begin{aligned} Y_{5,q^{30}}^{-1} = Y_{5,1} &\times A_{1,q^5} A_{1,q^7} A_{1,q^9} A_{1,q^{11}} A_{1,q^{13}} A_{1,q^{15}}^2 A_{1,q^{17}} A_{1,q^{19}} A_{1,q^{21}} A_{1,q^{23}} A_{1,q^{25}} \\ &\times A_{2,q^4} A_{2,q^6}^2 A_{2,q^8}^2 A_{2,q^{10}}^2 A_{2,q^{12}}^2 A_{2,q^{14}}^3 A_{2,q^{16}}^3 A_{2,q^{18}}^2 A_{2,q^{20}}^2 A_{2,q^{22}}^2 A_{2,q^{24}}^2 A_{2,q^{26}} \\ &\times A_{3,q^3} A_{3,q^5}^2 A_{3,q^7}^3 A_{3,q^9}^3 A_{3,q^{11}}^3 A_{3,q^{13}}^4 A_{3,q^{15}}^4 A_{3,q^{17}}^4 A_{3,q^{19}}^3 A_{3,q^{21}}^3 A_{3,q^{23}}^3 A_{3,q^{25}}^2 A_{3,q^{27}} \\ &\times A_{4,q^2} A_{4,q^4}^2 A_{4,q^6}^3 A_{4,q^8}^4 A_{4,q^{10}}^4 A_{4,q^{12}}^5 A_{4,q^{14}}^5 A_{4,q^{16}}^5 A_{4,q^{18}}^5 A_{4,q^{20}}^4 A_{4,q^{22}}^4 A_{4,q^{24}}^3 A_{4,q^{26}}^2 A_{4,q^{28}} \end{aligned}$$

$$\times A_{6,q^2} A_{6,q^4}^2 A_{6,q^6}^2 A_{6,q^8}^3 A_{6,q^{10}}^4 A_{6,q^{12}}^4 A_{6,q^{14}}^4 A_{6,q^{16}}^4 A_{6,q^{18}}^4 A_{6,q^{20}}^4 A_{6,q^{22}}^3 A_{6,q^{24}}^2 A_{6,q^{26}}^2 A_{6,q^{28}}$$
$$\times A_{7,q^3} A_{7,q^5} A_{7,q^7} A_{7,q^9}^2 A_{7,q^{11}}^2 A_{7,q^{13}}^2 A_{7,q^{15}}^2 A_{7,q^{17}}^2 A_{7,q^{19}}^2 A_{7,q^{21}}^2 A_{7,q^{23}} A_{7,q^{25}} A_{7,q^{27}}$$
$$\times A_{8,q^2} A_{8,q^4} A_{8,q^6}^2 A_{8,q^8}^2 A_{8,q^{10}}^3 A_{8,q^{12}}^3 A_{8,q^{14}}^3 A_{8,q^{16}}^3 A_{8,q^{18}}^3 A_{8,q^{20}}^3 A_{8,q^{22}}^2 A_{8,q^{24}}^2 A_{8,q^{26}} A_{8,q^{28}}.$$

Any other monomial is given equal to $Y_{5,1}$ multiplied by a part of A_{i,q^k}'s appeared above. We record the monomial as a sequence of A_{i,q^k}^m's, where i runs 1 to 8, k runs from 1 to 29 and m runs from 1 to 6. We can store the triple (i,k,m) in a single `short int`, i.e., 16bit of memory. The length of the sequence is at most 106, which is the length for $Y_{5,q^{30}}^{-1}$. A naive count gives $6899079264 \times 106 \times 16\text{bit} >$ 1,300 Gbyte. This is too large. Therefore, we use the following trick: Noticing that many monomials share the same sequences of A_{i,q^k}^m's, we store the data into a tree so that we do not need to repeat the common part. By this trick, it becomes uncertain how much size we need in advance, as we mentioned above.

Next let us turn to colouring. By [17], $\chi_{q,t}(L(P)) = \sum_m a_m(t)m$ is given by the Poincaré polynomials of various graded quiver varieties corresponding to m. Therefore, the degree of the coefficient $a_m(t)$ is equal to the (real) dimension of the variety corresponding to m. On the other hand, the dimension of the graded quiver variety is bounded by half of the ordinary quiver variety containing it. For the fifth fundamental representation, the maximum (among various connected components) of the dimension is equal to 60. Therefore, the maximum of the degree is 30. As $a_{m,i}(t)$ is given by a virtual Hodge polynomial of a certain stratum of the graded quiver variety, the degree is also less than or equal to 30. As $a_m(t)$, $a_{m,i}(t)$ are polynomials in t^2, we have $30/2+1 = 16$ coefficients. Therefore, we must record 16×8 integers for each monomial. We did not know how large integers were in advance. As a result of our calculation, it turns out we can store it into a `short int`. Then we would need $16 \times 8 \times 16\text{bit} = 256\text{byte}$ for each monomial. This is huge size, though it could be handled by our computer probably. However, we note that many monomials m have coefficient $a_m(t) = 1$. We store $a_{m,i}(t)$ for those monomials in a special format to save the size of data. As we do not need $a_{m,i}(t)$ for the final result, they are not included. (As a result of our calculation, we find that 4639565354 among 6899079264 monomials have this property.)

We have explained the total size of the data so far. In practice, it is more important to know how much memory is required in the course of the calculation. For the simplicity of the program, we replace the ordering $<$ among monomials by more manageable ordering given by

$$\operatorname{depth} m \stackrel{\text{def.}}{=} \sum_{i,a} v_{i,a}(m, m_P).$$

Therefore, the l-highest weight vector has depth 0, $Y_{5,1}A_{5,q}^{-1}$ has depth 1, etc. We expand the monomial of depth 0, then monomials with depth 1, monomials with depth 2, and so on. When we expand all monomials of given depth, we store all obtained monomials together with colouring in memory. As a single monomial appears many times in the expansions at various monomials, it is not practical to save the data in the hard disk. Therefore, the most crucial point is to save the size of data so that the program requires, in a fixed depth, up to 200 Gbyte of memory, which is the limit of the supercomputer. We estimated the memory requirement by that for fourth l-fundamental representation as above, and we guessed that the calculation was possible. This turns out to be true fortunately.

4 Results

We only consider the fifth l-fundamental representation of $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ with $\mathfrak{g} = E_8$.

As the final result is a huge polynomial, we cannot give it here. So we only give a part of the information. The monomial whose coefficient with the highest degree t^{30} is

$$\begin{aligned}&(1 + 4t^2 + 10t^4 + 20t^6 + 33t^8 + 47t^{10} + 59t^{12} + 66t^{14}\\ &\quad +66t^{16} + 59t^{18} + 47t^{20} + 33t^{22} + 20t^{24} + 10t^{26} + 4t^{28} + t^{30})\\ &\quad \times Y_{1,q^{14}} Y_{1,q^{16}}^{-1} Y_{3,q^{14}}^{2} Y_{3,q^{16}}^{-2} Y_{5,q^{14}}^{3} Y_{5,q^{16}}^{-3} Y_{7,q^{14}} Y_{7,q^{16}}^{-1}.\end{aligned}$$

The coefficient is the Poincaré polynomial of a certain graded quiver variety.

We define the t-graded character by

$$\mathrm{ch}_t(L(P)) = \widetilde{\chi_{q,t}}(L(P))\big|_{Y_{i,a}\to y_i}.$$

If we put $t = 1$, it becomes the ordinary character of the restriction of $L(P)$ to $\mathbf{U}_q(\mathfrak{g})$. It is also equal to the generating function of the Poincaré polynomials of the quiver varieties, where the degree 0 is corresponds to the middle degree. For example, the coefficient of the weight 0 is

$$\begin{aligned}&1357104+2232771t^2+2002423t^4+1317308t^6+716312t^8+342421t^{10}+148512t^{12}\\ &\quad +59490t^{14}+22162t^{16}+7687t^{18}+2463t^{20}+726t^{22}+192t^{24}+44t^{26}+8t^{28}+t^{30}.\end{aligned}$$

Let $V(\lambda)$ denote the irreducible highest weight representation of $\mathbf{U}_q(\mathfrak{g})$ with the highest weight λ. Let $\mathrm{ch}\, V(\lambda)$ be its character. If we write

$$\mathrm{ch}_t\, L(P) = \sum_{\lambda} M(P,\lambda,t)\, \mathrm{ch}\, V(\lambda),$$

the coefficient $M(P,\lambda,t)$ is specialized to the multiplicity of $V(\lambda)$ in the restriction of $L(P)$ at $t = 1$. The fermionic formula mentioned in the Introduction is a conjectural expression of $M(P,\lambda,t)$ (for P corresponding to the Kirillov–Reshetikhin modules).

As we have computed $\widetilde{\chi_{q,t}}(L(P))$, $M(P, \lambda, t)$ can be given if we compute $V(\lambda)$. Let us compute $V(\lambda)$ by the method in [14, 7.1.1], i.e.,

$$V(\lambda) = \widetilde{\chi_{q,t}}(L(Q))\big|_{Y_{i,a}\to y_i, t\to 0},$$

where Q corresponding to λ is given as follows: We choose an orientation for each edge of the Dynkin diagram and choose a function $m\colon I \to \mathbb{Z}$ such that $m(i) - m(j) = 1$ for an oriented edge $i \to j$. Then we take

$$Q_i(u) = (1 - uq^{m(i)})^{\langle \lambda, h_i \rangle}.$$

For this choice of Q, it is known that $\operatorname{ch}_t(L(Q)) = \widetilde{\chi_{q,t}}(L(Q))|_{Y_{i,a}\to y_i}$ is equal to the generating function of shifted Poincaré polynomial of the quiver variety as above. In particular, it is independent of the choice of the orientation. For each dominant weight λ appearing in $\operatorname{ch}_t L(P)$, we choose $Q = Q_\lambda$ as above and define matrices $P(t) = (P_{\lambda\mu}(t))$ and $IC(t) = (IC_{\lambda\mu}(t))$ by

$$\operatorname{ch}_t L(Q_\lambda) = \sum_\mu P_{\lambda\mu}(t) e^\mu + \text{non dominant terms},$$
$$\operatorname{ch}_t L(Q_\lambda) = \sum_\mu IC_{\lambda\mu}(t) \operatorname{ch} V(\mu).$$

Then we have

$$IC(t) = P(t)P(0)^{-1}.$$

By [14, 17], $IC_{\lambda\mu}(t)$ is the Poincaré polynomial of the stalk of the intersection cohomology sheaf of a stratum of the quiver variety corresponding to λ at a point in the stratum corresponding to μ. In our case, it is given by

$$IC(t) = \left(\begin{array}{c|c|c} \text{Table 1} & \multirow{2}{*}{Table 3} & \multirow{2}{*}{Table 4} \\ \hline \multirow{2}{*}{0} & & \\ \cline{2-3} & 0 & \text{Table 2} \end{array}\right),$$

where $y_i = e^{\varpi_i}$. The first row gives $\operatorname{ch}_t(L(P))$ for the fifth l-fundamental representation $L(P)$. We see that it coincides with the conjectural formula in [6]. The same assertion for other l-fundamental representations can be proved by invoking other rows. The same can be proved for types E_6, E_7 in the same manner.

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Table 1

	ϖ_5	$\varpi_3+\varpi_7$	$\varpi_2+\varpi_8$	$\varpi_1+2\varpi_7$	$\varpi_1+\varpi_6$	$2\varpi_2$	$\varpi_7+\varpi_8$	$\varpi_1+\varpi_3$	ϖ_4	$2\varpi_1+\varpi_7$
ϖ_5	1	t^2	t^2+t^4	t^4	$t^2+t^4+t^6$	t^6	$t^2+2t^4+2t^6+t^8$	$2t^4+t^6+t^8$	$t^2+2t^4+2t^6+t^8+t^{10}$	$2t^6+t^8+t^{10}$
$\varpi_3+\varpi_7$	0	1	t^2	t^2	t^2+t^4	t^4	$t^2+2t^4+t^6$	$t^2+t^4+t^6$	$t^2+2t^4+t^6+t^8$	$2t^4+t^6+t^8$
$\varpi_2+\varpi_8$	0	0	1	0	t^2	t^2	t^2+t^4	t^2+t^4	$t^2+t^4+t^6$	t^4+t^6
$\varpi_1+2\varpi_7$	0	0	0	1	t^2	0	t^2+t^4	t^4	t^4+t^6	$t^2+t^4+t^6$
$\varpi_1+\varpi_6$	0	0	0	0	1	0	t^2	t^2	t^2+t^4	t^2+t^4
$2\varpi_2$	0	0	0	0	0	1	0	t^2	t^4	t^4
$\varpi_7+\varpi_8$	0	0	0	0	0	0	1	0	t^2	0
$\varpi_1+\varpi_3$	0	0	0	0	0	0	0	1	t^2	t^2
ϖ_4	0	0	0	0	0	0	0	0	1	0
$2\varpi_1+\varpi_7$	0	0	0	0	0	0	0	0	0	1

Table 2

	ϖ_8	$2\varpi_1$	ϖ_2	ϖ_7	ϖ_1	0
ϖ_8	1	0	t^2	t^2+t^4	t^4+t^6	t^8
$2\varpi_1$	0	1	t^2	t^4	$t^2+t^4+t^6$	t^4+t^8
ϖ_2	0	0	1	t^2	t^2+t^4	t^6
ϖ_7	0	0	0	1	t^2	t^4
ϖ_1	0	0	0	0	1	t^2
0	0	0	0	0	0	1

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Table 3

	$\varpi_2+\varpi_7$	$\varpi_1+\varpi_8$	$2\varpi_7$	ϖ_6	$3\varpi_1$	$\varpi_1+\varpi_2$	ϖ_3	$\varpi_1+\varpi_7$
ϖ_5	$3t^4+4t^6+4t^8+2t^{10}+t^{12}$	$2t^4+5t^6+5t^8+3t^{10}+2t^{12}+t^{14}$	$3t^6+2t^8+3t^{10}+t^{12}+t^{14}$	$2t^4+4t^6+6t^8+4t^{10}+3t^{12}+t^{14}+t^{16}$	t^8+t^{12}	$2t^6+5t^8+5t^{10}+3t^{12}+2t^{14}+t^{16}$	$5t^6+5t^8+7t^{10}+4t^{12}+3t^{14}+t^{16}+t^{18}$	$2t^6+9t^8+10t^{10}+10t^{12}+6t^{14}+4t^{16}+2t^{18}+t^{20}$
$\varpi_3+\varpi_7$	$t^2+3t^4+4t^6+2t^8+t^{10}$	$3t^4+5t^6+3t^8+2t^{10}+t^{12}$	$2t^4+2t^6+3t^8+t^{10}+t^{12}$	$2t^4+5t^6+4t^8+3t^{10}+t^{12}+t^{14}$	t^6+t^{10}	$t^4+4t^6+5t^8+3t^{10}+2t^{12}+t^{14}$	$2t^4+4t^6+7t^8+4t^{10}+3t^{12}+t^{14}+t^{16}$	$6t^6+9t^8+10t^{10}+6t^{12}+4t^{14}+2t^{16}+t^{18}$
$\varpi_2+\varpi_8$	$t^2+3t^4+2t^6+t^8$	$t^2+3t^4+3t^6+2t^8+t^{10}$	$t^4+2t^6+t^8+t^{10}$	$3t^4+3t^6+3t^8+t^{10}+t^{12}$	t^8	$2t^4+4t^6+3t^8+2t^{10}+t^{12}$	$2t^4+5t^6+4t^8+3t^{10}+t^{12}+t^{14}$	$t^4+6t^6+8t^8+6t^{10}+4t^{12}+2t^{14}+t^{16}$
$\varpi_1+2\varpi_7$	$t^2+2t^4+2t^6+t^8$	$2t^4+3t^6+2t^8+t^{10}$	$t^2+t^4+2t^6+t^8+t^{10}$	$2t^4+3t^6+3t^8+t^{10}+t^{12}$	t^4+t^8	$t^4+3t^6+3t^8+2t^{10}+t^{12}$	$4t^6+4t^8+3t^{10}+t^{12}+t^{14}$	$2t^4+4t^6+8t^8+6t^{10}+4t^{12}+2t^{14}+t^{16}$
$\varpi_1+\varpi_6$	$t^2+2t^4+t^6$	$t^2+3t^4+2t^6+t^8$	$t^4+t^6+t^8$	$t^2+2t^4+3t^6+t^8+t^{10}$	t^6	$2t^4+3t^6+2t^8+t^{10}$	$3t^4+4t^6+3t^8+t^{10}+t^{12}$	$2t^4+6t^6+6t^8+4t^{10}+2t^{12}+t^{14}$
$2\varpi_2$	$t^2+t^4+t^6$	$t^4+2t^6+t^8$	t^4+t^8	$2t^6+t^8+t^{10}$	t^6	$t^2+2t^4+2t^6+2t^8+t^{10}$	$2t^4+2t^6+3t^8+t^{10}+t^{12}$	$t^4+4t^6+4t^8+4t^{10}+2t^{12}+t^{14}$
$\varpi_7+\varpi_8$	t^2+t^4	$t^2+t^4+t^6$	$t^2+t^4+t^6$	$t^2+2t^4+t^6+t^8$	0	$t^4+t^6+t^8$	$2t^4+2t^6+t^8+t^{10}$	$2t^4+4t^6+3t^8+2t^{10}+t^{12}$
$\varpi_1+\varpi_3$	t^2+t^4	$t^2+2t^4+t^6$	t^6	$2t^4+t^6+t^8$	t^4	$t^2+2t^4+2t^6+t^8$	$t^2+2t^4+3t^6+t^8+t^{10}$	$3t^4+4t^6+4t^8+2t^{10}+t^{12}$
ϖ_4	t^2	t^2+t^4	t^4	$t^2+t^4+t^6$	0	t^4+t^6	$t^2+2t^4+t^6+t^8$	$2t^4+3t^6+2t^8+t^{10}$
$2\varpi_1+\varpi_7$	t^2	t^2+t^4	t^4	t^4+t^6	t^2	$t^2+2t^4+t^6$	$2t^4+t^6+t^8$	$t^2+2t^4+4t^6+2t^8+t^{10}$
$\varpi_2+\varpi_7$	1	t^2	t^2	t^2+t^4	0	t^2+t^4	$t^2+t^4+t^6$	$t^2+3t^4+2t^6+t^8$
$\varpi_1+\varpi_8$	0	1	0	t^2	0	t^2	t^2+t^4	$t^2+2t^4+t^6$
$2\varpi_7$	0	0	1	t^2	0	0	t^4	$t^2+t^4+t^6$
ϖ_6	0	0	0	1	0	0	t^2	t^2+t^4
$3\varpi_1$	0	0	0	0	1	t^2+t^4	t^6	$t^4+t^6+t^8$
$\varpi_1+\varpi_2$	0	0	0	0	0	1	t^2	t^2+t^4
ϖ_3	0	0	0	0	0	0	1	t^2
$\varpi_1+\varpi_7$	0	0	0	0	0	0	0	1

Table 4

	ϖ_8	$2\varpi_1$	ϖ_2	ϖ_7	ϖ_1	0
ϖ_5	$t^6+5t^8+8t^{10}+7t^{12}+6t^{14}+4t^{16}+2t^{18}+t^{20}+t^{22}$	$5t^{10}+4t^{12}+6t^{14}+3t^{16}+3t^{18}+t^{20}+t^{22}$	$3t^8+6t^{10}+11t^{12}+8t^{14}+7t^{16}+4t^{18}+3t^{20}+t^{22}+t^{24}$	$5t^{10}+6t^{12}+9t^{14}+6t^{16}+6t^{18}+3t^{20}+2t^{22}+t^{24}+t^{26}$	$4t^{12}+5t^{14}+8t^{16}+5t^{18}+5t^{20}+3t^{22}+2t^{24}+t^{26}+t^{28}$	$t^{14}+3t^{18}+t^{20}+2t^{22}+t^{24}+t^{26}+t^{30}$
$\varpi_3+\varpi_7$	$2t^6+7t^8+7t^{10}+6t^{12}+4t^{14}+2t^{16}+t^{18}+t^{20}$	$4t^8+4t^{10}+6t^{12}+3t^{14}+3t^{16}+t^{18}+t^{20}$	$t^6+4t^8+10t^{10}+8t^{12}+7t^{14}+4t^{16}+3t^{18}+t^{20}+t^{22}$	$3t^8+5t^{10}+9t^{12}+6t^{14}+6t^{16}+3t^{18}+2t^{20}+t^{22}+t^{24}$	$3t^{10}+4t^{12}+8t^{14}+5t^{16}+5t^{18}+3t^{20}+2t^{22}+t^{24}+t^{26}$	$t^{12}+3t^{16}+t^{18}+2t^{20}+t^{22}+t^{24}+t^{28}$
$\varpi_2+\varpi_8$	$4t^6+5t^8+6t^{10}+4t^{12}+2t^{14}+t^{16}+t^{18}$	$t^6+3t^8+5t^{10}+3t^{12}+3t^{14}+t^{16}+t^{18}$	$t^6+7t^8+7t^{10}+7t^{12}+4t^{14}+3t^{16}+t^{18}+t^{20}$	$3t^8+6t^{10}+6t^{12}+6t^{14}+3t^{16}+2t^{18}+t^{20}+t^{22}$	$3t^{10}+6t^{12}+5t^{14}+5t^{16}+3t^{18}+2t^{20}+t^{22}+t^{24}$	$2t^{14}+t^{16}+2t^{18}+t^{20}+t^{22}+t^{26}$
$\varpi_1+2\varpi_7$	$2t^6+5t^8+6t^{10}+4t^{12}+2t^{14}+t^{16}+t^{18}$	$2t^6+2t^8+5t^{10}+3t^{12}+3t^{14}+t^{16}+t^{18}$	$t^6+5t^8+7t^{10}+7t^{12}+4t^{14}+3t^{16}+t^{18}+t^{20}$	$t^6+2t^8+6t^{10}+6t^{12}+6t^{14}+3t^{16}+2t^{18}+t^{20}+t^{22}$	$2t^8+2t^{10}+6t^{12}+5t^{14}+5t^{16}+3t^{18}+2t^{20}+t^{22}+t^{24}$	$t^{10}+2t^{14}+t^{16}+2t^{18}+t^{20}+t^{22}+t^{26}$
$\varpi_1+\varpi_6$	$t^4+4t^6+6t^8+4t^{10}+2t^{12}+t^{14}+t^{16}$	$t^6+4t^8+3t^{10}+3t^{12}+t^{14}+t^{16}$	$3t^6+6t^8+7t^{10}+4t^{12}+3t^{14}+t^{16}+t^{18}$	$t^6+4t^8+6t^{10}+6t^{12}+3t^{14}+2t^{16}+t^{18}+t^{20}$	$t^8+4t^{10}+5t^{12}+5t^{14}+3t^{16}+2t^{18}+t^{20}+t^{22}$	$t^{12}+t^{14}+2t^{16}+t^{18}+t^{20}+t^{24}$
$2\varpi_2$	$t^6+4t^8+4t^{10}+2t^{12}+t^{14}+t^{16}$	$t^4+t^6+4t^8+2t^{10}+3t^{12}+t^{14}+t^{16}$	$3t^6+3t^8+6t^{10}+4t^{12}+3t^{14}+t^{16}+t^{18}$	$3t^8+3t^{10}+6t^{12}+3t^{14}+2t^{16}+t^{18}+t^{20}$	$t^8+4t^{10}+3t^{12}+5t^{14}+3t^{16}+2t^{18}+t^{20}+t^{22}$	$2t^{12}+2t^{16}+t^{18}+t^{20}+t^{24}$
$\varpi_7+\varpi_8$	$t^4+3t^6+3t^8+2t^{10}+t^{12}+t^{14}$	$t^6+2t^8+2t^{10}+t^{12}+t^{14}$	$2t^6+4t^8+3t^{10}+3t^{12}+t^{14}+t^{16}$	$t^6+3t^8+4t^{10}+3t^{12}+2t^{14}+t^{16}+t^{18}$	$t^8+3t^{10}+3t^{12}+3t^{14}+2t^{16}+t^{18}+t^{20}$	$t^{12}+t^{14}+t^{16}+t^{18}+t^{22}$
$\varpi_1+\varpi_3$	$t^4+4t^6+4t^8+2t^{10}+t^{12}+t^{14}$	$3t^6+2t^8+3t^{10}+t^{12}+t^{14}$	$t^4+3t^6+6t^8+4t^{10}+3t^{12}+t^{14}+t^{16}$	$2t^6+3t^8+6t^{10}+3t^{12}+2t^{14}+t^{16}+t^{18}$	$3t^8+3t^{10}+5t^{12}+3t^{14}+2t^{16}+t^{18}+t^{20}$	$t^{10}+2t^{14}+t^{16}+t^{18}+t^{22}$
ϖ_4	$2t^4+3t^6+2t^8+t^{10}+t^{12}$	$t^6+2t^8+t^{10}+t^{12}$	$3t^6+3t^8+3t^{10}+t^{12}+t^{14}$	$t^6+4t^8+3t^{10}+2t^{12}+t^{14}+t^{16}$	$t^8+3t^{10}+3t^{12}+2t^{14}+t^{16}+t^{18}$	$t^{12}+t^{14}+t^{16}+t^{20}$
$2\varpi_1+\varpi_7$	$t^4+3t^6+2t^8+t^{10}+t^{12}$	$2t^4+2t^6+3t^8+t^{10}+t^{12}$	$t^4+4t^6+4t^8+3t^{10}+t^{12}+t^{14}$	$t^4+t^6+5t^8+3t^{10}+2t^{12}+t^{14}+t^{16}$	$2t^6+2t^8+5t^{10}+3t^{12}+2t^{14}+t^{16}+t^{18}$	$t^8+2t^{12}+t^{14}+t^{16}+t^{20}$
$\varpi_2+\varpi_7$	$2t^4+2t^6+t^8+t^{10}$	$t^4+2t^6+t^8+t^{10}$	$2t^4+3t^6+3t^8+t^{10}+t^{12}$	$3t^6+3t^8+2t^{10}+t^{12}+t^{14}$	$t^6+3t^8+3t^{10}+2t^{12}+t^{14}+t^{16}$	$t^{10}+t^{12}+t^{14}+t^{18}$
$\varpi_1+\varpi_8$	$t^2+2t^4+t^6+t^8$	$t^4+t^6+t^8$	$2t^4+3t^6+t^8+t^{10}$	$t^4+3t^6+2t^8+t^{10}+t^{12}$	$t^6+3t^8+2t^{10}+t^{12}+t^{14}$	$t^{10}+t^{12}+t^{16}$
$2\varpi_7$	$t^4+t^6+t^8$	t^4+t^8	$2t^6+t^8+t^{10}$	$t^4+t^6+2t^8+t^{10}+t^{12}$	$t^6+t^8+2t^{10}+t^{12}+t^{14}$	$t^8+t^{12}+t^{16}$
ϖ_6	$t^2+t^4+t^6$	t^6	$2t^4+t^6+t^8$	$t^4+2t^6+t^8+t^{10}$	$t^6+2t^8+t^{10}+t^{12}$	$t^{10}+t^{14}$
$3\varpi_1$	t^8+t^{10}	$t^2+t^4+2t^6+t^8+t^{10}$	$t^4+2t^6+2t^8+t^{10}+t^{12}$	$t^6+t^8+2t^{10}+t^{12}+t^{14}$	$t^4+t^6+3t^8+2t^{10}+2t^{12}+t^{14}+t^{16}$	$t^6+t^{10}+t^{12}+t^{14}+t^{18}$
$\varpi_1+\varpi_2$	t^4+t^6	$t^2+t^4+t^6$	$t^2+2t^4+t^6+t^8$	$t^4+2t^6+t^8+t^{10}$	$t^4+2t^6+2t^8+t^{10}+t^{12}$	$t^8+t^{10}+t^{14}$
ϖ_3	t^2+t^4	t^4	$t^2+t^4+t^6$	$2t^4+t^6+t^8$	$2t^6+t^8+t^{10}$	t^8+t^{12}
$\varpi_1+\varpi_7$	t^2	t^2	t^2+t^4	$t^2+t^4+t^6$	$2t^4+t^6+t^8$	t^6+t^{10}

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Ultra-Discretization of the $G_2^{(1)}$-Geometric Crystals to the $D_4^{(3)}$-Perfect Crystals

Toshiki Nakashima

Abstract We obtain the affirmative answer to the conjecture in [14]. More precisely, let $\chi := (\mathcal{V}, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ be the affine geometric crystal of type $G_2^{(1)}$ in [14] and $\mathcal{UD}(\chi, T, \theta)$ a ultra-discretization of χ with respect to a certain positive structure θ. Then we show that $\mathcal{UD}(\chi, T, \theta)$ is isomorphic to the limit of coherent family of perfect crystals of type $D_4^{(3)}$ in [9].

Keywords Geometric crystal · Perfect crystal · Ultra-discretization

Mathematics Subject Classifications (2000): Primary 17B37; 17B67; Secondary 22E65; 14M15

1 Introduction

In [4], we introduced the notion of perfect crystal, which holds several nice properties, e.g., the existence of the isomorphism of crystals:

$$B(\lambda) \cong B(\sigma(\lambda)) \otimes B,$$

where B is a perfect crystal of level $l \in \mathbb{Z}_{>0}$, $B(\lambda)$ is the crystal of the integrable highest weight module of a quantum affine group with the level l highest weight λ and σ is a certain bijection on dominant weights. Iterating this isomorphism, one can get the so-called Kyoto path model for $B(\lambda)$, which plays a crucial role in calculating the one-point functions for vertex-type lattice models [4, 5].

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In [5], perfect crystals with arbitrary level have been constructed explicitly for affine Kac–Moody algebra of type $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $D_{n+1}^{(2)}$, $A_{2n-1}^{(2)}$ and $A_{2n}^{(2)}$. In [16], the $G_2^{(1)}$ case has been accomplished. But so far, the other cases except $D_4^{(3)}$ have not yet been obtained. In a recent work [9], they constructed the perfect crystal of type $D_4^{(3)}$ with arbitrary level explicitly. A coherent family of perfect crystals is defined in [6], and it has been shown that the perfect crystals in [5] constitute a coherent family. A coherent family $\{B_l\}_{l\geq 1}$ of perfect crystals B_l possesses a limit B_∞ which still keeps a structure of crystal. This has a similar property to B_l, that is, there exists the isomorphism of crystals:

$$B(\infty) \cong B(\infty) \otimes B_\infty,$$

where $B(\infty)$ is the crystal of the nilpotent subalgebra $U_q^-(\mathfrak{g})$ of a quantum affine algebra $U_q(\mathfrak{g})$. An iteration of the isomorphism also produces a path model of $B(\infty)$ [6]. It is shown in [9] that the obtained perfect crystals consists of a coherent family, and the structure of the limit B_∞ has been described explicitly.

Geometric crystal is an object defined over certain algebraic (or ind-) variety which seems to be a kind of geometric lifting of Kashiwara's crystal. It is defined in [1] for reductive algebraic groups and is extended to general Kac–Moody cases in [12]. For a fixed Cartan data $(A, \{\alpha_i\}_{i\in I}, \{\alpha_i^\vee\}_{i\in I})$, a geometric crystal consists of an ind-variety X over the complex number $\mathbb{C}$, a rational $\mathbb{C}^\times$-action $e_i : \mathbb{C}^\times \times X \longrightarrow X$ and rational functions $\gamma_i, \varepsilon_i : X \longrightarrow \mathbb{C}$ $(i \in I)$, which satisfy the conditions as in Definition 2.1. It has many similarity to the theory of crystals, e.g., some product structure, Weyl group actions and R-matrices. Moreover, one has a direct connection between geometric crystals and free crystals, called tropicalization/ultra-discretization procedure (see Sect. 2). Here let us explain this procedure. For an algebraic torus T' and a birational morphism $\theta : T' \to X$, the pair (T', θ) is positive if it satisfies the conditions as in Sect. 2, roughly speaking: Through the morphism θ, we can induce a geometric crystal structure on T' from X and express the data e_i^c, γ_i and ε_i $(i \in I)$ using the coordinate of T' explicitly. In case each of them is expressed as a ratio of positive polynomials, it is said that (T', θ) is a positive structure of the geometric crystal $(X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$. Then using a map $v : \mathbb{C}(c) \setminus \{0\} \to \mathbb{Z}$ $(v(f) := \deg(f))$, we can define a morphism $T' \to \mathbb{Z}^m$ $(m = \dim T' = \dim X)$, which defines the so-called ultra-discretization functor. If $\theta : T' \to X$ is a positive structure on X, then we obtain a Kashiwara's crystal from X by applying the ultra-discretization functor [1].

Let G (resp. $\mathfrak{g} = \langle \mathfrak{t}, e_i, f_i \rangle_{i\in I}$) be the affine Kac–Moody group (resp. algebra) associated with a generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$. Let $B^\pm$ be fixed Borel subgroups and T the maximal torus such that $B^+ \cap B^- = T$. Set $y_i(c) := \exp(cf_i)$, and let $\alpha_i^\vee(c) \in T$ be the image of $c \in \mathbb{C}^\times$ by the group morphism $\mathbb{C}^\times \to T$ induced by the simple coroot $\alpha_i^\vee$ as in 2.1. We set $Y_i(c) := y_i(c^{-1})\,\alpha_i^\vee(c) = \alpha_i^\vee(c)\,y_i(c)$. Let W (resp. $\widetilde{W}$) be the Weyl group (resp. the extended Weyl group) associated with $\mathfrak{g}$. The Schubert cell $X_w := BwB/B$ ($w = s_{i_1} \cdots s_{i_k} \in W$) is birationally isomorphic to the variety

$$B_w^- := \left\{ Y_{i_1}(x_1) \cdots Y_{i_k}(x_k) \mid x_1, \dots, x_k \in \mathbb{C}^\times \right\} \subset B^-,$$

and X_w has a natural geometric crystal structure [1, 12].

We choose $0 \in I$ as in [7, 8], and let $\{\varpi_i\}_{i \in I \setminus \{0\}}$ be the set of level 0 fundamental weights. Let $W(\varpi_i)$ be the fundamental representation of $U_q(\mathfrak{g})$ with ϖ_i as an extremal weight [7, 8]. Let us denote its specialization at $q = 1$ by the same notation $W(\varpi_i)$. It is a finite-dimensional $\mathfrak{g}$-module. Let $\mathbb{P}(\varpi_i)$ be the projective space $(W(\varpi_i) \setminus \{0\})/\mathbb{C}^\times$.

For any $i \in I$, define $c_i^\vee := \max(1, \frac{2}{(\alpha_i, \alpha_i)})$. Then the translation $t(c_i^\vee \varpi_i)$ belongs to $\widetilde{W}$ (see [10]). For a subset J of I, let us denote by $\mathfrak{g}_J$ the subalgebra of $\mathfrak{g}$ generated by $\{e_i, f_i\}_{i \in J}$. For an integral weight μ, define $I(\mu) := \left\{ j \in I \mid \langle \alpha_j^\vee, \mu \rangle \geq 0 \right\}$.

Here we state the conjecture given in [10]:

Conjecture 1.1 ([10]). For any $i \in I$, there exist a unique variety X endowed with a positive $\mathfrak{g}$-geometric crystal structure and a rational mapping $\pi : X \longrightarrow \mathbb{P}(\varpi_i)$ satisfying the following property:

(i) For an arbitrary extremal vector $u \in W(\varpi_i)_\mu$, writing the translation $t(c_i^\vee \mu)$ as $\iota w \in \widetilde{W}$ with a Dynkin diagram automorphism ι and $w = s_{i_1} \cdots s_{i_k}$, there exists a birational mapping $\xi : B_w^- \longrightarrow X$ such that ξ is a morphism of $\mathfrak{g}_{I(\mu)}$-geometric crystals, and that the composition $\pi \circ \xi : B_w^- \to \mathbb{P}(\varpi_i)$ coincides with $Y_{i_1}(x_1) \cdots Y_{i_k}(x_k) \mapsto Y_{i_1}(x_1) \cdots Y_{i_k}(x_k)\overline{u}$, where $\overline{u}$ is the line including u,

(ii) The ultra-discretization (see Sect. 2) of X is isomorphic to the crystal $B_\infty(\varpi_i)$ of the Langlands dual $\mathfrak{g}^L$.

In [10], the cases $i = 1$ and $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}$ have been resolved, that is, certain positive geometric crystal $\mathcal{V}(\mathfrak{g})$ associated with the fundamental representation $W(\varpi_1)$ for the above affine Lie algebras has been constructed, and it was shown that the ultra-discretization limit of $\mathcal{V}(\mathfrak{g})$ is isomorphic to the limit of the coherent family of perfect crystals as above for $\mathfrak{g}^L$ the Langlands dual of $\mathfrak{g}$. In [14] for the case $i = 1$ and $\mathfrak{g} = G_2^{(1)}$, a positive geometric crystal $\mathcal{V}$ was constructed. However, the ultra-discretization of the geometric crystal has not been given there, though it was conjectured that the ultra-discretization of $\mathcal{V}$ is isomorphic to B_∞ as in [9].

In this chapter, we shall describe the structure of the crystal obtained by ultra-discretization process from the geometric crystal $\mathcal{V}$ for $\mathfrak{g} = G_2^{(1)}$ in [14]. Finally, we shall show that the crystal is isomorphic to B_∞ as in [9].

2 Geometric Crystals

In this section, we review Kac–Moody groups and geometric crystals following [1, 11, 15].

2.1 Kac–Moody Algebras and Kac–Moody Groups

Fix a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$ with a finite index set I. Let $(\mathfrak{t}, \{\alpha_i\}_{i\in I}, \{\alpha_i^\vee\}_{i\in I})$ be the associated root data, where $\mathfrak{t}$ is a vector space over $\mathbb{C}$ and $\{\alpha_i\}_{i\in I} \subset \mathfrak{t}^*$ and $\{\alpha_i^\vee\}_{i\in I} \subset \mathfrak{t}$ are linearly independent satisfying $\alpha_j(\alpha_i^\vee) = a_{ij}$.

The Kac–Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ associated with A is the Lie algebra over $\mathbb{C}$ generated by $\mathfrak{t}$, the Chevalley generators e_i and f_i $(i \in I)$ with the usual defining relations [3, 15]. There is the root space decomposition $\mathfrak{g} = \bigoplus_{\alpha\in\mathfrak{t}^*} \mathfrak{g}_\alpha$. Denote the set of roots by $\Delta := \{\alpha \in \mathfrak{t}^* \mid \alpha \neq 0,\ \mathfrak{g}_\alpha \neq (0)\}$. Set $Q = \sum_i \mathbb{Z}\alpha_i$, $Q_+ = \sum_i \mathbb{Z}_{\geq 0}\alpha_i$, $Q^\vee := \sum_i \mathbb{Z}\alpha_i^\vee$ and $\Delta_+ := \Delta \cap Q_+$. An element of Δ_+ is called a *positive root*. Let $P \subset \mathfrak{t}^*$ be a weight lattice such that $\mathbb{C} \otimes P = \mathfrak{t}^*$, whose element is called a weight.

Define simple reflections $s_i \in \mathrm{Aut}(\mathfrak{t})$ $(i \in I)$ by $s_i(h) := h - \alpha_i(h)\alpha_i^\vee$, which generate the Weyl group W. It induces the action of W on $\mathfrak{t}^*$ by $s_i(\lambda) := \lambda - \lambda(\alpha_i^\vee)\alpha_i$. Set $\Delta^{\mathrm{re}} := \{w(\alpha_i) \mid w \in W,\ i \in I\}$, whose element is called a real root.

Let $\mathfrak{g}'$ be the derived Lie algebra of $\mathfrak{g}$ and let G be the Kac–Moody group associated with $\mathfrak{g}'$([15]). Let $U_\alpha := \exp \mathfrak{g}_\alpha$ $(\alpha \in \Delta^{\mathrm{re}})$ be the one-parameter subgroup of G. The group G is generated by U_α $(\alpha \in \Delta^{\mathrm{re}})$. Let $U^\pm$ be the subgroup generated by $U_{\pm\alpha}$ $(\alpha \in \Delta^{\mathrm{re}}_+ = \Delta^{\mathrm{re}} \cap Q_+)$, i.e., $U^\pm := \langle U_{\pm\alpha} \mid \alpha \in \Delta^{\mathrm{re}}_+\rangle$.

For any $i \in I$, there exists a unique homomorphism; $\phi_i : SL_2(\mathbb{C}) \to G$ such that

$$\phi_i\left(\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}\right) = c^{\alpha_i^\vee},\ \phi_i\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) = \exp(te_i),\ \phi_i\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) = \exp(tf_i),$$

where $c \in \mathbb{C}^\times$ and $t \in \mathbb{C}$. Set $\alpha_i^\vee(c) := c^{\alpha_i^\vee}$, $x_i(t) := \exp(te_i)$, $y_i(t) := \exp(tf_i)$, $G_i := \phi_i(SL_2(\mathbb{C}))$, $T_i := \phi_i(\{\mathrm{diag}(c, c^{-1}) \mid c \in \mathbb{C}^\vee\})$ and $N_i := N_{G_i}(T_i)$. Let T (resp. N) be the subgroup of G with the Lie algebra $\mathfrak{t}$ (resp. generated by the N_i's), which is called a *maximal torus* in G, and let $B^\pm = U^\pm T$ be the Borel subgroup of G. We have the isomorphism $\phi : W \xrightarrow{\sim} N/T$ defined by $\phi(s_i) = N_i T/T$. An element $\overline{s}_i := x_i(-1)y_i(1)x_i(-1) = \phi_i\left(\begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix}\right)$ is in $N_G(T)$, which is a representative of $s_i \in W = N_G(T)/T$.

2.2 Geometric Crystals

Let W be the Weyl group associated with $\mathfrak{g}$. Define $R(w)$ for $w \in W$ by

$$R(w) := \{(i_1, i_2, \ldots, i_l) \in I^l \mid w = s_{i_1} s_{i_2} \cdots s_{i_l}\},$$

where l is the length of w. Then $R(w)$ is the set of reduced words of w.

Let X be an ind-variety, $\gamma_i : X \to \mathbb{C}$ and $\varepsilon_i : X \longrightarrow \mathbb{C}$ $(i \in I)$ rational functions on X, and $e_i : \mathbb{C}^\times \times X \longrightarrow X$ $((c,x) \mapsto e_i^c(x))$ a rational $\mathbb{C}^\times$-action.

For a word $\mathbf{i} = (i_1, \dots, i_l) \in R(w)$ $(w \in W)$, set $\alpha^{(j)} := s_{i_l} \cdots s_{i_{j+1}}(\alpha_{i_j})$ $(1 \le j \le l)$ and

$$
\begin{aligned}
e_{\mathbf{i}} : T \times X &\to X \\
(t,x) &\mapsto e_{\mathbf{i}}^t(x) := e_{i_1}^{\alpha^{(1)}(t)} e_{i_2}^{\alpha^{(2)}(t)} \cdots e_{i_l}^{\alpha^{(l)}(t)}(x).
\end{aligned}
$$

Definition 2.1. A quadruple $(X, \{e_i\}_{i\in I}, \{\gamma_i,\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ is a G (or $\mathfrak{g}$)-*geometric crystal* if

(i) $\{1\} \times X \subset \mathrm{dom}(e_i)$ for any $i \in I$.
(ii) $\gamma_j(e_i^c(x)) = c^{a_{ij}}\gamma_j(x)$.
(iii) $e_{\mathbf{i}} = e_{\mathbf{i}'}$ for any $w \in W$, $\mathbf{i}, \mathbf{i}' \in R(w)$.
(iv) $\varepsilon_i(e_i^c(x)) = c^{-1}\varepsilon_i(x)$.

Note that the condition (iii) as above is equivalent to the following so-called *Verma relations*:

$$
\begin{aligned}
&e_i^{c_1} e_j^{c_2} = e_j^{c_2} e_i^{c_1} && \text{if } a_{ij} = a_{ji} = 0,\\
&e_i^{c_1} e_j^{c_1c_2} e_i^{c_2} = e_j^{c_2} e_i^{c_1c_2} e_j^{c_1} && \text{if } a_{ij} = a_{ji} = -1,\\
&e_i^{c_1} e_j^{c_1^2c_2} e_i^{c_1c_2} e_j^{c_2} = e_j^{c_2} e_i^{c_1c_2} e_j^{c_1^2c_2} e_i^{c_1} && \text{if } a_{ij} = -2,\ a_{ji} = -1,\\
&e_i^{c_1} e_j^{c_1^3c_2} e_i^{c_1^2c_2} e_j^{c_1^3c_2^2} e_i^{c_1c_2} e_j^{c_2} = e_j^{c_2} e_i^{c_1c_2} e_j^{c_1^3c_2^2} e_i^{c_1^2c_2} e_j^{c_1^3c_2} e_i^{c_1} && \text{if } a_{ij} = -3,\ a_{ji} = -1.
\end{aligned}
$$

Note that the last formula is different from the one in [1, 12, 13] which seems to be incorrect. The formula here may be correct.

2.3 Geometric Crystal on Schubert Cell

Let $w \in W$ be a Weyl group element and take a reduced expression $w = s_{i_1} \cdots s_{i_l}$. Let $X := G/B$ be the flag variety, which is an ind-variety and $X_w \subset X$ the Schubert cell associated with w, which has a natural geometric crystal structure [1, 12]. For $\mathbf{i} := (i_1, \dots, i_k)$, set

$$
B_{\mathbf{i}}^- := \{Y_{\mathbf{i}}(c_1, \dots, c_k) := Y_{i_1}(c_1) \cdots Y_{i_l}(c_k) \mid c_1, \dots, c_k \in \mathbb{C}^\times\} \subset B^-, \tag{2.1}
$$

which has a geometric crystal structure [12] isomorphic to X_w. The explicit forms of the action e_i^c, the rational function ε_i and γ_i on $B_{\mathbf{i}}^-$ are given by

$$
e_i^c(Y_{i_1}(c_1) \cdots Y_{i_l}(c_k)) = Y_{i_1}(\mathcal{C}_1) \cdots Y_{i_l}(\mathcal{C}_k),
$$

where

$$\mathcal{C}_j := c_j \cdot \frac{\displaystyle\sum_{1\le m\le j, i_m=i} \frac{c}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m} + \sum_{j<m\le k, i_m=i} \frac{1}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m}}{\displaystyle\sum_{1\le m<j, i_m=i} \frac{c}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m} + \sum_{j\le m\le k, i_m=i} \frac{1}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m}}, \quad (2.2)$$

$$\varepsilon_i(Y_{i_1}(c_1)\cdots Y_{i_l}(c_k)) = \sum_{1\le m\le k, i_m=i} \frac{1}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m}, \quad (2.3)$$

$$\gamma_i(Y_{i_1}(c_1)\cdots Y_{i_l}(c_k)) = c_1^{a_{i_1,i}} \cdots c_k^{a_{i_k,i}}. \quad (2.4)$$

2.4 Positive Structure, Ultra-Discretizations and Tropicalizations

Let us recall the notions of positive structure, ultra-discretization and tropicalization.

The setting below is same as [10]. Let $T = (\mathbb{C}^\times)^l$ be an algebraic torus over $\mathbb{C}$ and $X^*(T) := \mathrm{Hom}(T, \mathbb{C}^\times) \cong \mathbb{Z}^l$ (resp. $X_*(T) := \mathrm{Hom}(\mathbb{C}^\times, T) \cong \mathbb{Z}^l$) be the lattice of characters (resp. co-characters) of T. Set $R := \mathbb{C}(c)$ and define

$$\begin{array}{rcl} v : R \setminus \{0\} & \longrightarrow & \mathbb{Z} \\ f(c) & \mapsto & \deg(f(c)), \end{array}$$

where deg is the degree of poles at $c = \infty$. Here note that for $f_1, f_2 \in R \setminus \{0\}$, we have

$$v(f_1 f_2) = v(f_1) + v(f_2), \quad v\left(\frac{f_1}{f_2}\right) = v(f_1) - v(f_2). \quad (2.5)$$

A non-zero rational function on an algebraic torus T is called *positive* if it is written as g/h, where g and h are a positive linear combination of characters of T.

Definition 2.2. Let $f : T \to T'$ be a rational morphism between two algebraic tori T and T'. We say that f is *positive*, if $\chi \circ f$ is positive for any character $\chi : T' \to \mathbb{C}$.

Denote by $\mathrm{Mor}^+(T, T')$ the set of positive rational morphisms from T to T'.

Lemma 2.3 ([1]). *For any $f \in \mathrm{Mor}^+(T_1, T_2)$ and $g \in \mathrm{Mor}^+(T_2, T_3)$, the composition $g \circ f$ is well-defined and belongs to $\mathrm{Mor}^+(T_1, T_3)$.*

By Lemma 2.3, we can define a category $\mathcal{T}_+$ whose objects are algebraic tori over $\mathbb{C}$ and arrows are positive rational morphisms.

Let $f : T \to T'$ be a positive rational morphism of algebraic tori T and T'. We define a map $\widehat{f} : X_*(T) \to X_*(T')$ by

$$\langle \chi, \widehat{f}(\xi) \rangle = v(\chi \circ f \circ \xi),$$

where $\chi \in X^*(T')$ and $\xi \in X_*(T)$.

Lemma 2.4 ([1]). *For any algebraic tori T_1, T_2, T_3, and positive rational morphisms $f \in \mathrm{Mor}^+(T_1, T_2)$, $g \in \mathrm{Mor}^+(T_2, T_3)$, we have $\widehat{g \circ f} = \widehat{g} \circ \widehat{f}$.*

By this lemma, we obtain a functor

$$\begin{array}{cccc} \mathcal{UD}: & \mathcal{T}_+ & \longrightarrow & \mathfrak{Set} \\ & T & \mapsto & X_*(T) \\ & (f : T \to T') & \mapsto & (\widehat{f} : X_*(T) \to X_*(T')) \end{array}$$

Definition 2.5 ([1]). Let $\chi = (X, \{e_i\}_{i\in I}, \{\mathrm{wt}_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ be a geometric crystal, T' an algebraic torus and $\theta : T' \to X$ a birational isomorphism. The isomorphism θ is called *positive structure* on χ if it satisfies

(i) For any $i \in I$ the rational functions $\gamma_i \circ \theta : T' \to \mathbb{C}$ and $\varepsilon_i \circ \theta : T' \to \mathbb{C}$ are positive.
(ii) For any $i \in I$, the rational morphism $e_{i,\theta} : \mathbb{C}^\times \times T' \to T'$ defined by $e_{i,\theta}(c, t) := \theta^{-1} \circ e_i^c \circ \theta(t)$ is positive.

Let $\theta : T \to X$ be a positive structure on a geometric crystal $\chi = (X, \{e_i\}_{i\in I}, \{\mathrm{wt}_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$. Applying the functor $\mathcal{UD}$ to positive rational morphisms $e_{i,\theta} : \mathbb{C}^\times \times T' \to T'$ and $\gamma \circ \theta : T' \to T$ (the notations are as above), we obtain

$$\begin{aligned} \tilde{e}_i &:= \mathcal{UD}(e_{i,\theta}) : \mathbb{Z} \times X_*(T) \to X_*(T), \\ \mathrm{wt}_i &:= \mathcal{UD}(\gamma_i \circ \theta) : X_*(T') \to \mathbb{Z}, \\ \varepsilon_i &:= \mathcal{UD}(\varepsilon_i \circ \theta) : X_*(T') \to \mathbb{Z}. \end{aligned}$$

Now, for given positive structure $\theta : T' \to X$ on a geometric crystal $\chi = (X, \{e_i\}_{i\in I}, \{\mathrm{wt}_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$, we associate the quadruple $(X_*(T'), \{\tilde{e}_i\}_{i\in I}, \{\mathrm{wt}_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ with a free pre-crystal structure (see [1, 2.2]) and denote it by $\mathcal{UD}_{\theta,T'}(\chi)$. We have the following theorem:

Theorem 2.6 ([1, 12]). *For any geometric crystal $\chi = (X, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ and positive structure $\theta : T' \to X$, the associated pre-crystal $\mathcal{UD}_{\theta,T'}(\chi) = (X_*(T'), \{e_i\}_{i\in I}, \{\mathrm{wt}_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ is a crystal* (see [1, 2.2]).

Now, let $\mathcal{GC}^+$ be a category whose object is a triplet (χ, T', θ) where $\chi = (X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ is a geometric crystal and $\theta : T' \to X$ is a positive structure on χ, and whose morphism $f : (\chi_1, T'_1, \theta_1) \longrightarrow (\chi_2, T'_2, \theta_2)$ is given by a morphism $\varphi : X_1 \longrightarrow X_2$ ($\chi_i = (X_i, \ldots)$) such that

$$f := \theta_2^{-1} \circ \varphi \circ \theta_1 : T'_1 \longrightarrow T'_2,$$

is a positive rational morphism. Let $\mathcal{CR}$ be a category of crystals. Then by the theorem above, we have

Corollary 2.7. *$\mathcal{UD}_{\theta,T'}$ as above defines a functor*

$$\begin{array}{cccc} \mathcal{UD}: & \mathcal{GC}^+ & \longrightarrow & \mathcal{CR}, \\ & (\chi, T', \theta) & \mapsto & X_*(T'), \\ & (f : (\chi_1, T_1', \theta_1) \to (\chi_2, T_2', \theta_2)) & \mapsto & (\widehat{f} : X_*(T_1') \to X_*(T_2')). \end{array}$$

We call the functor $\mathcal{UD}$ *ultra-discretization* as [12, 13] instead of "tropicalization" as in [1]. And for a crystal B, if there exists a geometric crystal χ and a positive structure $\theta : T' \to X$ on χ such that $\mathcal{UD}(\chi, T', \theta) \cong B$ as crystals, we call an object (χ, T', θ) in $\mathcal{GC}^+$ a *tropicalization* of B, where it is not known that this correspondence is a functor.

3 Limit of Perfect Crystals

We review limit of perfect crystals following [6]. (See also [4, 5]).

3.1 Crystals

First, we review the theory of crystals, which is the notion obtained by abstracting the combinatorial properties of crystal bases. Let $(A, \{\alpha_i\}_{i\in I}, \{\alpha_i^\vee\}_{i\in I})$ be a Cartan data.

Definition 3.1. A *crystal* B is a set endowed with the following maps:

$$\begin{aligned} &\mathrm{wt} : B \longrightarrow P, \\ &\varepsilon_i : B \longrightarrow \mathbb{Z} \sqcup \{-\infty\}, \quad \varphi_i : B \longrightarrow \mathbb{Z} \sqcup \{-\infty\} \quad \text{for} \quad i \in I, \\ &\tilde{e}_i : B \sqcup \{0\} \longrightarrow B \sqcup \{0\}, \quad \tilde{f}_i : B \sqcup \{0\} \longrightarrow B \sqcup \{0\} \quad \text{for} \quad i \in I, \\ &\tilde{e}_i(0) = \tilde{f}_i(0) = 0. \end{aligned}$$

Those maps satisfy the following axioms: for all $b, b_1, b_2 \in B$, we have

$$\begin{aligned} &\varphi_i(b) = \varepsilon_i(b) + \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle, \\ &\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i \text{ if } \tilde{e}_i b \in B, \\ &\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i \text{ if } \tilde{f}_i b \in B, \\ &\tilde{e}_i b_2 = b_1 \Longleftrightarrow \tilde{f}_i b_1 = b_2 \ (b_1, b_2 \in B), \\ &\varepsilon_i(b) = -\infty \Longrightarrow \tilde{e}_i b = \tilde{f}_i b = 0. \end{aligned}$$

The following tensor product structure is one of the most crucial properties of crystals.

Theorem 3.2. *Let B_1 and B_2 be crystals. Set $B_1 \otimes B_2 := \{b_1 \otimes b_2;\ b_j \in B_j\ (j = 1, 2)\}$. Then we have*

(i) *$B_1 \otimes B_2$ is a crystal.*
(ii) *For $b_1 \in B_1$ and $b_2 \in B_2$, we have*

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \le \varepsilon_i(b_2), \end{cases}$$

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \\ \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \ge \varepsilon_i(b_2). \end{cases}$$

Definition 3.3. Let B_1 and B_2 be crystals. A *strict morphism* of crystals $\psi : B_1 \longrightarrow B_2$ is a map $\psi : B_1 \sqcup \{0\} \longrightarrow B_2 \sqcup \{0\}$ satisfying: $\psi(0) = 0$, $\psi(B_1) \subset B_2$, ψ commutes with all $\tilde{e}_i$ and $\tilde{f}_i$ and

$$\mathrm{wt}(\psi(b)) = \mathrm{wt}(b), \quad \varepsilon_i(\psi(b)) = \varepsilon_i(b), \quad \varphi_i(\psi(b)) = \varphi_i(b) \text{ for any } b \in B_1.$$

In particular, a bijective strict morphism is called an *isomorphism of crystals*.

Example 3.4. If (L, B) is a crystal base, then B is a crystal. Hence, for the crystal base $(L(\infty), B(\infty))$ of the nilpotent subalgebra $U_q^-(\mathfrak{g})$ of the quantum algebra $U_q(\mathfrak{g})$, $B(\infty)$ is a crystal.

Example 3.5. For $\lambda \in P$, set $T_\lambda := \{t_\lambda\}$. We define a crystal structure on T_λ by

$$\tilde{e}_i(t_\lambda) = \tilde{f}_i(t_\lambda) = 0, \quad \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty, \quad \mathrm{wt}(t_\lambda) = \lambda.$$

Definition 3.6. For a crystal B, a colored oriented graph structure is associated with B by

$$b_1 \xrightarrow{i} b_2 \Longleftrightarrow \tilde{f}_i b_1 = b_2.$$

We call this graph a *crystal graph* of B.

3.2 Affine Weights

Let $\mathfrak{g}$ be an affine Lie algebra. The sets $\mathfrak{t}$, $\{\alpha_i\}_{i \in I}$ and $\{\alpha_i^\vee\}_{i \in I}$ be as in Sect. 2.1. We take $\dim \mathfrak{t} = \sharp I + 1$. Let $\delta \in Q_+$ be the unique element satisfying $\{\lambda \in Q \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \text{ for any } i \in I\} = \mathbb{Z}\delta$, and $\mathbf{c} \in \mathfrak{g}$ be the canonical central element satisfying $\{h \in Q^\vee \mid \langle h, \alpha_i \rangle = 0 \text{ for any } i \in I\} = \mathbb{Z}c$. We write [2, 6.1]

$$\mathbf{c} = \sum_i a_i^\vee \alpha_i^\vee, \qquad \delta = \sum_i a_i \alpha_i.$$

Let $(\ ,\ )$ be the non-degenerate W-invariant symmetric bilinear form on $\mathfrak{t}^*$ normalized by $(\delta, \lambda) = \langle \mathbf{c}, \lambda \rangle$ for $\lambda \in \mathfrak{t}^*$. Let us set $\mathfrak{t}^*_{\mathrm{cl}} := \mathfrak{t}^*/\mathbb{C}\delta$ and let $\mathrm{cl} : \mathfrak{t}^* \longrightarrow \mathfrak{t}^*_{\mathrm{cl}}$

be the canonical projection. Here we have $\mathfrak{t}_{\mathrm{cl}}^* \cong \bigoplus_i (\mathbb{C}\alpha_i^\vee)^*$. Set $\mathfrak{t}_0^* := \{\lambda \in \mathfrak{t}^* \mid \langle \mathbf{c}, \lambda\rangle = 0\}$, $(\mathfrak{t}_{\mathrm{cl}}^*)_0 := \mathrm{cl}(\mathfrak{t}_0^*)$. Since $(\delta, \delta) = 0$, we have a positive-definite symmetric form on $\mathfrak{t}_{\mathrm{cl}}^*$ induced by the one on $\mathfrak{t}^*$. Let $\Lambda_i \in \mathfrak{t}_{\mathrm{cl}}^*$ $(i \in I)$ be a classical weight such that $\langle \alpha_i^\vee, \Lambda_j\rangle = \delta_{i,j}$, which is called a fundamental weight. We choose P so that $P_{\mathrm{cl}} := \mathrm{cl}(P)$ coincides with $\bigoplus_{i\in I} \mathbb{Z}\Lambda_i$ and we call P_{cl} a *classical weight lattice*.

3.3 Definitions of Perfect Crystal and Its Limit

Let $\mathfrak{g}$ be an affine Lie algebra, P_{cl} be a classical weight lattice as above and set $(P_{\mathrm{cl}})_l^+ := \{\lambda \in P_{\mathrm{cl}} \mid \langle c, \lambda\rangle = l,\ \langle \alpha_i^\vee, \lambda\rangle \geq 0\}$ $(l \in \mathbb{Z}_{>0})$.

Definition 3.7. A crystal B is a *perfect* of level l if

(i) $B \otimes B$ is connected as a crystal graph.
(ii) There exists $\lambda_0 \in P_{\mathrm{cl}}$ such that

$$\mathrm{wt}(B) \subset \lambda_0 + \sum_{i\neq 0} \mathbb{Z}_{\leq 0}\mathrm{cl}(\alpha_i), \qquad \sharp B_{\lambda_0} = 1.$$

(iii) There exists a finite-dimensional $U_q'(\mathfrak{g})$-module V with a crystal pseudo-base B_{ps} such that $B \cong B_{\mathrm{ps}}/\pm 1$.
(iv) The maps $\varepsilon, \varphi : B^{\min} := \{b \in B \mid \langle c, \varepsilon(b)\rangle = l\} \longrightarrow (P_{\mathrm{cl}}^+)_l$ are bijective, where $\varepsilon(b) := \sum_i \varepsilon_i(b)\Lambda_i$ and $\varphi(b) := \sum_i \varphi_i(b)\Lambda_i$.

Let $\{B_l\}_{l\geq 1}$ be a family of perfect crystals of level l and set $J := \{(l, b) \mid l > 0, b \in B_l^{\min}\}$.

Definition 3.8. A crystal B_∞ with an element b_∞ is called a *limit of* $\{B_l\}_{l\geq 1}$ if

(i) $\mathrm{wt}(b_\infty) = \varepsilon(b_\infty) = \varphi(b_\infty) = 0$.
(ii) For any $(l, b) \in J$, there exists an embedding of crystals:

$$\begin{aligned} f_{(l,b)} : T_{\varepsilon(b)} \otimes B_l \otimes T_{-\varphi(b)} &\hookrightarrow B_\infty \\ t_{\varepsilon(b)} \otimes b \otimes t_{-\varphi(b)} &\mapsto b_\infty \end{aligned}$$

(iii) $B_\infty = \bigcup_{(l,b)\in J} \mathrm{Im}\, f_{(l,b)}$.

As for the crystal T_λ, see Example 3.5. If a limit exists for a family $\{B_l\}$, we say that $\{B_l\}$ is a *coherent family* of perfect crystals.

The following is one of the most important properties of limit of perfect crystals.

Proposition 3.9. *Let $B(\infty)$ be the crystal as in Example 3.4. Then we have the following isomorphism of crystals:*

$$B(\infty) \otimes B_\infty \xrightarrow{\sim} B(\infty).$$

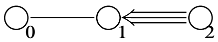
4 Perfect Crystals of Type $D_4^{(3)}$

In this section, we review the family of perfect crystals of type $D_4^{(3)}$ and its limit [9].

We fix the data for $D_4^{(3)}$. Let $\{\alpha_0, \alpha_1, \alpha_2\}$, $\{\alpha_0^\vee, \alpha_1^\vee, \alpha_2^\vee\}$ and $\{\Lambda_0, \Lambda_1, \Lambda_2\}$ be the set of simple roots, simple coroots and fundamental weights, respectively. The Cartan matrix $A = (a_{ij})_{i,j=0,1,2}$ is given by

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix},$$

and its Dynkin diagram is as follows:



The standard null root δ and the canonical central element c are given by

$$\delta = \alpha_0 + 2\alpha_1 + \alpha_2 \quad \text{and} \quad c = \alpha_0^\vee + 2\alpha_1^\vee + 3\alpha_2^\vee,$$

where $\alpha_0 = 2\Lambda_0 - \Lambda_1 + \delta, \quad \alpha_1 = -\Lambda_0 + 2\Lambda_1 - \Lambda_2, \quad \alpha_2 = -3\Lambda_1 + 2\Lambda_2.$

For a positive integer l, we introduce $D_4^{(3)}$-crystals B_l and B_∞ as

$$B_l = \left\{ b = (b_1, b_2, b_3, \bar{b}_3, \bar{b}_2, \bar{b}_1) \in (\mathbb{Z}_{\geq 0})^6 \;\middle|\; \begin{array}{l} b_3 \equiv \bar{b}_3 \ (\mathrm{mod}\ 2), \\ \sum_{i=1,2}(b_i + \bar{b}_i) + \frac{b_3 + \bar{b}_3}{2} \leq l \end{array} \right\},$$

$$B_\infty = \left\{ b = (b_1, b_2, b_3, \bar{b}_3, \bar{b}_2, \bar{b}_1) \in (\mathbb{Z})^6 \;\middle|\; \begin{array}{l} b_3 \equiv \bar{b}_3 \ (\mathrm{mod}\ 2), \\ \sum_{i=1,2}(b_i + \bar{b}_i) + \frac{b_3 + \bar{b}_3}{2} \in \mathbb{Z} \end{array} \right\}.$$

Now we describe the explicit crystal structures of B_l and B_∞. Indeed, most of them coincide with each other except for ε_0 and φ_0. In the rest of this section, we use the following convention: $(x)_+ = \max(x, 0)$,

$$\tilde{e}_1 b = \begin{cases} (\ldots, \bar{b}_2 + 1, \bar{b}_1 - 1) & \text{if } \bar{b}_2 - \bar{b}_3 \geq (b_2 - b_3)_+, \\ (\ldots, b_3 + 1, \bar{b}_3 - 1, \ldots) & \text{if } \bar{b}_2 - \bar{b}_3 < 0 \leq b_3 - b_2, \\ (b_1 + 1, b_2 - 1, \ldots) & \text{if } (\bar{b}_2 - \bar{b}_3)_+ < b_2 - b_3, \end{cases}$$

$$\tilde{f}_1 b = \begin{cases} (b_1 - 1, b_2 + 1, \ldots) & \text{if } (\bar{b}_2 - \bar{b}_3)_+ \leq b_2 - b_3, \\ (\ldots, b_3 - 1, \bar{b}_3 + 1, \ldots) & \text{if } \bar{b}_2 - \bar{b}_3 \leq 0 < b_3 - b_2, \\ (\ldots, \bar{b}_2 - 1, \bar{b}_1 + 1) & \text{if } \bar{b}_2 - \bar{b}_3 > (b_2 - b_3)_+, \end{cases}$$

$$\tilde{e}_2 b = \begin{cases} (\ldots, \bar{b}_3 + 2, \bar{b}_2 - 1, \ldots) & \text{if } \bar{b}_3 \geq b_3, \\ (\ldots, b_2 + 1, b_3 - 2, \ldots) & \text{if } \bar{b}_3 < b_3, \end{cases}$$

$$\tilde{f}_2 b = \begin{cases} (\ldots, b_2 - 1, b_3 + 2, \ldots) & \text{if } \bar{b}_3 \le b_3, \\ (\ldots, \bar{b}_3 - 2, \bar{b}_2 + 1, \ldots) & \text{if } \bar{b}_3 > b_3, \end{cases}$$

$$\varepsilon_1(b) = \bar{b}_1 + (\bar{b}_3 - \bar{b}_2 + (b_2 - b_3)_+)_+, \quad \varphi_1(b) = b_1 + (b_3 - b_2 + (\bar{b}_2 - \bar{b}_3)_+)_+,$$

$$\varepsilon_2(b) = \bar{b}_2 + \frac{1}{2}(b_3 - \bar{b}_3)_+, \qquad \varphi_2(b) = b_2 + \frac{1}{2}(\bar{b}_3 - b_3)_+,$$

$$\varepsilon_0(b) = \begin{cases} l - s(b) + \max A - (2z_1 + z_2 + z_3 + 3z_4) & b \in B_l, \\ -s(b) + \max A - (2z_1 + z_2 + z_3 + 3z_4) & b \in B_\infty, \end{cases}$$

$$\varphi_0(b) = \begin{cases} l - s(b) + \max A & b \in B_l, \\ -s(b) + \max A & b \in B_\infty, \end{cases}$$

where

$$s(b) = b_1 + b_2 + \frac{b_3 + \bar{b}_3}{2} + \bar{b}_2 + \bar{b}_1, \tag{4.1}$$

$$z_1 = \bar{b}_1 - b_1, \quad z_2 = \bar{b}_2 - \bar{b}_3, \quad z_3 = b_3 - b_2, \quad z_4 = (\bar{b}_3 - b_3)/2, \tag{4.2}$$

$$A = (0, z_1, z_1 + z_2, z_1 + z_2 + 3z_4, z_1 + z_2 + z_3 + 3z_4, 2z_1 + z_2 + z_3 + 3z_4). \tag{4.3}$$

For $b \in B_l$, if $\tilde{e}_i b$ or $\tilde{f}_i b$ does not belong to B_l, namely, if b_j or $\bar{b}_j$ for some j becomes negative, we understand it to be 0.

Let us see the actions of $\tilde{e}_0$ and $\tilde{f}_0$. We shall consider the conditions (E_1)–(E_6) and (F_1)–(F_6) [9].

$$\begin{array}{ll} (E_1) & z_1 + z_2 + z_3 + 3z_4 < 0, z_1 + z_2 + 3z_4 < 0, z_1 + z_2 < 0, z_1 < 0, \\ (E_2) & z_1 + z_2 + z_3 + 3z_4 < 0, z_2 + 3z_4 < 0, z_2 < 0, z_1 \ge 0, \\ (E_3) & z_1 + z_3 + 3z_4 < 0, z_3 + 3z_4 < 0, z_4 < 0, z_2 \ge 0, z_1 + z_2 \ge 0, \\ (E_4) & z_1 + z_2 + 3z_4 \ge 0, z_2 + 3z_4 \ge 0, z_4 \ge 0, z_3 < 0, z_1 + z_3 < 0, \\ (E_5) & z_1 + z_2 + z_3 + 3z_4 \ge 0, z_3 + 3z_4 \ge 0, z_3 \ge 0, z_1 < 0, \\ (E_6) & z_1 + z_2 + z_3 + 3z_4 \ge 0, z_1 + z_3 + 3z_4 \ge 0, z_1 + z_3 \ge 0, z_1 \ge 0. \end{array}$$

(F_i) $(1 \le i \le 6)$ is obtained from (E_i) by replacing $\ge$ (resp. $<$) with $>$ (resp. $\le$). We define

$$\tilde{e}_0 b = \begin{cases} \mathscr{E}_1 b := (b_1 - 1, \ldots) & \text{if } (E_1), \\ \mathscr{E}_2 b := (\ldots, b_3 - 1, \bar{b}_3 - 1, \ldots, \bar{b}_1 + 1) & \text{if } (E_2), \\ \mathscr{E}_3 b := (\ldots, b_3 - 2, \ldots, \bar{b}_2 + 1, \ldots) & \text{if } (E_3), \\ \mathscr{E}_4 b := (\ldots, b_2 - 1, \ldots, \bar{b}_3 + 2, \ldots) & \text{if } (E_4), \\ \mathscr{E}_5 b := (b_1 - 1, \ldots, b_3 + 1, \bar{b}_3 + 1, \ldots) & \text{if } (E_5), \\ \mathscr{E}_6 b := (\ldots, \bar{b}_1 + 1) & \text{if } (E_6), \end{cases}$$

$$\tilde{f}_0 b = \begin{cases} \mathscr{F}_1 b := (b_1+1,\ldots) & \text{if } (F_1), \\ \mathscr{F}_2 b := (\ldots, b_3+1, \bar{b}_3+1, \ldots, \bar{b}_1-1) & \text{if } (F_2), \\ \mathscr{F}_3 b := (\ldots, b_3+2, \ldots, \bar{b}_2-1, \ldots) & \text{if } (F_3), \\ \mathscr{F}_4 b := (\ldots, b_2+1, \ldots, \bar{b}_3-2, \ldots) & \text{if } (F_4), \\ \mathscr{F}_5 b := (b_1+1, \ldots, b_3-1, \bar{b}_3-1, \ldots) & \text{if } (F_5), \\ \mathscr{F}_6 b := (\ldots, \bar{b}_1-1) & \text{if } (F_6). \end{cases}$$

The following is one of the main results in [9]:

Theorem 4.1 ([9]).

(i) *The $D_4^{(3)}$-crystal B_l is a perfect crystal of level l.*
(ii) *The family of the perfect crystals $\{B_l\}_{l\geq 1}$ forms a coherent family and the crystal B_∞ is its limit with the vector $b_\infty = (0,0,0,0,0,0)$.*

As was shown in [9], the minimal elements are given

$$(B_l)_{\min} = \{(\alpha,\beta,\beta,\beta,\beta,\alpha) \mid \alpha,\beta \in \mathbb{Z}_{\geq 0}, 2\alpha+3\beta \leq l\}.$$

Let $J = \{(l,b) \mid l \in \mathbb{Z}_{\geq 1}, b \in (B_l)_{\min}\}$ and the maps $\varepsilon, \varphi : (B_l)_{\min} \to (P_{\mathrm{cl}}^+)_l$ be as in Sect. 3. Then we have $\mathrm{wt}\, b_\infty = 0$ and $\varepsilon_i(b_\infty) = \varphi_i(b_\infty) = 0$ for $i = 0,1,2$.

For $(l,b_0) \in J$, since $\varepsilon(b_0) = \varphi(b_0)$, one can set $\lambda = \varepsilon(b_0) = \varphi(b_0)$. For $b = (b_1,b_2,b_3,\bar{b}_3,\bar{b}_2,\bar{b}_1) \in B_l$, we define a map

$$f_{(l,b_0)} : T_\lambda \otimes B_l \otimes B_{-\lambda} \longrightarrow B_\infty$$

by

$$f_{(l,b_0)}(t_\lambda \otimes b \otimes t_{-\lambda}) = b' = (\nu_1,\nu_2,\nu_3,\bar{\nu}_3,\bar{\nu}_2,\bar{\nu}_1),$$

where $b_0 = (\alpha,\beta,\beta,\beta,\beta,\alpha)$, and

$$\nu_1 = b_1 - \alpha, \qquad \bar{\nu}_1 = \bar{b}_1 - \alpha,$$
$$\nu_j = b_j - \beta, \qquad \bar{\nu}_j = \bar{b}_j - \beta \ (j = 2,3).$$

Finally, we obtain $B_\infty = \bigcup_{(l,b)\in J} \mathrm{Im}\, f_{(l,b)}$.

5 Fundamental Representation for $G_2^{(1)}$

5.1 Fundamental Representation $W(\varpi_1)$

Let $c = \sum_i a_i^\vee \alpha_i^\vee$ be the canonical central element in an affine Lie algebra $\mathfrak{g}$ (see [2, 6.1]), $\{\Lambda_i \mid i \in I\}$ the set of fundamental weight as in the previous section and $\varpi_1 := \Lambda_1 - a_1^\vee \Lambda_0$ the (level 0) fundamental weight. Let $W(\varpi_1)$ be the fundamental representation of $U_q'(\mathfrak{g})$ associated with ϖ_1 [7,8].

By [7, Theorem 5.17], $W(\varpi_1)$ is a finite-dimensional irreducible integrable $U'_q(\mathfrak{g})$-module and has a global basis with a simple crystal. Thus, we can consider the specialization $q = 1$ and obtain the finite-dimensional $\mathfrak{g}$-module $W(\varpi_1)$, which we call a fundamental representation of $\mathfrak{g}$ and use the same notation as above.

We shall present the explicit form of $W(\varpi_1)$ for $\mathfrak{g} = G_2^{(1)}$.

5.2 $W(\varpi_1)$ for $G_2^{(1)}$

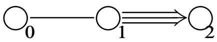
The Cartan matrix $A = (a_{i,j})_{i,j=0,1,2}$ of type $G_2^{(1)}$ is

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}.$$

Then the simple roots are

$$\alpha_0 = 2\Lambda_0 - \Lambda_1 + \delta, \quad \alpha_1 = -\Lambda_0 + 2\Lambda_1 - 3\Lambda_2, \quad \alpha_2 = -\Lambda_1 + 2\Lambda_2,$$

and the Dynkin diagram is:



The $\mathfrak{g}$-module $W(\varpi_1)$ is a 15-dimensional module with the basis,

$$\{\boxed{i}, \boxed{\bar{i}}, \emptyset, \boxed{0_1}, \boxed{0_2} \mid i = 1, \dots, 6\}.$$

The following description of $W(\varpi_1)$ slightly differs from [16]:

$$\mathrm{wt}(\boxed{1}) = \Lambda_1 - 2\Lambda_0, \ \mathrm{wt}(\boxed{2}) = -\Lambda_0 - \Lambda_1 + 3\Lambda_2, \ \mathrm{wt}(\boxed{3}) = -\Lambda_0 + \Lambda_2,$$
$$\mathrm{wt}(\boxed{4}) = -\Lambda_0 + \Lambda_1 - \Lambda_2, \ \mathrm{wt}(\boxed{5}) = -\Lambda_1 + 2\Lambda_2, \ \mathrm{wt}(\boxed{6}) = -\Lambda_0 + 2\Lambda_1 - 3\Lambda_2,$$
$$\mathrm{wt}(\boxed{\bar{i}}) = -\mathrm{wt}(\boxed{i}) \ (i = 1, \dots, 6), \ \mathrm{wt}(\boxed{0_1}) = \mathrm{wt}(\boxed{0_2}) = \mathrm{wt}(\emptyset) = 0.$$

The actions of e_i and f_i on these basis vectors are given as follows:

$$f_0\left(\boxed{0_2}, \boxed{\bar{6}}, \boxed{\bar{4}}, \boxed{\bar{3}}, \boxed{\bar{2}}, \boxed{\bar{1}}, \emptyset\right) = \left(\boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}, \boxed{6}, \emptyset, 2\boxed{1}\right),$$
$$e_0\left(\boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}, \boxed{6}, \boxed{0_2}, \emptyset\right) = \left(\emptyset, \boxed{\bar{6}}, \boxed{\bar{4}}, \boxed{\bar{3}}, \boxed{\bar{2}}, \boxed{\bar{1}}, 2\boxed{\bar{1}}\right),$$
$$f_1\left(\boxed{1}, \boxed{4}, \boxed{6}, \boxed{0_1}, \boxed{0_2}, \boxed{\bar{5}}, \boxed{\bar{2}}, \emptyset\right) = \left(\boxed{2}, \boxed{5}, \boxed{0_2}, 3\boxed{\bar{6}}, 2\boxed{\bar{6}}, \boxed{\bar{4}}, \boxed{\bar{1}}, \boxed{\bar{6}}\right),$$
$$e_1\left(\boxed{2}, \boxed{5}, \boxed{0_1}, \boxed{0_2}, \boxed{\bar{6}}, \boxed{\bar{4}}, \boxed{\bar{1}}, \emptyset\right) = \left(\boxed{1}, \boxed{4}, 3\boxed{6}, 2\boxed{6}, \boxed{0_2}, \boxed{\bar{5}}, \boxed{\bar{2}}, \boxed{6}\right),$$

$$f_2\left(\boxed{2},\boxed{3},\boxed{4},\boxed{5},\boxed{0_1},\boxed{0_2},\boxed{\bar{6}},\boxed{\bar{4}},\boxed{\bar{3}}\right)$$
$$=\left(\boxed{3},2\boxed{4},3\boxed{6},\boxed{0_1},2\boxed{\bar{5}},\boxed{\bar{5}},\boxed{\bar{4}},2\boxed{\bar{3}},3\boxed{\bar{2}}\right),$$
$$e_2\left(\boxed{3},\boxed{4},\boxed{6},\boxed{0_1},\boxed{0_2},\boxed{\bar{5}},\boxed{\bar{4}},\boxed{\bar{3}},\boxed{\bar{2}}\right)$$
$$=\left(3\boxed{2},2\boxed{3},\boxed{4},2\boxed{5},\boxed{5},\boxed{0_1},3\boxed{\bar{6}},2\boxed{\bar{4}},\boxed{\bar{3}}\right),$$

where we give non-trivial actions only.

6 Affine Geometric Crystal $\mathcal{V}_1(G_2^{(1)})$

Let us review the construction of the affine geometric crystal $\mathcal{V}(G_2^{(1)})$ in $W(\varpi_1)$ following [14].

For $\xi \in (\mathfrak{t}^*_{\mathrm{cl}})_0$, let $t(\xi)$ be the translation as in [7, Sect. 4]. Then we have

$$t(\widetilde{\varpi}_1) = s_0 s_1 s_2 s_1 s_2 s_1 =: w_1,$$
$$t(\mathrm{wt}(\boxed{\bar{2}})) = s_2 s_1 s_2 s_1 s_0 s_1 =: w_2.$$

Associated with these Weyl group elements w_1 and w_2, we define algebraic varieties $\mathcal{V}_1 = \mathcal{V}_1(G_2^{(1)})$ and $\mathcal{V}_2 = \mathcal{V}_2(G_2^{(1)}) \subset W(\varpi_1)$, respectively:

$$\mathcal{V}_1 := \{v_1(x) := Y_0(x_0)Y_1(x_1)Y_2(x_2)Y_1(x_3)Y_2(x_4)Y_1(x_5)\boxed{1} \mid x_i \in \mathbb{C}^\times, (0 \le i \le 5)\},$$
$$\mathcal{V}_2 := \{v_2(y) := Y_2(y_2)Y_1(y_1)Y_2(y_4)Y_1(y_3)Y_0(y_0)Y_1(y_5)\boxed{\bar{2}} \mid y_i \in \mathbb{C}^\times, (0 \le i \le 5)\}.$$

Owing to the explicit forms of f_i's on $W(\varpi_1)$ as above, we have $f_0^3 = 0$, $f_1^3 = 0$ and $f_2^4 = 0$ and then

$$Y_i(c) = \left(1 + \frac{f_i}{c} + \frac{f_i^2}{2c^2}\right)\alpha_i^\vee(c) \ (i = 0, 1), \quad Y_2(c) = \left(1 + \frac{f_2}{c} + \frac{f_2^2}{2c^2} + \frac{f_2^3}{6c^3}\right)\alpha_2^\vee(c).$$

We get explicit forms of $v_1(x) \in \mathcal{V}_1$ and $v_2(y) \in \mathcal{V}_2$ as in [14]:

$$v_1(x) = \sum_{1 \le i \le 6}\left(X_i\boxed{i} + X_{\bar{i}}\boxed{\bar{i}}\right) + X_{0_1}\boxed{0_1} + X_{0_2}\boxed{0_2} + X_\emptyset \emptyset,$$
$$v_2(y) = \sum_{1 \le i \le 6}\left(Y_i\boxed{i} + Y_{\bar{i}}\boxed{\bar{i}}\right) + Y_{0_1}\boxed{0_1} + Y_{0_2}\boxed{0_2} + Y_\emptyset \emptyset,$$

where the rational functions X_i's and Y_i's are all positive (as for their explicit forms, see [14]) and then we get the positive birational isomorphism $\overline{\sigma} : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ $(v_1(x) \mapsto v_2(y))$ and its inverse $\overline{\sigma}^{-1}$ is also positive. The actions of e_0^c

on $v_2(y)$ (respectively $\gamma_0(v_2(y))$ and $\varepsilon_0(v_2(y))$) are induced from the ones on $Y_2(y_2)Y_1(y_1)Y_2(y_4)Y_1(y_3)Y_0(y_0)Y_1(y_5)$ as an element of the geometric crystal $\mathcal{V}_2$. We define the action e_0^c on $v_1(x)$ by

$$e_0^c v_1(x) := \overline{\sigma}^{-1} \circ e_0^c \circ \overline{\sigma}(v_1(x)). \tag{6.1}$$

We also define $\gamma_0(v_1(x))$ and $\varepsilon_0(v_1(x))$ by

$$\gamma_0(v_1(x)) := \gamma_0(\overline{\sigma}(v_1(x))), \qquad \varepsilon_0(v_1(x)) := \varepsilon_0(\overline{\sigma}(v_1(x))). \tag{6.2}$$

Theorem 6.1 ([14]). *Together with (6.1) and (6.2) on $\mathcal{V}_1$, we obtain a positive affine geometric crystal $\chi := (\mathcal{V}_1, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ $(I = \{0,1,2\})$, whose explicit form is as follows: first we have e_i^c, γ_i and ε_i for $i = 1, 2$ from the formula (2.2)–(2.4).*

$$\begin{aligned} e_1^c(v_1(x)) &= v_1(x_0, \mathcal{C}_1 x_1, x_2, \mathcal{C}_3 x_3, x_4, \mathcal{C}_5 x_5),\\ e_2^c(v_1(x)) &= v_1(x_0, x_1, \mathcal{C}_2 x_2, x_3, \mathcal{C}_4 x_4, x_5), \end{aligned}$$

where

$$\mathcal{C}_1 = \frac{\frac{c\,x_0}{x_1} + \frac{x_0\,{x_2}^3}{{x_1}^2\,x_3} + \frac{x_0\,{x_2}^3\,{x_4}^3}{{x_1}^2\,{x_3}^2\,x_5}}{\frac{x_0}{x_1} + \frac{x_0\,{x_2}^3}{{x_1}^2\,x_3} + \frac{x_0\,{x_2}^3\,{x_4}^3}{{x_1}^2\,{x_3}^2\,x_5}}, \quad \mathcal{C}_3 = \frac{\frac{c\,x_0}{x_1} + \frac{c\,x_0\,{x_2}^3}{{x_1}^2\,x_3} + \frac{x_0\,{x_2}^3\,{x_4}^3}{{x_1}^2\,{x_3}^2\,x_5}}{\frac{c\,x_0}{x_1} + \frac{x_0\,{x_2}^3}{{x_1}^2\,x_3} + \frac{x_0\,{x_2}^3\,{x_4}^3}{{x_1}^2\,{x_3}^2\,x_5}},$$

$$\mathcal{C}_5 = \frac{c\left(\frac{x_0}{x_1} + \frac{x_0\,{x_2}^3}{{x_1}^2\,x_3} + \frac{x_0\,{x_2}^3\,{x_4}^3}{{x_1}^2\,{x_3}^2\,x_5}\right)}{\frac{c\,x_0}{x_1} + \frac{c\,x_0\,{x_2}^3}{{x_1}^2\,x_3} + \frac{x_0\,{x_2}^3\,{x_4}^3}{{x_1}^2\,{x_3}^2\,x_5}}, \mathcal{C}_2 = \frac{\frac{c\,x_1}{x_2} + \frac{x_1\,x_3}{{x_2}^2\,x_4}}{\frac{x_1}{x_2} + \frac{x_1\,x_3}{{x_2}^2\,x_4}}, \mathcal{C}_4 = \frac{c\left(\frac{x_1}{x_2} + \frac{x_1\,x_3}{{x_2}^2\,x_4}\right)}{\frac{c\,x_1}{x_2} + \frac{x_1\,x_3}{{x_2}^2\,x_4}},$$

$$\varepsilon_1(v_1(x)) = \frac{x_0}{x_1} + \frac{x_0\,{x_2}^3}{{x_1}^2\,x_3} + \frac{x_0\,{x_2}^3\,{x_4}^3}{{x_1}^2\,{x_3}^2\,x_5}, \quad \varepsilon_2(v_1(x)) = \frac{x_1}{x_2} + \frac{x_1\,x_3}{{x_2}^2\,x_4},$$

$$\gamma_1(v_1(x)) = \frac{x_1^2 x_3^2 x_5^2}{x_0 x_2^3 x_4^3}, \quad \gamma_2(v_1(x)) = \frac{x_2^2 x_4^2}{x_1 x_3 x_5}.$$

We also have e_0^c, ε_0 and γ_0 on $v_1(x)$ as:

$$e_0^c(v_1(x)) = v_1\left(\frac{D}{c\cdot E}x_0, \frac{F}{c\cdot E}x_1, \frac{G}{c\cdot E}x_2, \frac{D\cdot H}{c^2\cdot E\cdot F}x_3, \frac{D}{c\cdot G}x_4, \frac{D}{c\cdot H}x_5\right),$$

$$\varepsilon_0(v_1(x)) = \frac{E}{{x_0}^3\,{x_2}^3\,x_3}, \qquad \gamma_0(v_1(x)) = \frac{x_0^2}{x_1 x_3 x_5},$$

where

$$\begin{aligned} D = {}& c^2\,{x_0}^2\,{x_2}^3\,x_3 + x_1\,{x_2}^3\,{x_3}^2\,x_5 + c\,x_0\Big(x_1\,{x_3}^3 + 3\,x_1\,x_2\,{x_3}^2\,x_4\\ &+3\,x_1\,{x_2}^2\,x_3\,{x_4}^2 + {x_2}^3\left({x_3}^2 + x_1\,{x_4}^3 + x_1\,x_3\,x_5\right)\Big), \end{aligned}$$

$$
\begin{aligned}
E &= {x_0}^2 {x_2}^3 x_3 + x_1 {x_2}^3 {x_3}^2 x_5 + x_0 \Big(x_1 {x_3}^3 + 3 x_1 x_2 {x_3}^2 x_4 \\
&\quad + 3 x_1 {x_2}^2 x_3 {x_4}^2 + {x_2}^3 \left({x_3}^2 + x_1 {x_4}^3 + x_1 x_3 x_5 \right) \Big), \\
F &= c {x_0}^2 {x_2}^3 x_3 + x_1 {x_2}^3 {x_3}^2 x_5 + x_0 \Big(c x_1 {x_3}^3 + 3 c x_1 x_2 {x_3}^2 x_4 \\
&\quad + 3 c x_1 {x_2}^2 x_3 {x_4}^2 + {x_2}^3 \Big({x_3}^2 + c x_1 {x_4}^3 + c x_1 x_3 x_5 \Big) \Big), \\
G &= c {x_0}^2 {x_2}^3 x_3 + x_1 {x_2}^3 {x_3}^2 x_5 + x_0 \Big(x_1 {x_3}^3 + (2+c)\, x_1 x_2 {x_3}^2 x_4 \\
&\quad + (1+2c)\, x_1 {x_2}^2 x_3 {x_4}^2 + {x_2}^3 \left({x_3}^2 + c x_1 {x_4}^3 + c x_1 x_3 x_5 \right) \Big), \\
H &= c {x_0}^2 {x_2}^3 x_3 + x_1 {x_2}^3 {x_3}^2 x_5 + x_0 \Big(x_1 {x_3}^3 + 3 x_1 x_2 {x_3}^2 x_4 \\
&\quad + 3 x_1 {x_2}^2 x_3 {x_4}^2 + {x_2}^3 \left({x_3}^2 + x_1 {x_4}^3 + c x_1 x_3 x_5 \right) \Big).
\end{aligned}
$$

7 Ultra-Discretization

We denote the positive structure on χ as in the previous section by $\theta : T' := (\mathbb{C}^\times)^6 \longrightarrow \mathcal{V}_1$. Then by Corollary 2.7, we obtain the ultra-discretization $\mathcal{UD}(\chi, T', \theta)$, which is a Kashiwara's crystal. Now we show that the conjecture in [14] is correct and it turns out to be the following theorem.

Theorem 7.1. *The crystal $\mathcal{UD}(\chi, T', \theta)$ as above is isomorphic to the crystal B_∞ of type $D_4^{(3)}$ as in Sect. 4.*

In order to show the theorem, we shall see the explicit crystal structure on $\mathcal{X} := \mathcal{UD}(\chi, T', \theta)$. Note that $\mathcal{UD}(\chi) = \mathbb{Z}^6$ as a set. Here as for variables in $\mathcal{X}$, we use the same notations $c, x_0, x_1, \ldots, x_5$ as for χ.

For $x = (x_0, x_1, \ldots, x_5) \in \mathcal{X}$, it follows from the results in the previous section that the functions wt_i and ε_i $(i = 0, 1, 2)$ are given as

$$
\begin{aligned}
&\mathrm{wt}_0(x) = 2x_0 - x_1 - x_3 - x_5, \;\; \mathrm{wt}_1(x) = 2(x_1 + x_3 + x_5) - x_0 - 3x_2 - 3x_4, \\
&\mathrm{wt}_2(x) = 2(x_2 + x_4) - x_1 - x_3 - x_5.
\end{aligned}
$$

Set

$$
\begin{aligned}
&\alpha := 2x_0 + 3x_2 + x_3, \;\; \beta := x_1 + 3x_2 + 2x_3 + x_5, \;\; \gamma := x_0 + x_1 + 3x_3, \\
&\delta := x_0 + x_1 + x_2 + 2x_3 + x_4, \;\; \epsilon := x_0 + x_1 + 2x_2 + x_3 + 2x_4, \\
&\phi := x_0 + 3x_2 + 2x_3, \;\; \psi := x_0 + x_1 + 3x_2 + 3x_4, \\
&\xi := x_0 + x_1 + 3x_2 + x_3 + x_5. \qquad (7.1)
\end{aligned}
$$

Indeed, from the explicit form of E as in the previous section, we have

$$
\mathcal{UD}(E) = \max(\alpha, \beta, \gamma, \delta, \epsilon, \phi, \psi, \xi),
$$

and then

$$\begin{aligned}
\varepsilon_0(x) &= \max(\alpha,\beta,\gamma,\delta,\epsilon,\phi,\psi,\xi) - (3x_0 + 3x_2 + x_3),\\
\varepsilon_1(x) &= \max(x_0-x_1, x_0+3x_2-2x_1-x_3, x_0+3x_2+3x_4-2x_1-2x_3-x_5),\\
\varepsilon_2(x) &= \max(x_1 - x_2, x_1 + x_3 - 2x_2 - x_4).
\end{aligned} \tag{7.2}$$

Next, we describe the actions of $\tilde{e}_i$ $(i = 0, 1, 2)$. Set $\Xi_j := \mathcal{UD}(\mathcal{C}_j)|_{c=1}$ $(j = 1, \dots, 5)$. Then we have

$$\begin{aligned}
\Xi_1 &= \max(1+x_0-x_1, x_0+3x_2-2x_1-x_3, x_0+3x_2+3x_4-2x_1-2x_3-x_5)\\
&\quad - \max(x_0-x_1, x_0+3x_2-2x_1-x_3, x_0+3x_2+3x_4-2x_1-2x_3-x_5),\\
\Xi_3 &= \max(1+x_0-x_1, 1+x_0+3x_2-2x_1-x_3, x_0+3x_2+3x_4-2x_1-2x_3-x_5)\\
&\quad - \max(1+x_0-x_1, x_0+3x_2-2x_1-x_3, x_0+3x_2+3x_4-2x_1-2x_3-x_5),\\
\Xi_5 &= \max(1+x_0-x_1, 1+x_0+3x_2-2x_1-x_3, 1+x_0+3x_2+3x_4-2x_1-2x_3-x_5)\\
&\quad - \max(1+x_0-x_1, 1+x_0+3x_2-2x_1-x_3, x_0+3x_2+3x_4-2x_1-2x_3-x_5),\\
\Xi_2 &= \max(1+x_1-x_2, x_1+x_3-2x_2-x_4) - \max(x_1-x_2, x_1+x_3-2x_2-x_4),\\
\Xi_4 &= \max(1+x_1-x_2, 1+x_1+x_3-2x_2-x_4) - \max(1+x_1-x_2, x_1+x_3-2x_2-x_4).
\end{aligned}$$

Therefore, for $x \in \mathcal{X}$ we have

$$\begin{aligned}
\tilde{e}_1(x) &= (x_0, x_1 + \Xi_1, x_2, x_3 + \Xi_3, x_4, x_5 + \Xi_5),\\
\tilde{e}_2(x) &= (x_0, x_1, x_2 + \Xi_2, x_3, x_4 + \Xi_4, x_5).
\end{aligned}$$

We obtain the action $\tilde{f}_i$ $(i = 1, 2)$ by setting $c = -1$ in $\mathcal{UD}(\mathcal{C}_i)$.

Finally, we describe the action of $\tilde{e}_0$. Set

$$\begin{aligned}
\Psi_0 &:= \max(2+\alpha, \beta, 1+\gamma, 1+\delta, 1+\epsilon, 1+\phi, 1+\psi, 1+\xi)\\
&\quad - \max(\alpha,\beta,\gamma,\delta,\epsilon,\phi,\psi,\xi) - 1,\\
\Psi_1 &:= \max(1+\alpha, \beta, 1+\gamma, 1+\delta, 1+\epsilon, \phi, 1+\psi, 1+\xi)\\
&\quad - \max(\alpha,\beta,\gamma,\delta,\epsilon,\phi,\psi,\xi) - 1,\\
\Psi_2 &:= \max(1+\alpha, \beta, \gamma, 1+\delta, 1+\epsilon, \phi, 1+\psi, 1+\xi)\\
&\quad - \max(\alpha,\beta,\gamma,\delta,\epsilon,\phi,\psi,\xi) - 1,\\
\Psi_3 &:= \max(2+\alpha, \beta, 1+\gamma, 1+\delta, 1+\epsilon, 1+\phi, 1+\psi, 1+\xi)\\
&\quad + \max(1+\alpha, \beta, \gamma, \delta, \epsilon, \phi, \psi, 1+\xi) - \max(1+\alpha, \beta, \gamma, \delta, \epsilon, \phi, \psi, 1+\xi)\\
&\quad - \max(1+\alpha, \beta, \gamma, 1+\delta, 1+\epsilon, \phi, 1+\psi, 1+\xi) - 2,\\
\Psi_4 &:= \max(2+\alpha, \beta, 1+\gamma, 1+\delta, 1+\epsilon, 1+\phi, 1+\psi, 1+\xi)\\
&\quad - \max(1+\alpha, \beta, \gamma, 1+\delta, 1+\epsilon, \phi, 1+\psi, 1+\xi) - 1,\\
\Psi_5 &:= \max(2+\alpha, \beta, 1+\gamma, 1+\delta, 1+\epsilon, 1+\phi, 1+\psi, 1+\xi)\\
&\quad - \max(1+\alpha, \beta, \gamma, \delta, \epsilon, \phi, \psi, 1+\xi) - 1,
\end{aligned}$$

where $\alpha, \beta, \ldots, \xi$ are as in (7.1). Therefore, by the explicit form of e_0^c as in the previous section, we have

$$\tilde{e}_0(x) = (x_0 + \Psi_0, x_1 + \Psi_1, x_2 + \Psi_2, x_3 + \Psi_3, x_4 + \Psi_4, x_5 + \Psi_5). \tag{7.3}$$

Now, let us show the theorem.

Proof of Theorem 7.1. Define the map

$$\begin{array}{rccc} \Omega: & \mathcal{X} & \longrightarrow & B_\infty, \\ & (x_0, \ldots, x_5) & \mapsto & (b_1, b_2, b_3, \overline{b}_3, \overline{b}_2, \overline{b}_1), \end{array}$$

by

$$b_1 = x_5,\ b_2 = x_4 - x_5,\ b_3 = x_3 - 2x_4,\ \overline{b}_3 = 2x_2 - x_3,\ \overline{b}_2 = x_1 - x_2,\ \overline{b}_1 = x_0 - x_1,$$

and Ω^{-1} is given by

$$x_0 = b_1 + b_2 + \frac{b_3 + \overline{b}_3}{2} + \overline{b}_2 + \overline{b}_1, \quad x_1 = b_1 + b_2 + \frac{b_3 + \overline{b}_3}{2} + \overline{b}_2,$$
$$x_2 = b_1 + b_2 + \frac{b_3 + \overline{b}_3}{2},\ x_3 = 2b_1 + 2b_2 + b_3,\ x_4 = b_1 + b_2,\ x_5 = b_1,$$

which means that Ω is bijective. Here note that $\frac{b_3 + \overline{b}_3}{2} \in \mathbb{Z}$ by the definition of B_∞. We shall show that Ω is commutative with actions of $\tilde{e}_i$ and preserves the functions wt_i and ε_i, that is,

$$\tilde{e}_i(\Omega(x)) = \Omega(\tilde{e}_i x), \quad \mathrm{wt}_i(\Omega(x)) = \mathrm{wt}_i(x), \quad \varepsilon_i(\Omega(x)) = \varepsilon_i(x) \quad (i = 0, 1, 2).$$

First, let us check wt_i: Set $b = \Omega(x)$. By the explicit forms of wt_i on $\mathcal{X}$ and B_∞, we have

$$\begin{aligned}
\mathrm{wt}_0(\Omega(x)) &= \varphi_0(\Omega(x)) - \varepsilon_0(\Omega(x)) = 2z_1 + z_2 + z_3 + 3z_4 \\
&= 2(\overline{b}_1 - b_1) + (\overline{b}_2 - \overline{b}_3) + (b_3 - b_2) + \frac{3}{2}(\overline{b}_3 - b_3) \\
&= 2(\overline{b}_1 - b_1) + \overline{b}_2 - b_2 + \frac{\overline{b}_3 - b_3}{2} = 2x_0 - x_1 - x_3 - x_5 = \mathrm{wt}_0(x), \\
\mathrm{wt}_1(\Omega(x)) &= \varphi_1(\Omega(x)) - \varepsilon_1(\Omega(x)) \\
&= b_1 + (b_3 - b_2 + (\bar{b}_2 - \bar{b}_3)_+)_+ - (\bar{b}_1 + (\bar{b}_3 - \bar{b}_2 - (b_2 - b_3)_+)_+) \\
&= b_1 - \overline{b}_1 - b_2 + \overline{b}_2 + b_3 - \overline{b}_3 = 2(x_1 + x_3 + x_5) - x_0 - 3x_2 - 3x_4 \\
&= \mathrm{wt}_1(x), \\
\mathrm{wt}_2(\Omega(x)) &= \varphi_2(\Omega(x)) - \varepsilon_2(\Omega(x)) = b_2 + \frac{1}{2}(\bar{b}_3 - b_3)_+ - \bar{b}_2 + \frac{1}{2}(b_3 - \bar{b}_3)_+ \\
&= b_2 - \overline{b}_2 + \frac{1}{2}(\overline{b}_3 - b_3) = 2(x_2 + x_4) - x_1 - x_3 - x_5 = \mathrm{wt}_2(x).
\end{aligned}$$

Next, we shall check ε_i:

$$\begin{aligned}
\varepsilon_1(\Omega(x)) &= \bar{b}_1 + (\bar{b}_3 - \bar{b}_2 + (b_2 - b_3)_+)_+ \\
&= \max(\bar{b}_1, \bar{b}_1 + \bar{b}_3 - \bar{b}_2, \bar{b}_1 + \bar{b}_3 - \bar{b}_2 + b_2 - b_3) \\
&= \max(x_0 - x_1, x_0 + 3x_2 - 2x_1 - x_3, x_0 + 3x_2 + 3x_4 - 2x_1 - 2x_3 - x_5) \\
&= \varepsilon_1(x), \\
\varepsilon_2(\Omega(x)) &= \bar{b}_2 + \frac{1}{2}(b_3 - \bar{b}_3)_+ = \max(\bar{b}_2, \bar{b}_2 + \frac{1}{2}(b_3 - \bar{b}_3)_+) \\
&= \max(x_1 - x_2, x_1 + x_3 - 2x_2 - x_4) = \varepsilon_2(x).
\end{aligned}$$

Before checking $\varepsilon_0(\Omega(x)) = \varepsilon_0(x)$, we see the following formula, which has been given in [12, Sect. 6].

Lemma 7.2. *For $m_1, \dots, m_k \in \mathbb{R}$ and $t_1, \dots, t_k \in \mathbb{R}_{\geq 0}$ such that $t_1 + \cdots + t_k = 1$, we have*

$$\max\left(m_1, \dots, m_k, \sum_{i=1}^{k} t_i m_i\right) = \max(m_1, \dots, m_k).$$

By the facts

$$\delta = \frac{2\gamma + \psi}{3}, \quad \epsilon = \frac{\gamma + 2\psi}{3}, \tag{7.4}$$

and Lemma 7.2, we have

$$\max(\alpha, \beta, \gamma, \delta, \epsilon, \phi, \psi, \xi) = \max(\alpha, \beta, \gamma, \phi, \psi, \xi). \tag{7.5}$$

Here let us see ε_0:

$$\begin{aligned}
\varepsilon_0(\Omega(x)) &= -s(b) + \max A - (2z_1 + z_2 + z_3 + 3z_4) \\
&= -x_0 + \max(0, z_1, z_1 + z_2, z_1 + z_2 + 3z_4, \\
&\qquad z_1 + z_2 + z_3 + 3z_4, 2z_1 + z_2 + z_3 + 3z_4) - (\alpha - \beta) \\
&= -x_0 + \max(-2x_0 + x_1 + x_3 + x_5, -x_0 + x_3, \\
&\quad - x_0 + x_1 - 3x_2 + 2x_3, -x_0 + x_1 - x_3 + 3x_4, -x_0 + x_1 + x_5, 0) \\
&= -(3x_0 + 3x_2 + x_3) + \max(x_1 + 3x_2 + 2x_3 + x_5, \\
&\qquad x_0 + 3x_2 + 2x_3, x_0 + x_1 + 3x_3, x_0 + x_1 + 3x_2 + 3x_4, \\
&\qquad x_0 + x_1 + 3x_2 + x_3 + x_5, 2x_0 + 3x_2 + x_3) \\
&= -(3x_0 + 3x_2 + x_3) + \max(\beta, \phi, \gamma, \psi, \xi, \alpha).
\end{aligned}$$

On the other hand, we have

$$\varepsilon_0(x) = -(3x_0 + 3x_2 + x_3) + \max(\alpha, \beta, \gamma, \delta, \epsilon, \phi, \psi, \xi).$$

Then by (7.5), we get $\varepsilon_0(\Omega(x)) = \varepsilon_0(x)$.

Let us show $\tilde{e}_i(\Omega(x)) = \Omega(\tilde{e}_i(x))$ $(x \in \mathcal{X},\ i = 0, 1, 2)$. As for $\tilde{e}_1$, set

$$A = x_0 - x_1,\ \ B = x_0 + 3x_2 - 2x_1 - x_3,\ \ C = x_0 + 3x_2 + 3x_4 - 2x_1 - 2x_3 - x_5.$$

Then we obtain $\Xi_1 = \max(A+1, B, C) - \max(A, B, C)$, $\Xi_3 = \max(A+1, B+1, C) - \max(A+1, B, C)$, $\Xi_5 = \max(A+1, B+1, C+1) - \max(A+1, B+1, C)$. Therefore, we have

$$\begin{aligned}
&\Xi_1 = 1,\ \ \Xi_3 = 0,\ \ \Xi_5 = 0,\ \text{if } A \geq B, C\\
&\Xi_1 = 0,\ \ \Xi_3 = 1,\ \ \Xi_5 = 0,\ \text{if } A < B \geq C\\
&\Xi_1 = 0,\ \ \Xi_3 = 0,\ \ \Xi_5 = 1,\ \text{if } A, B < C,
\end{aligned}$$

which implies

$$\tilde{e}_1(x) = \begin{cases} (x_0, x_1 + 1, x_2, \dots, x_5) & \text{if } A \geq B, C \\ (x_0, \dots, x_3 + 1, x_4, x_5) & \text{if } A < B \geq C \\ (x_0, \dots, x_4, x_5 + 1) & \text{if } A, B < C. \end{cases}$$

Since $A = \overline{b}_1$, $B = \overline{b}_1 + \overline{b}_3 - \overline{b}_2$ and $C = \overline{b}_1 + \overline{b}_3 - \overline{b}_2 + b_2 - b_3$, we get $(b = \Omega(x))$

$$\Omega(\tilde{e}_1(x)) = \begin{cases} (\dots, \bar{b}_2 + 1, \bar{b}_1 - 1) & \text{if } \bar{b}_2 - \bar{b}_3 \geq (b_2 - b_3)_+, \\ (\dots, b_3 + 1, \bar{b}_3 - 1, \dots) & \text{if } \bar{b}_2 - \bar{b}_3 < 0 \leq b_3 - b_2, \\ (b_1 + 1, b_2 - 1, \dots) & \text{if } (\bar{b}_2 - \bar{b}_3)_+ < b_2 - b_3, \end{cases}$$

which is the same as the action of $\tilde{e}_1$ on $b = \Omega(x)$ as in Sect. 4. Hence, we have $\Omega(\tilde{e}_1(x)) = \tilde{e}_1(\Omega(x))$.

Let us see $\Omega(\tilde{e}_2(x)) = \tilde{e}_2(\Omega(x))$. Set

$$L = x_1 - x_2, \quad M := x_1 + x_3 - 2x_2 - x_4.$$

Then $\Xi_2 = \max(1+L, M) - \max(L, M)$ and $\Xi_4 = \max(1+L, 1+M) - \max(1+L, M)$. Thus, one has

$$\begin{aligned}
&\Xi_2 = 1, \quad \Xi_4 = 0 \quad \text{if } L \geq M,\\
&\Xi_2 = 0, \quad \Xi_4 = 1 \quad \text{if } L < M,
\end{aligned}$$

which means

$$\tilde{e}_2(x) = \begin{cases} (x_0, x_1, x_2 + 1, x_3, x_4, x_5) & \text{if } L \geq M, \\ (x_0, x_1, x_2, x_3, x_4 + 1, x_5) & \text{if } L < M. \end{cases}$$

Since $L - M = x_2 - x_3 + x_4 = \frac{\overline{b}_3 - b_3}{2}$, one gets

$$\Omega(\tilde{e}_2(x)) = \begin{cases} (\ldots, \overline{b}_3 + 2, \overline{b}_2 - 1, \ldots) & \text{if } \overline{b}_3 \geq b_3, \\ (\ldots, b_2 + 1, b_3 - 2, \ldots) & \text{if } \overline{b}_3 < b_3, \end{cases}$$

where $b = \Omega(x)$. This action coincides with the one of $\tilde{e}_2$ on $b \in B_\infty$ as in Sect. 4. Therefore, we get $\Omega(\tilde{e}_2(x)) = \tilde{e}_2(\Omega(x))$.

Finally, we shall check $\tilde{e}_0(\Omega(x)) = \Omega(\tilde{e}_0(x))$. For the purpose, we shall estimate the values $\Psi_0, \ldots, \Psi_5$ explicitly.

First, the following cases are investigated:

$$\begin{aligned}
&\text{(e1)} \;\; \beta > \alpha, \gamma, \delta, \epsilon, \phi, \psi, \xi, \\
&\text{(e2)} \;\; \beta \leq \phi > \alpha, \gamma, \delta, \epsilon, \psi, \xi, \\
&\text{(e3)} \;\; \beta, \phi \leq \gamma > \alpha, \delta, \epsilon, \psi, \xi, \\
&\text{(e4)} \;\; \beta, \gamma, \delta, \epsilon, \phi \leq \psi > \alpha, \xi, \\
&\text{(e4}'\text{)} \;\; \beta, \gamma, \epsilon, \phi, \psi \leq \delta > \alpha, \xi, \\
&\text{(e4}''\text{)} \;\; \beta, \gamma, \delta, \phi, \psi \leq \epsilon > \alpha, \xi, \\
&\text{(e5)} \;\; \beta, \gamma, \delta, \epsilon, \phi, \psi \leq \xi > \alpha, \\
&\text{(e6)} \;\; \alpha \geq \beta, \gamma, \delta, \epsilon, \phi, \psi, \xi.
\end{aligned}$$

It is easy to see that each of these conditions are equivalent to the conditions (E_1)–(E_6) in Sect. 4, more precisely, we have (ei) $\Leftrightarrow$ (E_i) ($i = 1, 2, \ldots, 6$), and that (e1)–(e6) cover all cases and they have no intersection. Note that the cases (e4$'$) and (e4$''$) are included in the case (e4), thanks to (7.4).

Let us show (e1) $\Leftrightarrow$ (E_1): the condition (e1) means $\beta - \alpha = -(2z_1 + z_2 + z_3 + 3z_4) > 0$, $\beta - \gamma = -(z_1 + z_2) > 0$, $\beta - \delta = -(z_1 + z_2 + z_4) > 0$, $\beta - \epsilon = -(z_1 + z_2 + 2z_4) > 0$, $\beta - \phi = -z_1 > 0$, $\beta - \psi = -(z_1 + z_2 + 3z_4) > 0$ and $\beta - \xi = -(z_1 + z_2 + z_3 + 3z_4) > 0$, which is equivalent to the condition $z_1 + z_2 < 0$, $z_1 < 0$, $z_1 + z_2 + 3z_4 < 0$ and $z_1 + z_2 + z_3 + 3z_4 < 0$. This is just the condition (E_1). Other cases are shown similarly.

Under the condition (e1) ($\Leftrightarrow$ (E_1)), we have

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = \Psi_5 = -1, \quad \Psi_3 = -2,$$

which means $\tilde{e}_0(x) = (x_0 - 1, x_1 - 1, x_2 - 1, x_3 - 2, x_4 - 1, x_5 - 1)$. Thus, we have

$$\Omega(\tilde{e}_0(x)) = (b_1 - 1, b_2, \ldots, \overline{b}_1),$$

which coincides with the action of $\tilde{e}_0$ under (E_1) in Sect. 4. Similarly, we have

$$\begin{aligned}
\text{(e2)} \Rightarrow & \; (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (0, -1, -1, -1, 0, 0) \\
\Rightarrow & \quad \tilde{e}_0(x) = (x_0, x_1 - 1, x_2 - 1, x_3 - 1, x_4, x_5) \\
\Rightarrow & \; \Omega(\tilde{e}_0(x)) = (b_1, b_2, b_3 - 1, \overline{b}_3 - 1, \overline{b}_2, \overline{b}_1 + 1),
\end{aligned}$$

which coincides with the action of $\tilde{e}_0$ under (E_2) in Sect. 4.

$$\begin{aligned}
\text{(e3)} &\Rightarrow (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (0, 0, -1, -2, 0, 0) \\
&\Rightarrow \quad \tilde{e}_0(x) = (x_0, x_1, x_2 - 1, x_3 - 2, x_4, x_5) \\
&\Rightarrow \quad \Omega(\tilde{e}_0(x)) = (b_1, b_2, b_3 - 2, \overline{b}_3, \overline{b}_2 + 1, \overline{b}_1),
\end{aligned}$$

which coincides with the action of $\tilde{e}_0$ under (E_3) in Sect. 4.

$$\begin{aligned}
\text{(e4)} &\Rightarrow (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (0, 0, 0, -2, -1, 0) \\
&\Rightarrow \quad \tilde{e}_0(x) = (x_0, x_1, x_2, x_3 - 2, x_4 - 1, x_5) \\
&\Rightarrow \quad \Omega(\tilde{e}_0(x)) = (b_1, b_2 - 1, b_3, \overline{b}_3 + 2, \overline{b}_2, \overline{b}_1),
\end{aligned}$$

which coincides with the action of $\tilde{e}_0$ under (E_4) in Sect. 4.

$$\begin{aligned}
\text{(e5)} &\Rightarrow (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (0, 0, 0, -1, -1, -1) \\
&\Rightarrow \quad \tilde{e}_0(x) = (x_0, x_1, x_2, x_3 - 1, x_4 - 1, x_5 - 1) \\
&\Rightarrow \Omega(\tilde{e}_0(x)) = (b_1 - 1, b_2, b_3 + 1, \overline{b}_3 + 1, \overline{b}_2, \overline{b}_1),
\end{aligned}$$

which coincides with the action of $\tilde{e}_0$ under (E_5) in Sect. 4.

$$\begin{aligned}
\text{(e6)} &\Rightarrow (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (1, 0, 0, 0, 0, 0) \\
&\Rightarrow \quad \tilde{e}_0(x) = (x_0 + 1, x_1, x_2, x_3, x_4, x_5) \\
&\Rightarrow \quad \Omega(\tilde{e}_0(x)) = (b_1, b_2, b_3, \overline{b}_3, \overline{b}_2, \overline{b}_1 + 1),
\end{aligned}$$

which coincides with the action of $\tilde{e}_0$ under (E_6) in Sect. 4. Now, we have $\Omega(\tilde{e}_0(x)) = \tilde{e}_0(\Omega(x))$. Therefore, the proof of Theorem 7.1 has been completed.

□

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On Hecke Algebras Associated with Elliptic Root Systems

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Abstract We define the elliptic Hecke algebras for arbitrary marked elliptic root systems in terms of the corresponding elliptic Dynkin diagrams and make a "dictionary" between the elliptic Hecke algebras and the double affine Hecke algebras.

Keywords Elliptic root systems · Double affine Hecke algebras

Mathematics Subject Classifications (2000): Primary 17B35; Secondary 14D30, 16G20

1 Introduction

In a process of solving the Macdonald's inner product conjecture, Cherednik [C92, C95] introduced a new class of algebras, so-called the *double affine Hecke algebras*. Recently, it is known that the theory of double affine Hecke algebras gives a very powerful tool in the study of orthogonal polynomials.

Cherednik's construction is generalized to an important class of non-reduced root systems, $(C_n^\vee, C_n)$ by Noumi [No95], Noumi-Stokman [NS04] and Sahi [Sa99]. When $n = 1$ (rank 1 case), the corresponding orthogonal polynomials are the Askey-Wilson polynomial [AW85] which include as special and limiting cases all the classical families of orthogonal polynomials in one variable. In [Mc03], Macdonald formulated all the above results uniformly (see also [K97]).

On the other hand, motivated by the theory of elliptic singularities, Saito [Sa85] introduced a notion of the *marked elliptic root systems*, which is a generalization of finite or affine root systems. In that article, he also introduced a diagram which describes the structure of a marked elliptic root system, so-called the *elliptic Dynkin diagram*. This diagram consists of some vertices and edges. In

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the original motivation, these vertices correspond to vanishing cycles and edges describe intersection numbers of them. After that, he and Takebayashi studied the structure of the corresponding Weyl groups (elliptic Weyl groups) and found a new presentation of them by using the elliptic Dynkin diagrams [ST97]. As a generalization of the work of Saito and Takebayashi, Yamada [Y00] defined a q-analogue of elliptic Weyl groups called the *elliptic Hecke algebras* for "one-codimensional" marked elliptic root systems which have only one dotted line in their elliptic Dynkin diagrams. He also pointed out a connection between the elliptic Hecke algebras and the double affine Hecke algebras.

In this article we define the elliptic Hecke algebra for an arbitrary marked elliptic root system in terms of the corresponding elliptic Dynkin diagram and make a "dictionary" between elliptic Hecke algebras and double affine Hecke algebras.

Let us briefly summarize the content of this article. In Sect. 1, we review the theory of double affine Hecke algebra following [C05] and [Mc03]. In Sect. 2, after giving a brief introduction on elliptic root systems, we give a definition of elliptic Hecke algebras for marked elliptic root systems which belong to group **(A)**, **(B)**, **(C)** or **(D)** (c.f. 3.1). After that, we introduce another presentation of elliptic Hecke algebras by using Lusztig's relations. Lusztig's relations play on important role in the presentation of affine Hecke algebras. By using this presentation, we compare double affine Hecke algebras and elliptic Hecke algebras in Sect. 3.

All proofs of the results of this article are omitted. They will be given in a future publication.

Finally, we must refer to the results of Takebayashi. He already introduced a notion of elliptic Hecke algebras. More precisely, in [T02], he defined them for elliptic root systems of type $(1, 1)$ and compare them and double affine Hecke algebras. After that, in [T05], he defined them for arbitrary marked elliptic root systems except for the group **(D)** (c.f. 3.1), but he did not compare them and double affine Hecke algebras for arbitrary cases. In his definition, he use new diagrams which are called "completed elliptic Dynkin diagrams." But, as we mentioned above, the elliptic Dynkin diagram have a concrete meaning in geometrical setting. Therefore, in this article, we try to "re-define" elliptic Hecke algebras by using the original elliptic Dynkin diagrams, in stead of completed elliptic Dynkin diagrams and to make an explicit "dictionary" between the elliptic Hecke algebras and the double affine Hecke algebras for arbitrary cases.

2 Double Affine Hecke Algebras

2.1 Affine Root Systems and Affine Weyl Groups

Let V be an n-dimensional $\mathbb{R}$-vector space with a positive definite symmetric bilinear form $\langle \cdot, \cdot \rangle$ and $R_0 \subset V$ an irreducible reduced finite root system. We denote the root lattice by $Q(R_0)$, the weight lattice by $P(R_0)$ and the Weyl group by $W(R_0)$.

Fix $\{a_i\}_{i=1}^n$ a basis of R_0 and let $P(R_0)_+$ be the set of dominant weights with respect to this basis. Set $P(R_0)_- := -P(R_0)_+$. Let $R_0^\vee := \{a^\vee; a \in R_0\}$ be the dual root system of R_0. Here $a^\vee := 2a/\langle a, a\rangle$.

Let $\mathbb{F} := V \oplus \mathbb{R}c$. We think $u + rc \in \mathbb{F}$ as an affine linear function on V by the following way: $(u + rc)(v) = \langle u, v\rangle + r$ $(v \in V)$. We extend $\langle\cdot,\cdot\rangle$ to $\mathbb{F}$ as: $\langle u + rc, u' + r'c\rangle := \langle u, u'\rangle$. Set $a_0 := -\theta + c$, where θ is the highest root of R_0. Then $c \in \mathbb{F}$ can be written in the following form: $c = \sum_{i=0}^n n_i a_i$ $(n_i \in \mathbb{Z}_{>0})$. It is known that $S(R_0) := \{a + rc; a \in R_0, r \in \mathbb{Z}\}$ is an irreducible reduced affine root system with a basis $\{a_i\}_{i=0}^n$. Set $S(R_0)^\vee := \{a^\vee; a \in S(R_0)\}$. It is also an irreducible reduced affine root system with a basis $\{a_i^\vee\}_{i=0}^n$. Any irreducible affine root system S is isomorphic to either $S(R_0)$ or $S(R_0)^\vee$.

Assume that S is an irreducible reduced affine root system. Namely, $S = S(R_0)$ or $S(R_0)^\vee$. In this case, we can consider S as a subset of $\mathbb{F}$. The Weyl group $W(S)$ is generated by reflections w_f $(f \in S)$ where $w_f(g) = g - \langle g, f^\vee\rangle f$ for $g \in \mathbb{F}$. Since $w_f = w_{f^\vee}$, $W(S(R_0)) = W(S(R_0)^\vee)$. Therefore, when we discuss on $W(S)$, we may assume that $S = S(R_0)$. For simplicity, we denote $w_{a_i} = w_{a_i^\vee}$ by w_i. $W(S)$ is a Coxeter group generated by $\{w_i\}_{i=0}^n$. Define an action of $v \in V$ on $\mathbb{F}$ as $t(v) : f \mapsto f - \langle f, v\rangle c$. It is known that $W(S) = W(R_0) \ltimes t(Q(R_0^\vee))$. Let $\widetilde{W}(S) := W(R_0) \ltimes t(P(R_0^\vee))$ be an extended affine Weyl group. For $w \in \widetilde{W}(S)$, define the length of w as $l(w) := |S^+ \cap w^{-1}S^-|$. Here $S^+ \subset S$ is the set of positive roots, and $S^- := -S^+$. Set $\Omega := \{w \in \widetilde{W}(S); l(w) = 0\}$. Then we have $\widetilde{W}(S) \cong \Omega \ltimes W(S)$ and $\Omega \cong P(R_0^\vee)/Q(R_0^\vee)$. Let $\{\omega_i\}_{i=1}^n$ denote the set of fundamental weights of R_0. Let v_i be the shortest element of $W(R_0)$ such that $v_i(\omega_i) \in P(R_0)_-$, and define $u_i \in \widetilde{W}(S)$ by $u_i = t(\omega_i)v_i^{-1}$. For the sake of convenience, we put $u_0 = 1$. Let $J := \{j; 0 \le j \le n, n_j = 1\}$. Then we have $\Omega = \{u_j; j \in J\}$.

In this exposition, an irreducible but non-reduced affine root system refers to a unique type of root systems. It is called type $(C_n^\vee, C_n)$ defined as: the set of roots $S = S(C_n)^\vee \cup S(C_n)$. The root lattice $Q((C_n^\vee, C_n))$ is the $\mathbb{Z}$-submodule of $\mathbb{F}$ generated by all the roots of $(C_n^\vee, C_n)$. The Weyl group $W((C_n^\vee, C_n))$ is just equal to $W(S(C_n))$. The number of $W((C_n^\vee, C_n))$-orbit in S is five.

2.2 *Affine Hecke Algebras and Double Affine Hecke Algebras*

Assume that $S = S(R_0)$ or $S(R_0)^\vee$. The braid group $\tilde{\mathcal{B}}$ of the extended affine Weyl group $\widetilde{W}(S)$ is the group with generators $T(w)$ $(w \in \widetilde{W}(S))$ and relations $T(v)T(w) = T(vw)$ if $l(v) + l(w) = l(vw)$. We shall write $T_i = T(w_i)$, $U_j = T(u_j)$. Then $\tilde{\mathcal{B}}$ is generated by T_i $(i = 0, \dots, n)$ and U_j $(j \in J)$. Let $\mathcal{B} \subset \tilde{\mathcal{B}}$ be the subgroup generated by T_i $(i = 0, \dots, n)$.

Let $\mathbb{Z}[\tau_0^{\pm 1}, \dots, \tau_n^{\pm 1}]$ be the Laurent polynomial ring in $\tau_0, \dots, \tau_n$ and $\tilde{\mathcal{I}} \subset \mathbb{Z}[\tau_0^{\pm 1}, \dots, \tau_n^{\pm 1}]$ (*resp.* $\mathcal{I}$) an ideal generated by $\tau_i - \tau_j$ where w_i and w_j are conjugate in $\widetilde{W}(S)$ (*resp.* $W(S)$). Let $\tilde{\mathcal{A}}_a := \mathbb{Z}[\tau_0^{\pm 1}, \dots, \tau_n^{\pm 1}]/\tilde{\mathcal{I}}$, and $\mathcal{A}_a := \mathbb{Z}[\tau_0^{\pm 1}, \dots, \tau_n^{\pm 1}]/\mathcal{I}$.

Definition 2.1. Let $\mathcal{H}(W(S))$ (*resp.* $\mathcal{H}(\widetilde{W}(S))$) be the quotient algebra of the group algebra $\mathcal{A}_a[\mathcal{B}]$ (*resp.* $\tilde{\mathcal{A}}_a[\tilde{\mathcal{B}}]$) by the ideal generated by the elements $(T_i - \tau_i)(T_i + \tau_i^{-1})$ for $i = 0, \ldots, n$. We call $\mathcal{H}(W(S))$ the affine Hecke algebra and $\mathcal{H}(\widetilde{W}(S))$ the extended affine Hecke algebra, respectively.

We introduce another presentation of $\mathcal{H}(\widetilde{W}(S))$. For $\lambda' \in P(R_0^\vee)_+$, define $Y^{\lambda'} = T(t(\lambda')) \in \mathcal{H}(\widetilde{W}(S))$. For any element $\lambda' \in P(R_0^\vee)$, we write $\lambda' = \mu' - \nu'$ by $\mu', \nu' \in P(R_0^\vee)_+$. Then we define $Y^{\lambda'} = Y^{\mu'}(Y^{\nu'})^{-1}$. We can easily check that this definition is unambiguous and the $Y^{\lambda'}$ are pairwise commutative. We define $Y^{\lambda'} \in \mathcal{H}(W(S))$ by replacing $P(R_0^\vee)$ by $Q(R_0^\vee)$ and $P(R_0^\vee)_+$ by $P(R_0^\vee)_+ \cap Q(R_0^\vee)$.

Let

$$\mathbf{b}(z_1, z_2; x) := \frac{z_1 - z_1^{-1} + (z_2 - z_2^{-1})x}{1 - x^2}.$$

Theorem 2.2. *$\mathcal{H}(\widetilde{W}(S))$ (resp. $\mathcal{H}(W(S))$) is an associative $\tilde{\mathcal{A}}_a$-algebra (resp. $\mathcal{A}_a$-algebra) generated by T_i ($i = 1, \ldots, n$), $Y^{\lambda'}$ ($\lambda' \in P(R_0^\vee)$) (resp. $Q(R_0^\vee)$) subject to the following relations:*

(A1) $(T_i - \tau_i)(T_i + \tau_i^{-1}) = 0$,
(A2) *For $i \neq j$ such that $w_i w_j$ has order m_{ij}, $T_i T_j T_i \ldots = T_j T_i T_j \ldots$ with m_{ij} factors on each side,*
(A3) $Y^{\lambda'} Y^{\mu'} = Y^{\lambda'+\mu'}$, $Y^{\lambda'} Y^{-\lambda'} = 1$,
(A4) $Y^{\lambda'} T_i - T_i Y^{w_i \lambda'} = \mathbf{b}(\tau_i, \tau_i; Y^{-a_i^\vee})(Y^{\lambda'} - Y^{w_i \lambda'})$.

We prepare to define double affine Hecke algebras. We consider following three types of triplet $\Xi = (R_0; S, \Lambda_s)$:

(type I) R_0 is a finite irreducible reduced root system, $S = S(R_0)$ or $S(R_0)^\vee$, $\Lambda_s = Q(S(R_0))$,
(type II) R_0 is a finite irreducible reduced root system, $S = S(R_0)$ or $S(R_0)^\vee$, $\Lambda_s = Q(S(R_0)^\vee)$,
(type III) R_0 is a root system of type C_n ($n \geq 1$), S is an affine root system of type $(C_n^\vee, C_n)$, $\Lambda_s = Q(S(R_0)^\vee)$. Here we denote $C_1 = A_1$.

We put

$$L = \begin{cases} P(R_0), & \text{if } \Xi \text{ is of type I}, \\ P(R_0^\vee), & \text{if } \Xi \text{ is of type II}, \\ Q(R_0^\vee), & \text{if } \Xi \text{ is of type III}, \end{cases} \qquad L' = \begin{cases} P(R_0^\vee), & \text{if } \Xi \text{ is of type I or II}, \\ Q(R_0^\vee), & \text{if } \Xi \text{ is of type III}. \end{cases}$$

Let $c_0 := e^{-1}c$, where e is the exponent of Ω. Set $\Lambda := L \oplus \mathbb{Z}c_0$. Then, Λ_s is a sublattice of Λ. We normalize $\langle \cdot, \cdot \rangle$ so that $\langle \theta, \theta \rangle = 2$ if Ξ is of type I or II, $\langle \theta, \theta \rangle = 4$ if Ξ is of type III. Let $a_i^\sharp = a_i$ if Ξ is of type I, $a_i^\vee$ if Ξ is of type II or III. We define $W(\Xi) := W(R_0) \ltimes t(L')$, and $W(\Xi)_s := W(R_0) \ltimes t(Q(R_0^\vee))$. Let $\mathcal{A} := \tilde{\mathcal{A}}_a$ if Ξ is type I or II, $\mathcal{A}_a[(\tau_0^\natural)^{\pm 1}, (\tau_n^\natural)^{\pm 1}]$ if Ξ is type III, where $\tau_0^\natural$ and $\tau_n^\natural$ are new indeterminates.

Definition 2.3. The double affine Hecke algebra $\mathcal{H}(\Xi)$ is an associative $\mathcal{A}$-algebra generated by $\mathcal{H}(W(\Xi))$ and X^λ $(\lambda \in \Lambda)$ subject to the following relations: For a triplet Ξ which is of type I or II,

(D1) $X^\lambda X^\mu = X^{\lambda+\mu}, X^\lambda X^{-\lambda} = 1 \quad (\lambda, \mu \in \Lambda)$,
(D2) $T_i X^{w_i\lambda} - X^{w_i\lambda} T_i = \mathbf{b}(\tau_i, \tau_i; X^{a_i^\sharp})(X^\lambda - X^{w_i\lambda}) \quad (i = 0, \dots, n)$,
(D3) $U_j X^\lambda U_j^{-1} X^{u_j\lambda} \quad (j \in J)$.

For a triplet Ξ which is of type III, we set $\tau_i^\natural := \tau_i$ for $i = 1, \dots, n-1$, $T_0^\natural := X^{-a_0^\vee} T_0^{-1}$ and $T_n^\natural := X^{-a_n^\vee} T_n^{-1}$. The defining relations are (D1) and

(D2′) $T_i X^{w_i\lambda} - X^{w_i\lambda} T_i = \mathbf{b}(\tau_i, \tau_i^\natural; X^{a_i^\sharp})(X^\lambda - X^{w_i\lambda}) \quad (i = 0, \dots, n)$,
(D4) $(T_k^\natural - \tau_k^\natural)(T_k^\natural + (\tau_k^\natural)^{-1}) = 0 \quad (k = 0, n)$.

The small double affine Hecke algebra $\mathcal{H}(\Xi)_s$ is subalgebra of $\mathcal{H}(\Xi)$ generated by T_i $(i = 0, \dots, n)$ and X^λ $(\lambda \in \Lambda_s)$.

Remark 2.4. If Ξ is of type III, then we have $\mathcal{H}(\Xi) = \mathcal{H}(\Xi)_s$. In this case, $\mathcal{H}(\Xi)$ just coincides with the algebra which is introduced by Noumi-Stokman [NS04] and Sahi [Sa99].

The next theorem is borrowed from [C05] and [Mc03].

Theorem 2.5. *The elements* $\{T(w)U_j X^\lambda; w \in W(S(R_0)), j \in J, \lambda \in \Lambda\}$ *form a* $\mathcal{A}$*-basis of* $\mathcal{H}(\Xi)$.

Similarly, we have the following theorem.

Theorem 2.6. *The elements* $\{T(w)X^\lambda; w \in W(S(R_0)), \lambda \in \Lambda_s\}$ *form a* $\mathcal{A}$*-basis of* $\mathcal{H}(\Xi)_s$.

Corollary 2.7. *There exists an isomorphism*

$$\mathcal{H}(\Xi) \cong \bigoplus_{j\in J, \lambda\in\Lambda/\Lambda_s} \mathcal{H}(\Xi)_s U_j X^\lambda$$

as $\mathcal{H}(\Xi)_s$*-algebras.*

3 Elliptic Hecke Algebras

3.1 Marked Elliptic Root Systems

Let F be an $(n+2)$-dimensional $\mathbb{R}$-vector space equipped with symmetric positive semi-definite bilinear form $I(\cdot,\cdot)$ with a two-dimensional radical. For $\alpha \in F$ such that $I(\alpha,\alpha) \neq 0$, we define $\alpha^\vee := 2\alpha/I(\alpha,\alpha)$, and $s_\alpha : F \ni u \mapsto u - I(u,\alpha^\vee)\alpha \in F$.

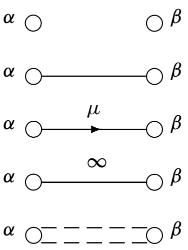
Definition 3.1. A subset $R \subset F$ is called elliptic root system, if the following conditions are satisfied:

(i) $I(\alpha,\alpha) \neq 0$ for any $\alpha \in R$,
(ii) $Q(R) \otimes_{\mathbb{Z}} \mathbb{R} = F$ where $Q(R)$ is the $\mathbb{Z}$-submodule of F generated by R,
(iii) $s_\alpha(R) = R$ for any $\alpha \in R$,
(iv) $I(\alpha,\beta^\vee) \in \mathbb{Z}$ for any $\alpha,\beta \in R$,
(v) R is irreducible, i.e., there is no partition of R into two non-empty subsets R_1 and R_2 such that $I(\alpha,\beta) = 0$ for all $\alpha \in R_1$ and $\beta \in R_2$.

We define the elliptic Weyl group as $W(R) := \langle s_\alpha \rangle_{\alpha \in R}$. Let G be a 1-dimensional subspace of $\operatorname{rad} I$ over $\mathbb{Q}$. We fix a pair (R,G) which is called a marked elliptic root system. Let (R,G) and (R',G') be two marked elliptic root systems. We assume R (*resp.* R') is a subset of an $\mathbb{R}$-vector space F (*resp.* F'). We say (R,G) and (R',G') are isomorphic if there exists a linear isomorphism $\phi : F \xrightarrow{\sim} F'$ which induces bijections $R \xrightarrow{\sim} R'$ and $G \xrightarrow{\sim} G'$.

Let δ_1 be a generator of the lattice $G \cap Q(R)$. For $\alpha \in R$, put $k_\alpha := \min\{k \in \mathbb{Z}_{>0}; \alpha + k\delta_1 \in R\}$, and $\alpha^* := \alpha + k_\alpha\delta_1$. Let $\pi_a : F \to F/G$, $\pi_f : F \to F/\operatorname{rad} I$ be the canonical projections and set $R_a := \pi_a(R)$, $R_f := \pi_f(R)$. Then R_a (*resp.* R_f) is an irreducible affine (*resp.* finite) root system. In this article, we always assume that both R_a and R_f are reduced. Let $\Gamma_a = \{\alpha_0,\ldots,\alpha_n\} \subset R$ such that $\pi_a(\Gamma_a)$ is a basis of R_a and $\pi_f(\{\alpha_1,\ldots,\alpha_n\})$ is a basis of R_f. Let θ_f be the corresponding highest root of R_f. We have $\theta_f = \sum_{i=1}^n n_i\pi_f(\alpha_i)$ ($n_i \in \mathbb{Z}_{>0}$). Let $\delta_2 := \alpha_0 + \sum_{i=1}^n n_i\alpha_i \in Q(R)$. Let I_R be a constant multiple of I normalized in the way that $\min\{I_R(\alpha,\alpha)\,;\alpha \in R\} = 2$. Put $m_i := \frac{I_R(\alpha_i,\alpha_i)n_i}{2k_{\alpha_i}}$, and $m_{\max} := \max\{m_i; 0 \le i \le n\}$. Set $\Gamma_{\max} := \{\alpha_i \in \Gamma_a; m_i = m_{\max}\}$, and $\Gamma^*_{\max} := \{\alpha_i^*; \alpha_i \in \Gamma_{\max}\}$. We introduce elliptic Dynkin diagrams.

Definition 3.2. The elliptic Dynkin diagram $\Gamma(R,G)$ of the marked elliptic root system (R,G) is a finite graph with its set of vertices $\Gamma := \Gamma_a \cup \Gamma^*_{\max}$ and edges are determined by the following convention (C): for $\alpha,\beta \in \Gamma$



if $I(\alpha,\beta) = I(\beta,\alpha) = 0$,

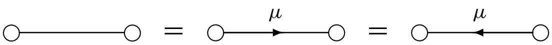
if $I(\alpha,\beta^\vee) = I(\beta,\alpha^\vee) = -1$,

if $I(\alpha,\beta^\vee) = -\mu$ and $I(\beta,\alpha^\vee) = -1$ for $\mu = 2,3$,

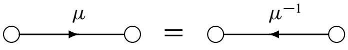
if $I(\alpha,\beta^\vee) = I(\beta,\alpha^\vee) = -2$,

if $I(\alpha,\beta^\vee) = I(\beta,\alpha^\vee) = 2$.

We shall use the conventions:



for $\mu = 1$,



for $\mu = 2^{\pm 1}, 3^{\pm 1}$.

For later use, we assume that both R_a and R_f are reduced. Under this assumption, the classification of the isomorphism classes of marked elliptic root systems are given by Saito [Sa85] as follows:

Let $\widehat{\Gamma}$ be one of diagrams which are listed in Appendix. If α is a vertex in the diagram $\widehat{\Gamma}$, we denote $\alpha \in \widehat{\Gamma}$. Consider the following data:

$\widehat{F}$:= the vector space spanned by the vertices of $\widehat{\Gamma}$ over $\mathbb{R}$,
$\widehat{I}$:= the symmetric bilinear form on $\widehat{F}$ defined (up to a positive constant factor) by the above convention (C),
$\widehat{s}_\alpha$:= the reflection with respect to the vertex $\alpha \in \widehat{\Gamma}$ on $(\widehat{F}, \widehat{I})$,
$\widehat{c} := \prod_{\alpha \in \widehat{\Gamma}} \widehat{s}_\alpha$, where $\widehat{s}_{\alpha^*}$ comes next to $\widehat{s}_\alpha$,
$\widehat{R} := \bigcup_{\alpha \in \widehat{\Gamma}} \widehat{W} \cdot \alpha$, where $\widehat{W}$ is the group which is generated by $\widehat{s}_\alpha$ for all $\alpha \in \widehat{\Gamma}$,
$\widehat{\Gamma}_{\max} := \{\alpha \in \widehat{\Gamma} \; ; \; \alpha^* \in \widehat{\Gamma}\}$ and $\widehat{\Gamma}^*_{\max} := \{\alpha^* \; ; \; \alpha \in \widehat{\Gamma}_{\max}\}$,
$\widehat{G}$:= the linear space spanned by $\alpha^* - \alpha$ for all $\alpha \in \widehat{\Gamma}_{\max}$,
$\widehat{l}_{\max} := \max\{\sharp$ of vertices in a connected component of $\widehat{\Gamma} \setminus (\widehat{\Gamma}_{\max} \cup \widehat{\Gamma}^*_{\max})\}$,
$m(\widehat{\Gamma}) := \widehat{l}_{\max} + 1$.

We remark that these data are determined by the given diagram $\widehat{\Gamma}$. Set

$$F_{\widehat{\Gamma}} := \widehat{F}/(\widehat{c}^{\,m(\widehat{\Gamma})} - \mathrm{Id}_{\widehat{F}})\widehat{F}, \quad (R_{\widehat{\Gamma}}, G_{\widehat{\Gamma}}) := \text{the image of } (\widehat{R}, \widehat{G}) \text{ in } F_{\widehat{\Gamma}}.$$

Theorem 3.3 (Saito [Sa85]).

(1) $(R_{\widehat{\Gamma}}, G_{\widehat{\Gamma}})$ *is a marked elliptic root system such that the corresponding elliptic Dynkin diagram* $\Gamma(R_{\widehat{\Gamma}}, G_{\widehat{\Gamma}})$ *just coincides with* $\widehat{\Gamma}$ *and both* $(R_{\widehat{\Gamma}})_a$ *and* $(R_{\widehat{\Gamma}})_f$ *are reduced.*
(2) *Conversely, for a marked elliptic root system* (R, G) *such that both* R_a *and* R_f *are reduced, there is the unique diagram* $\widehat{\Gamma}$ *in Appendix such that* (R, G) *is isomorphic to* $(R_{\widehat{\Gamma}}, G_{\widehat{\Gamma}})$.

Namely the isomorphism classes of marked elliptic root systems are completely classified by the diagrams listed in Appendix. We say (R, G) is of type $X_n^{(t_1,t_2)}$ if $\Gamma(R, G)$ is a diagram of type $X_n^{(t_1,t_2)}$. In that case, this diagram is called the elliptic Dynkin diagram of type $X_n^{(t_1,t_2)}$.

We form marked elliptic root systems into four groups (*cf.* Appendix).

(A) $A_n^{(1,1)}$ $(n \geq 1)$, $D_n^{(1,1)}$ $(n \geq 4)$, $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{1,1}$.
(B) $B_n^{(1,2)}$ $(n \geq 2)$, $B_n^{(2,2)}$ $(n \geq 2)$, $C_n^{(1,2)}$ $(n \geq 2)$, $C_n^{(2,2)}$ $(n \geq 3)$, $F_4^{(1,2)}$, $F_4^{(2,2)}$, $G_2^{(1,3)}$, $G_2^{(3,3)}$.
(C) $B_n^{(1,1)}$ $(n \geq 3)$, $B_n^{(2,1)}$ $(n \geq 2)$, $C_n^{(1,1)}$ $(n \geq 2)$, $C_n^{(2,1)}$ $(n \geq 2)$, $F_4^{(1,1)}$, $F_4^{(2,1)}$, $G_2^{(1,1)}$, $G_2^{(3,1)}$.
(D) $A_1^{(1,1)*}$, $B_n^{(2,2)*}$ $(n \geq 2)$, $C_n^{(1,1)*}$ $(n \geq 2)$.

We put $\alpha_i^\dagger := k_{\alpha_i} \alpha_i^\vee$ and set $Q((R, G)_a) := \bigoplus_{i=0}^n \mathbb{Z}\alpha_i^\dagger$.

Theorem 3.4 (Saito and Takebayashi [ST97]). *Let (R, G) be a marked elliptic root system which belongs to a group* **(A)**, **(B)** *or* **(C)**. *We have*

$$Q((R, G)_a) = \begin{cases} Q(R_a) = Q(R_a^\vee), & \textit{if } (R, G) \textit{ belongs to } \textbf{(A)}, \\ Q(R_a), & \textit{if } (R, G) \textit{ belongs to } \textbf{(B)}, \\ Q(R_a^\vee), & \textit{if } (R, G) \textit{ belongs to } \textbf{(C)}. \end{cases}$$

Namely, there is an irreducible reduced finite root system $R_f^{(0)}$ such that $(R, G)_a = S(R_f^{(0)})$ or $S(R_f^{(0)})^\vee$.

If (R, G) belongs to **(D)**, *then $Q((A_1^{(1,1)*})_a) = Q(S(A_1))$, $Q((B_n^{(2,2)*})_a) = Q((C_n^{(1,1)*})_a) = Q((C_n^\vee, C_n))$, respectively.*

Let us define the notion of *boundary side*. For $\alpha, \beta \in \Gamma$ which are connected as $\alpha_i \circ\!\!\xrightarrow{\ \mu\ }\!\!\circ\ \alpha_j$ for some $\mu = 2^{\pm 1}, 3^{\pm 1}$, set $k(\alpha; \beta) = k_\alpha / k_\beta$. By [Sa85], $k(\alpha; \beta)$ must be equal to 1 or μ.

Definition 3.5 (boundary side). Under the above setting, α is the boundary side (b-side for short) of the bond $\alpha_i \circ\!\!\xrightarrow{\ \mu\ }\!\!\circ\ \alpha_j$ if $k(\alpha; \beta) = \min\{1, \mu\}$.

3.2 Elliptic Hecke Algebras

Let (R, G) be a marked elliptic root system. Let $\mathbb{Z}[t_\alpha^{\pm 1}]_{\alpha \in \Gamma}$ be the Laurent polynomial ring with indeterminates t_α ($\alpha \in \Gamma$). Let $J \subset \mathbb{Z}[t_\alpha^{\pm 1}]_{\alpha \in \Gamma}$ be an ideal generated by $t_\alpha - t_\beta$ where α and β are in the same $W(R)$-orbit. Put $\mathbb{A} := \mathbb{Z}[t_\alpha^{\pm 1}]_{\alpha \in \Gamma} / J$.

Definition 3.6. The elliptic Hecke algebra $\mathbb{H}(R, G)$ is an associative $\mathbb{A}$-algebra generated by g_α ($\alpha \in \Gamma$) subject to the following relations:

(H0) $\quad {}^{\alpha}\bigcirc \qquad (g_\alpha - t_\alpha)(g_\alpha + t_\alpha^{-1}) = 0,$

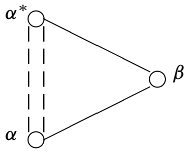
(H1-1) $\quad {}^{\alpha}\bigcirc \qquad \bigcirc^{\beta} \qquad g_\alpha g_\beta = g_\beta g_\alpha,$

(H1-2) $\quad {}^{\alpha}\bigcirc\!\!-\!\!-\!\!-\!\!\bigcirc^{\beta} \qquad g_\alpha g_\beta g_\alpha = g_\beta g_\alpha g_\beta,$

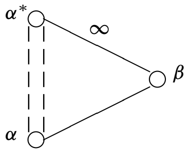
(H1-3) $\quad {}^{\alpha}\bigcirc\!\!\xrightarrow{\ 2^{\pm 1}\ }\!\!\bigcirc^{\beta} \qquad g_\alpha g_\beta g_\alpha g_\beta = g_\beta g_\alpha g_\beta g_\alpha,$

(H1-4) $\quad {}^{\alpha}\bigcirc\!\!\xrightarrow{\ 3^{\pm 1}\ }\!\!\bigcirc^{\beta} \qquad g_\alpha g_\beta g_\alpha g_\beta g_\alpha g_\beta = g_\beta g_\alpha g_\beta g_\alpha g_\beta g_\alpha,$

In the following diagrams ((H2-1)$\sim$(H4)), we always assume $\alpha, \beta \in \Gamma_a$. For $\alpha \in \Gamma_a$, we set $x_{\alpha^\dagger} := g_\alpha g_{\alpha^*}$.

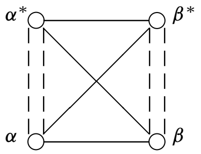
(H2-1) 

$$g_\beta x_{\alpha^\dagger} g_\beta x_{\alpha^\dagger} = x_{\alpha^\dagger} g_\beta x_{\alpha^\dagger} g_\beta,$$

(H2-2) 

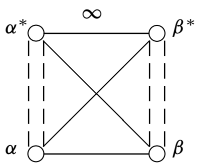
$$g_\beta x_{\alpha^\dagger} g_\beta x_{\alpha^\dagger} = x_{\alpha^\dagger} g_\beta x_{\alpha^\dagger} g_\beta,$$

$g_\beta x_{\alpha^\dagger} g_\beta x_{\alpha^\dagger}$ commutes with g_α, g_{α^*} and g_β,

(H2-3) 

$$x_{\beta^\dagger} x_{\alpha^\dagger} = g_\beta x_{\alpha^\dagger} g_\beta,$$

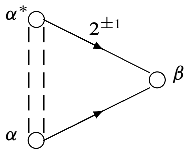
$$x_{\alpha^\dagger} x_{\beta^\dagger} = g_\alpha x_{\beta^\dagger} g_\alpha,$$

(H2-4) 

$$x_{\alpha^\dagger} x_{\beta^\dagger} = x_{\beta^\dagger} x_{\alpha^\dagger},$$

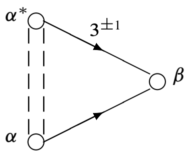
$x_{\alpha^\dagger} x_{\beta^\dagger}$ commutes with $g_\alpha, g_{\alpha^*}, g_\beta, g_{\beta^*}$.

In the next three diagrams, we assume α is b-side for the bond $\alpha \circ\!\xrightarrow{\mu}\!\circ \beta$ for $\mu = 2^{\pm 1}, 3^{\pm 1}$.

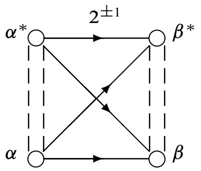
(H3-1) 

$$g_\beta x_{\alpha^\dagger} g_\beta x_{\alpha^\dagger} = x_{\alpha^\dagger} g_\beta x_{\alpha^\dagger} g_\beta,$$

$$g_\alpha g_\beta x_{\alpha^\dagger} g_\beta = g_\beta x_{\alpha^\dagger} g_\beta g_\alpha,$$

(H3-2) 

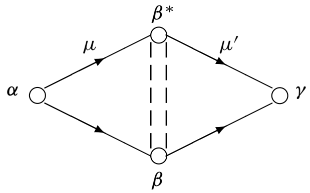
$$\begin{aligned} g_\beta x_{\alpha^\dagger} g_\beta x_{\alpha^\dagger} &= x_{\alpha^\dagger} g_\beta x_{\alpha^\dagger} g_\beta \\ &= g_\alpha g_\beta x_{\alpha^\dagger} g_\beta g_\alpha, \end{aligned}$$

(H3-3) 

$$x_{\beta^\dagger} x_{\alpha^\dagger} = g_\beta x_{\alpha^\dagger} g_\beta.$$

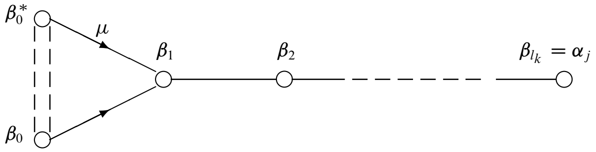
In the next diagram, we assume $\mu, \mu' = 1, 2^{\pm 1}, 3^{\pm 1}$ and $\alpha \notin \Gamma_{\max}$.

(H4)



$$x_{\beta^\dagger} g_\gamma g_\alpha x_{\beta^\dagger} g_\alpha = g_\alpha x_{\beta^\dagger} g_\alpha g_\gamma x_{\beta^\dagger}.$$

We introduce another presentation of $\mathbb{H}(R, G)$. For $\alpha_j \in \Gamma_a \setminus \Gamma_{\max}$, there is a subdiagram of $\Gamma(R, G)$ of the form:



$$\mu = 1, 2^{\pm 1}, 3^{\pm 1}.$$

We define $x_{\beta^\dagger_{l_{i+1}}} := g_{\beta_{l_{i+1}}} x_{\beta^\dagger_{l_i}} g_{\beta_{l_{i+1}}} x^{-1}_{\beta^\dagger_{l_i}}$ $(0 \leq i \leq k-1)$ inductively.

Proposition 3.7. *The elements $x_{\alpha_i^\dagger}$ $(\alpha_i \in \Gamma_a)$ are pairwise commutative.*

For $\lambda = \sum_{i=0}^n \lambda_i \alpha_i^\dagger$, we define $x_\lambda := (x_{\alpha_0^\dagger})^{\lambda_0} (x_{\alpha_1^\dagger})^{\lambda_1} \ldots (x_{\alpha_n^\dagger})^{\lambda_n}$. By Proposition 3.7, it is well-defined. In the below, we set $t_{\alpha_i^*} := t_{\alpha_i}$ $(\alpha_i \in \Gamma_a \setminus \Gamma_{\max})$.

Proposition 3.8. *For each $\alpha_i \in \Gamma_a$, and $\lambda \in Q((R, G)_a)$, we have*

$$g_{\alpha_i} x_\lambda - x_{s_{\alpha_i}(\lambda)} g_{\alpha_i} = \mathbf{b}(t_{\alpha_i}, t_{\alpha_i^*}; x_{\alpha_i^\dagger})(x_\lambda - x_{s_{\alpha_i}(\lambda)}).$$

Definition 3.9. Let (R, G) be a marked elliptic root system. $\hat{\mathbb{H}}(R, G)$ is an associative $\mathbb{A}$-algebra generated by $\hat{g}_{\alpha_i}$ $(\alpha_i \in \Gamma_a)$, $\hat{x}_\lambda$ $(\lambda \in Q((R, G)_a))$ subject to the following relations: Replacing g_α by $\hat{g}_{\alpha_i}$, the relations (H0) and (H1-1) $\sim$ (H1-4) in Definition 3.6 hold for $\hat{g}_{\alpha_i}$ $(\alpha_i \in \Gamma_a)$.

(H′2) $\quad \hat{x}_\lambda \hat{x}_\mu = \hat{x}_{\lambda+\mu}, \quad \hat{x}_\lambda \hat{x}_{-\lambda} = 1, \quad (\lambda, \mu \in Q((R, G)_a)),$

(H′3) $\quad \hat{g}_{\alpha_i} \hat{x}_\lambda - \hat{x}_{s_{\alpha_i}(\lambda)} \hat{g}_{\alpha_i} = \mathbf{b}(t_{\alpha_i}, t_{\alpha_i^*}; \hat{x}^{-1}_{\alpha_i^\dagger})(\hat{x}_\lambda - \hat{x}_{s_{\alpha_i}(\lambda)}), (\lambda \in Q((R, G)_a)).$

The next proposition can be checked by examining each defining relation.

Proposition 3.10. *We have an algebra isomorphism $\psi : \mathbb{H}(R, G) \to \hat{\mathbb{H}}(R, G)$, $g_{\alpha_i} \mapsto \hat{g}_{\alpha_i}$ $(\alpha_i \in \Gamma_a)$, $g_{\alpha_i^*} \mapsto \hat{g}_{\alpha_i}^{-1} \hat{x}_{\alpha_i^\dagger}$ $(\alpha_i \in \Gamma_{\max})$.*

4 Double Affine Hecke Algebras and Elliptic Hecke Algebras

Firstly, we give the correspondence between marked elliptic root systems which belong to the groups **(A)**, **(B)**, or **(C)** and double affine Hecke algebras of type I, II. Let (R, G) a marked elliptic root system which belongs to a group **(A)**, **(B)** or **(C)**. We normalize I so that $I(\theta_e, \theta_e) = 2$, where $\theta_e = \sum_{i=1}^{n} n_i \alpha_i$. By Theorem 3.4, there is an irreducible reduced finite root system $R_f^{(0)}$ such that $(R, G)_a = S(R_f^{(0)})$ or $S(R_f^{(0)})^\vee$. Then $\Xi(R, G) := (R_f^{(0)}, R_a; Q((R, G)_a))$ is a triplet of type I or II in the sense of Sect. 1.2. The explicit correspondence is the following:

$B_n^{(1,2)}, B_n^{(2,1)}, C_n^{(1,2)}, C_n^{(2,1)}, F_4^{(1,2)}, F_4^{(2,1)}, G_2^{(1,3)}, G_2^{(3,1)}$	I
$X_n^{(k,k)}\ (X = A \sim G)$	II

Remark 4.1. If (R, G) is of type $A_n^{(1,1)}, D_n^{(1,1)}, E_k^{(1,1)}$ $(k = 6, 7, 8)$, then the corresponding triplet may be regarded as of type II.

In each case we consider the corresponding double affine Hecke algebra $\mathcal{H}(\Xi(R, G))$ and the coordinate ring $\mathcal{A} = \mathbb{Z}[\tau_0^{\pm 1}, \ldots, \tau_n^{\pm 1}]/\tilde{\mathcal{I}}$. Let ϖ : $\mathbb{Z}[t_\alpha^{\pm 1}]_{\alpha \in \Gamma} \to \mathbb{Z}[\tau_0^{\pm 1}, \ldots, \tau_n^{\pm 1}]$ be a surjective homomorphism defined by $t_{\alpha_i} \mapsto \tau_i$ and $t_{\alpha_i^*} \mapsto \tau_i$ for $i = 0, \ldots, n$.

Lemma 4.2. *The map ϖ induces a surjective homomorphism $\mathbb{A} \to \mathcal{A}$.*

By this lemma, $\mathcal{A}$ can be regarded as an $\mathbb{A}$-algebra. By Definition 2.3 and Proposition 3.10, we obtain the following theorem which is one of the main theorems in this article.

Theorem 4.3. *There exists an isomorphism $\mathcal{A} \otimes_{\mathbb{A}} \mathbb{H}(R, G) \xrightarrow{\sim} \mathcal{H}_s(\Xi(R, G))$ which is defined by*

$$g_{\alpha_i} \mapsto T_i, \quad x_{\alpha_i^\dagger} \mapsto X^{a_i^\sharp} \quad (i = 0, \ldots, n).$$

By Corollary 2.7 and Theorem 4.3, we obtain a comparison between double affine Hecke algebras and elliptic Hecke algebras if marked elliptic root systems belong to the groups **(A)**, **(B)** and **(C)**.

Secondly, let (R, G) be a marked elliptic root system of type $A_1^{(1,1)}$, $C_n^{(1,1)}$ or $B_n^{(2,2)}$ $(n \geq 2)$. We normalize I so that $I(\theta_e, \theta_e) = 4$ where θ_e is as above. Then $\Upsilon(R, G) := (R_f, R_a \cup R_a^\vee; Q((R, G)_a))$ is a triplet of type III. As same as in the first case, let us consider the corresponding double affine Hecke algebra $\mathcal{H}(\Upsilon(R, G))$ and the coordinate ring $\mathcal{A} = \mathcal{A}_a[(\tau_0^\natural)^{\pm 1}, (\tau_n^\natural)^{\pm 1}]$, where $\mathcal{A}_a = \mathbb{Z}[\tau_0^{\pm 1}, \ldots, \tau_n^{\pm 1}]/\mathcal{I}$. Let $\varpi' : \mathbb{Z}[t_\alpha^{\pm 1}]_{\alpha \in \Gamma} \to \mathbb{Z}[\tau_0^{\pm 1}, \ldots, \tau_n^{\pm 1}, (\tau_0^\natural)^{\pm 1}, (\tau_n^\natural)^{\pm 1}]$ be a surjective homomorphism defined by $t_{\alpha_i} \mapsto \tau_i$ $(i = 0, \ldots, n)$, $t_{\alpha_j^*} \mapsto \tau_j$ $(j = 1, \ldots, n-1)$ and $t_{\alpha_k^*} \mapsto \tau_k^\natural$ $(k = 0, n)$. In this setting, Lemma 4.2 turns into

Lemma 4.4. *The map ϖ' induces an isomorphism $\mathbb{A} \xrightarrow{\sim} \mathcal{A}$.*

By a similar argument as in the first case, we have the following statement.

Theorem 4.5. *There exists an algebra isomorphism $\mathbb{H}(R,G) \xrightarrow{\sim} \mathcal{H}(\Upsilon(R,G))$ which is defined by*

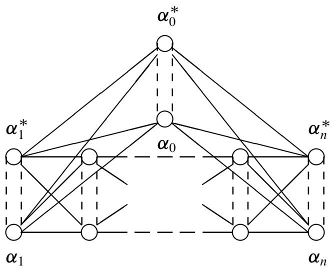
$$g_{\alpha_i} \mapsto T_i, \quad x_{\alpha_i^\dagger} \mapsto X^{a_i^\sharp} \quad (i = 0, \ldots, n).$$

Since $\Upsilon(R,G)$ is a triplet of type III, as we already mentioned in Remark 1.4, the corresponding double affine Hecke algebra $\mathcal{H}(\Upsilon(R,G))$ is nothing but the algebra which is introduced by Noumi-Stokman [NS04] and Sahi [Sa99]. Therefore the above theorem says that the elliptic Hecke algebra $\mathbb{H}(R,G)$ of type $A_1^{(1,1)}$, $C_n^{(1,1)}$ or $B_n^{(2,2)}$ $(n \geq 2)$ is isomorphic to the algebra which is introduced by Noumi-Stokman and Sahi.

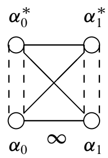
Appendix

Table 1 Elliptic Dynkin diagrams for marked elliptic root systems

$A_n^{(1,1)}$

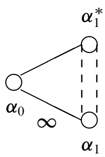


$(n \geq 2)$

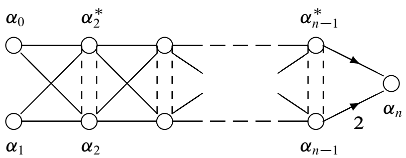


$(n = 1)$

$A_1^{(1,1)*}$



$B_n^{(1,1)}$

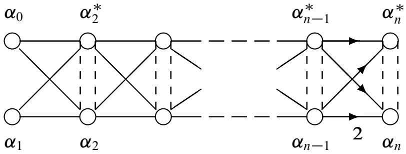


$(n \geq 3)$

(continued)

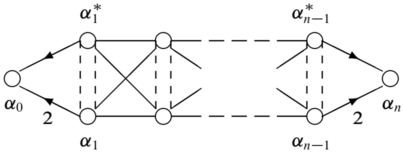
Table 1 (continued)

$B_n^{(1,2)}$



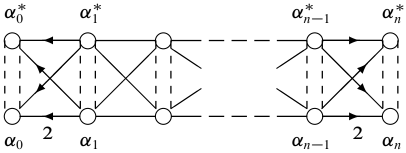
$(n \geq 3)$

$B_n^{(2,1)}$



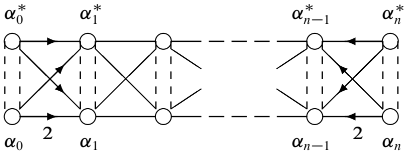
$(n \geq 2)$

$B_n^{(2,2)}$



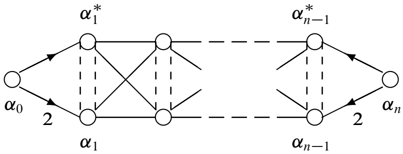
$(n \geq 2)$

$C_n^{(1,1)}$



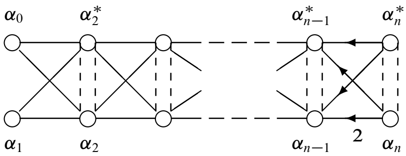
$(n \geq 2)$

$C_n^{(1,2)}$



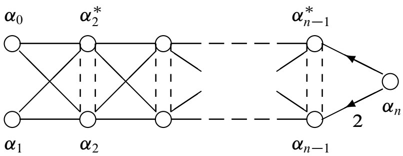
$(n \geq 2)$

$C_n^{(2,1)}$



$(n \geq 3)$

$C_n^{(2,2)}$

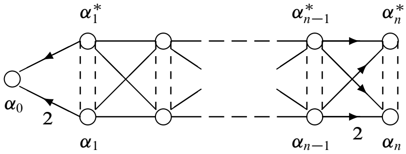


$(n \geq 3)$

(continued)

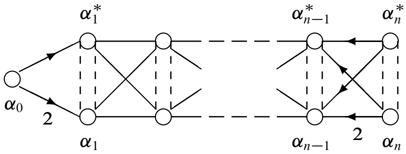
Table 1 (continued)

$B_n^{(2,2)*}$



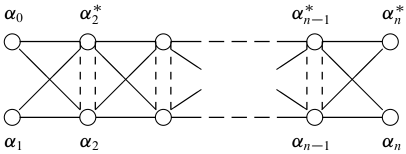
$(n \geq 2)$

$C_n^{(1,1)*}$



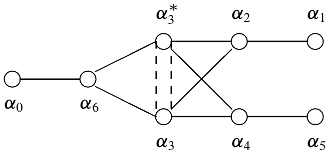
$(n \geq 2)$

$D_n^{(1,1)}$

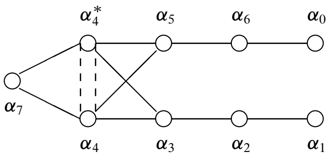


$(n \geq 4)$

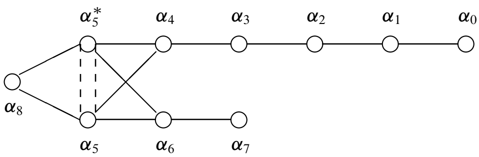
$E_6^{(1,1)}$



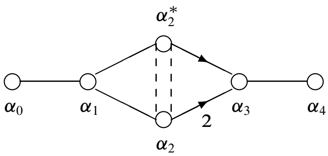
$E_7^{(1,1)}$



$E_8^{(1,1)}$



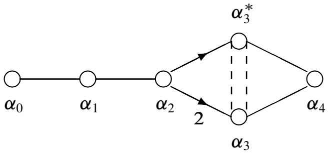
$F_4^{(1,1)}$



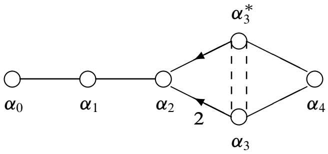
(continued)

Table 1 (continued)

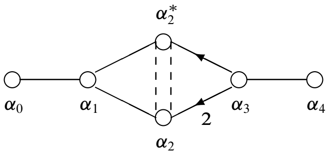
$F_4^{(1,2)}$



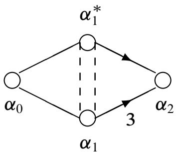
$F_4^{(2,1)}$



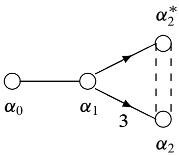
$F_4^{(2,2)}$



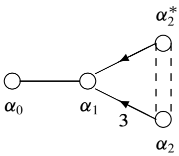
$G_2^{(1,1)}$



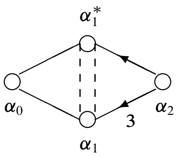
$G_2^{(1,3)}$



$G_2^{(3,1)}$



$G_2^{(3,3)}$



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Green's Formula with $\mathbb{C}^*$-Action and Caldero–Keller's Formula for Cluster Algebras

Jie Xiao and Fan Xu

Abstract It is known that Green's formula over finite fields gives rise to the comultiplications of Ringel–Hall algebras and quantum groups (see [Invent. Math. **120** (1995), 361–377], see also [J. Amer. Math. Soc. **4** (1991), 365–421]). In this chapter, we prove a projective version of Green's formula in a geometric way. Then following the method of Hubery in [Hubery, *Acyclic cluster algebras via Ringel–Hall algebras* (preprint)], we apply this formula to proving Caldero–Keller's multiplication formula for acyclic cluster algebras of arbitrary type.

Keywords Green's formula · Cluster algebra · $\mathbb{C}^*$-action

Mathematics Subject Classifications (2000): 14M99, 16G20, 16G70, 17B35

1 Introduction

1.1 Green's Theorem and Acyclic Cluster Algebras

Green in [Gre] found a homological counting formula for hereditary abelian categories over finite fields. It leads to the comultiplication formula for Ringel–Hall algebras, and as a generalization of the result of Ringel in [Rin1], it gives a realization of the positive part of the quantized enveloping algebra for arbitrary type symmetrizable Kac–Moody algebra. It is coincident with the comultiplication defined by the restriction functor in the geometric realization of the positive part of the quantized enveloping algebra (see [Lu]). In [DXX], we gave Green's formula over the complex numbers $\mathbb{C}$ via Euler characteristic and applied it to realizing comultiplication of the universal enveloping algebra. However, one should notice that many nonzero terms in the original formula vanish when we consider it over the

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complex numbers $\mathbb{C}$. In the following, we show that the geometric correspondence in the proof of Green's formula admits a canonical $\mathbb{C}^*$-action. Then we obtain a new formula, which can be regarded as the projective version of Green's formula.

Our motivation comes from cluster algebras. Cluster algebras were introduced by Fomin and Zelevinsky [FZ]. In [BMRRT], the authors categorified a lot of cluster algebras by defining and studying the cluster categories related to clusters and seeds. Under the framework of cluster categories, Caldero and Keller realized the acyclic cluster algebras of simply laced finite type by proving a cluster multiplication theorem [CK]. At the same time, Hubery researched on realizing acyclic cluster algebras (including non simply laced case) via Ringel–Hall algebras for valued graphs over finite fields [Hu2]. He counted the corresponding Hall numbers and then deduced the Caldero–Keller multiplication when evaluating at $q = 1$ where q is the order of the finite field. It seems that his method only works for the case of tame hereditary algebras [Hu3], due to the difficulty of the existence of Hall polynomials. In this chapter, we realize that the whole thing is independent of that over finite fields. By counting the Euler characteristics of the corresponding varieties and constructible sets with *pushforward* functors and geometric quotients, we show that the projective version of Green's theorem and the "higher order" associativity of Hall multiplication imply that Caldero–Keller's multiplication formula holds for acyclic cluster algebras of arbitrary type. We remark here that for the elements in the dual semicanonical basis which are given by certain constructible functions on varieties of nilpotent modules over a preprojective algebra of arbitrary type, a similar multiplication formula has been obtained in [GLS].

1.2 The chapter is organized as follows. In Sect. 2, we recall the general theory of algebraic geometry needed in this chapter. This is followed in Sect. 3 by a short survey of Green's formula over finite fields without proof. In particular, we consider many variants of Green's formula under various group actions. These variants can be viewed as the counterparts over finite field of the projective version of Green's formula. We give the main result in Sect. 4. Two geometric versions of Green's formula are proved. As an application, in Sect. 5 we prove Caldero–Keller multiplication formula following Hubery's method [Hu2], and also we give an example using the Kronecker quiver.

2 Preliminaries

2.1 Module Varieities

Let $Q = (Q_0, Q_1, s, t)$ be a quiver, where Q_0, also denoted by I, and Q_1 are the sets of vertices and arrows, respectively, and $s, t : Q_1 \to Q_0$ are maps such that any arrow α starts at $s(\alpha)$ and terminates at $t(\alpha)$. For any dimension vector $\underline{d} = \sum_i a_i i \in \mathbb{N}I$, we consider the affine space over $\mathbb{C}$

$$\mathbb{E}_{\underline{d}}(Q) = \bigoplus_{\alpha \in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{a_{s(\alpha)}}, \mathbb{C}^{a_{t(\alpha)}}).$$

Any element $x = (x_\alpha)_{\alpha\in Q_1}$ in $\mathbb{E}_{\underline{d}}(Q)$ defines a representation $M(x)$ with $\underline{\dim}\, M(x) = \underline{d}$ in a natural way. For any $\alpha \in Q_1$, we denote the vector space at $s(\alpha)$ (resp. $t(\alpha)$) of the representation M by $M_{s(\alpha)}$ (resp. $M_{t(\alpha)}$) and the linear map from $M_{s(\alpha)}$ to $M_{t(\alpha)}$ by M_α. A relation in Q is a linear combination $\sum_{i=1}^r \lambda_i p_i$, where $\lambda_i \in \mathbb{C}$ and p_i are paths of length at least two with $s(p_i) = s(p_j)$ and $t(p_i) = t(p_j)$ for all $1 \le i, j \le r$. For any $x = (x_\alpha)_{\alpha\in Q_1} \in \mathbb{E}_{\underline{d}}$ and any path $p = \alpha_m \cdots \alpha_2\alpha_1$ in Q, we set $x_p = x_{\alpha_m}\cdots x_{\alpha_2}x_{\alpha_1}$. Then x satisfies a relation $\sum_{i=1}^r \lambda_i p_i$ if $\sum_{i=1}^r \lambda_i x_{p_i} = 0$. If R is a set of relations in Q, then let $\mathbb{E}_{\underline{d}}(Q,R)$ be the closed subvariety of $\mathbb{E}_{\underline{d}}(Q)$ which consists of all elements satisfying all relations in R. Any element $x = (x_\alpha)_{\alpha\in Q_1}$ in $\mathbb{E}_{\underline{d}}(Q,R)$ defines in a natural way a representation $M(x)$ of $A = \mathbb{C}Q/J$ with $\underline{\dim}\, M(x) = \underline{d}$, where J is the admissible ideal generated by R. We consider the algebraic group

$$G_{\underline{d}}(Q) = \prod_{i\in I} GL(a_i,\mathbb{C}),$$

which acts on $\mathbb{E}_{\underline{d}}(Q)$ by $(x_\alpha)^g = (g_{t(\alpha)}x_\alpha g_{s(\alpha)}^{-1})$ for $g \in G_{\underline{d}}$ and $(x_\alpha) \in \mathbb{E}_{\underline{d}}$. It naturally induces an action of $G_{\underline{d}}(Q)$ on $\mathbb{E}_{\underline{d}}(Q,R)$. The induced orbit space is denoted by $\mathbb{E}_{\underline{d}}(Q,R)/G_{\underline{d}}(Q)$. There is a natural bijection between the set $\mathcal{M}(A,\underline{d})$ of isomorphism classes of $\mathbb{C}$-representations of A with dimension vector $\underline{d}$ and the set of orbits of $G_{\underline{d}}(Q)$ in $\mathbb{E}_{\underline{d}}(Q,R)$. So we may identify $\mathcal{M}(A,\underline{d})$ with $\mathbb{E}_{\underline{d}}(Q,R)/G_{\underline{d}}(Q)$.

The intersection of an open subset and a close subset in $\mathbb{E}_{\underline{d}}(Q,R)$ is called a locally closed subset. A subset in $\mathbb{E}_{\underline{d}}(Q,R)$ is called constructible if and only if it is a disjoint union of finitely many locally closed subsets. Obviously, an open set and a closed set are both constructible sets. A function f on $\mathbb{E}_{\underline{d}}(Q,R)$ is called constructible if $\mathbb{E}_{\underline{d}}(Q,R)$ can be divided into finitely many constructible sets such that f is constant on each such constructible set. Write $M(X)$ for the $\mathbb{C}$-vector space of constructible functions on some complex algebraic variety X.

Let $\mathcal{O}$ be a constructible set as defined above. Let $1_{\mathcal{O}}$ be the characteristic function of $\mathcal{O}$, defined by $1_{\mathcal{O}}(x) = 1$, for any $x \in \mathcal{O}$ and $1_{\mathcal{O}}(x) = 0$, for any $x \notin \mathcal{O}$. It is clear that $1_{\mathcal{O}}$ is the simplest constructible function, and any constructible function is a linear combination of characteristic functions. For any constructible subset $\mathcal{O}$ in $\mathbb{E}_{\underline{d}}(Q,R)$, we call $\mathcal{O}$ $G_{\underline{d}}$-invariant if $G_{\underline{d}}\cdot\mathcal{O} = \mathcal{O}$.

In the following, we will always assume constructible sets and functions to be $G_{\underline{d}}$-invariant unless particular stated.

2.2 *Euler Characterisitcs and Pushforward Functor*

Let χ denote Euler characteristic in compactly supported cohomology. Let X be an algebraic variety and $\mathcal{O}$ a constructible subset which is the disjoint union of finitely many locally closed subsets X_i for $i = 1,\ldots,m$. Define $\chi(\mathcal{O}) = \sum_{i=1}^m \chi(X_i)$. Note that it is well defined. We will use the following properties:

Proposition 2.1 ([Rie] and [Joy]). *Let X, Y be algebraic varieties over $\mathbb{C}$. Then*

(1) *If the algebraic variety X is the disjoint union of finitely many constructible sets $X_1, \ldots, X_r$, then*

$$\chi(X) = \sum_{i=1}^{r} \chi(X_i).$$

(2) *If $\varphi : X \longrightarrow Y$ is a morphism with the property that all fibers have the same Euler characteristic χ, then $\chi(X) = \chi \cdot \chi(Y)$. In particular, if φ is a locally trivial fibration in the analytic topology with fibre F, then $\chi(X) = \chi(F) \cdot \chi(Y)$.*

(3) *$\chi(\mathbb{C}^n) = 1$ and $\chi(\mathbb{P}^n) = n + 1$ for all $n \geq 0$.*

We recall the definition *pushforward* functor from the category of algebraic varieties over $\mathbb{C}$ to the category of $\mathbb{Q}$-vector spaces (see [Mac] and [Joy]). Let $\phi : X \to Y$ be a morphism of varieties. For $f \in M(X)$ and $y \in Y$, define

$$\phi_*(f)(y) = \sum_{c \in \mathbb{Q}} c\chi(f^{-1}(c) \cap \phi^{-1}(y)).$$

Theorem 2.2 ([Di], [Joy]). *Let X, Y and Z be algebraic varieties over $\mathbb{C}$, ϕ: $X \to Y$ and $\psi : Y \to Z$ be morphisms of varieties, and $f \in M(X)$. Then $\phi_*(f)$ is constructible, $\phi_* : M(X) \to M(Y)$ is a $\mathbb{Q}$-linear map and $(\psi \circ \phi)_* = (\psi)_* \circ (\phi)_*$ as $\mathbb{Q}$-linear maps from $M(X)$ to $M(Z)$.*

In order to deal with orbit spaces, we need to consider geometric quotients.

Definition 2.3. Let G be an algebraic group acting on a variety X and $\phi : X \to Y$ be a G-invariant morphism, i.e. a morphism constant on orbits. The pair (Y, ϕ) is called a geometric quotient if ϕ is open and for any open subset U of Y, the associated comorphism identifies the ring $\mathcal{O}_Y(U)$ of regular functions on U with the ring $\mathcal{O}_X(\phi^{-1}(U))^G$ of G-invariant regular functions on $\phi^{-1}(U)$.

The following result due to Rosenlicht [Ro] is essential to us.

Lemma 2.4. *Let X be a G-variety, then there exists an open and dense G-stable subset which has a geometric G-quotient.*

By this lemma, we can construct a finite stratification over X. Let U_1 be an open and dense G-stable subset of X as in Lemma 2.4. Then $\dim_{\mathbb{C}}(X - U_1) < \dim_{\mathbb{C}} X$. We can use the above lemma again, and there exists a dense open G-stable subset U_2 of $X - U_1$ which has a geometric G-quotient. Inductively, we get a finite stratification $X = \bigcup_{i=1}^{l} U_i$, where U_i is a G-invariant locally closed subset and has a geometric quotient, $l \leq \dim_{\mathbb{C}} X$. We denote by ϕ_{U_i} the geometric quotient map on U_i. Define the quasi Euler–Poincaré characteristic of X/G by $\chi(X/G) := \sum_i \chi(\phi_{U_i}(U_i))$. If $\{U'_i\}$ is another choice in the definition of $\chi(X/G)$, then $\chi(\phi_{U_i}(U_i)) = \sum_j \chi(\phi_{U_i \cap U'_j}(U_i \cap U'_j))$ and $\chi(\phi_{U'_j}(U'_j)) = \sum_i \chi(\phi_{U_i \cap U'_j}(U_i \cap U'_j))$. Thus $\sum_i \chi(\phi_{U_i}(U_i)) = \sum_i \chi(\phi_{U'_i}(U'_i))$ and $\chi(X/G)$ is well defined (see [XXZ]). Similarly, $\chi(\mathcal{O}/G) := \sum_i \chi(\phi_{U_i}(\mathcal{O} \cap U_i))$ is well defined for any G-invariant constructible subset $\mathcal{O}$ of X.

2.3 Quasi Euler Characteristics

We also introduce the following notation. Let f be a constructible function over a variety X, and it is natural to define

$$\int_{x\in X} f(x) := \sum_{m\in\mathbb{C}} m\chi(f^{-1}(m)). \tag{1}$$

Comparing with Proposition 2.1, we also have the following (see [XXZ]).

Proposition 2.5. *Let X, Y be algebraic varieties over $\mathbb{C}$ under the actions of the algebraic groups G and H, respectively. Then*

(1) *If the algebraic variety X is the disjoint union of finitely many G-invariant constructible sets $X_1, \dots, X_r$, then*

$$\chi(X/G) = \sum_{i=1}^{r} \chi(X_i/G).$$

(2) *If a morphism $\varphi : X \longrightarrow Y$ induces a quotient map $\phi : X/G \to Y/H$ whose fibers all have the same Euler characteristic χ, then $\chi(X/G) = \chi \cdot \chi(Y/H)$.*

Moreover, if there exists an action of an algebraic group G on X as in Definition 2.3, and f is a G-invariant constructible function over X, we define

$$\int_{x\in X/G} f(x) := \sum_{m\in\mathbb{C}} m\chi(f^{-1}(m)/G). \tag{2}$$

In particular, we frequently use the following corollary.

Corollary 2.6. *Let X, Y be algebraic varieties over $\mathbb{C}$ under the actions of an algebraic group G. These actions naturally induce an action of G on $X \times Y$. Then*

$$\chi(X \times_G Y) = \int_{y\in Y/G} \chi(X/G_y),$$

where G_y is the stabilizer in G of $y \in Y$ and $X \times_G Y$ is the orbit space of $X \times Y$ under the action of G.

3 Green's Formula Over Finite Fields

3.1 Green's Theorem

In this section, we recall Green's formula over finite fields ([Gre], [Rin2]). Let k be a finite field and Λ a hereditary finitary k-algebra, i.e. $\mathrm{Ext}^1(M, N)$ is a finite set and $\mathrm{Ext}^2(M, N) = 0$ for any Λ-modules M, N. Let $\mathcal{P}$ be the set of isomorphism classes of finite Λ-modules. Let $\mathcal{H}(\Lambda)$ be the Ringel–Hall algebra associated with $\mathrm{mod}\,\Lambda$. Green introduced on $\mathcal{H}$ a comultiplication so that $\mathcal{H}$ becomes a bialgebra up to a twist on $\mathcal{H}\otimes\mathcal{H}$. His proof of the compatibility between the multiplication and the comultiplication completely depends on the following Green's formula.

Given $\alpha \in \mathcal{P}$, let V_α be a representative in α, and $a_\alpha = |\operatorname{Aut}_\Lambda V_\alpha|$. Given ξ, η and λ in $\mathcal{P}$, let $g^{\lambda}_{\xi\eta}$ be the number of submodules Y of V_λ such that Y and V_λ/Y belong to η and ξ, respectively.

Theorem 3.1. *Let k be a finite field and Λ a hereditary finitary k-algebra. Let $\xi, \eta, \xi', \eta' \in \mathcal{P}$. Then*

$$a_\xi a_\eta a_{\xi'} a_{\eta'} \sum_{\lambda} g^{\lambda}_{\xi\eta} g^{\lambda}_{\xi'\eta'} a_\lambda^{-1} = \sum_{\alpha,\beta,\gamma,\delta} \frac{|\operatorname{Ext}^1(V_\gamma, V_\beta)|}{|\operatorname{Hom}(V_\gamma, V_\beta)|} g^{\xi}_{\gamma\alpha} g^{\xi'}_{\gamma\delta} g^{\eta}_{\delta\beta} g^{\eta'}_{\alpha\beta} a_\alpha a_\beta a_\delta a_\gamma.$$

Suppose $X \in \xi, Y \in \eta, M \in \xi', N \in \eta'$ and $A \in \gamma, C \in \alpha, B \in \delta, D \in \beta$, $E \in \lambda$. Set $h^{\xi\eta}_{\lambda} := |\operatorname{Ext}^1(X,Y)_E|$, where $\operatorname{Ext}^1(X,Y)_E$ is the subset of $\operatorname{Ext}^1(X,Y)$ consisting of elements ω such that the middle term of an exact sequence represented by ω is isomorphic to E. Then the above formula can be rewritten as ([DXX], [Hu2])

$$\sum_{\lambda} g^{\lambda}_{\xi\eta} h^{\xi'\eta'}_{\lambda} = \sum_{\alpha,\beta,\gamma,\delta} \frac{|\operatorname{Ext}^1(A,D)||\operatorname{Hom}(M,N)|}{|\operatorname{Hom}(A,D)||\operatorname{Hom}(A,C)||\operatorname{Hom}(B,D)|} g^{\xi'}_{\gamma\delta} g^{\eta'}_{\alpha\beta} h^{\gamma\alpha}_{\xi} h^{\delta\beta}_{\eta}.$$

3.2 Counting the Crossings Under Group Actions

For fixed kQ-modules X, Y, M, N with $\underline{\dim}\, X + \underline{\dim}\, Y = \underline{\dim}\, M + \underline{\dim}\, N$, we fix a Q_0-graded k-space E such that $\underline{\dim}\, E = \underline{\dim}\, X + \underline{\dim}\, Y$. Let (E, m) be the kQ-module structure on E given by an algebraic morphism $m : \Lambda \to \operatorname{End}_k E$. Let $Q(E, m)$ be the set of (a, b, a', b') such that the row and the column of the following diagram are exact:

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ & & Y \arrow[d, "a'"] & & \\ 0 \arrow[r] & N \arrow[r, "a"] & (E,m) \arrow[r, "b"] \arrow[d, "b'"] & M \arrow[r] & 0 \\ & & X \arrow[d] & & \\ & & 0 & & \end{tikzcd} \tag{3}$$

Let

$$Q(X,Y,M,N) = \bigcup_{m:\Lambda\to\mathrm{End}_k E} Q(E,m).$$

It is clear that

$$|Q(E,m)| = g^{\lambda}_{\xi\eta} g^{\lambda}_{\xi'\eta'} a_\xi a_\eta a_{\xi'} a_{\eta'},$$

where $\lambda \in \mathcal{P}$ satisfies $(E,m) \in \lambda$, or simply write $m \in \lambda$. Hence,

$$|Q(X,Y,M,N)| = \sum_{\lambda} \frac{|\operatorname{Aut}_k E|}{a_\lambda} g^{\lambda}_{\xi\eta} g^{\lambda}_{\xi'\eta'} a_\xi a_\eta a_{\xi'} a_{\eta'}.$$

There is an action of $\operatorname{Aut}_\Lambda(E,m)$ on $Q(E,m)$ given by

$$g.(a,b,a',b') = (ga, bg^{-1}, ga', b'g^{-1}).$$

This induces an orbit space of $Q(E,m)$, denoted by $Q(E,m)^*$. The orbit of (a,b,a',b') in $Q(E,m)^*$ is denoted by $(a,b,a',b')^*$. We have

$$\begin{aligned}&|Q(X,Y,M,N)|\\ &\quad = |\operatorname{Aut}_k E| \sum_{\lambda\in\mathcal{P}} \sum_{(a,b,a',b')^*\in Q(E,m)^*, m\in\lambda} \frac{1}{|\operatorname{Hom}(\operatorname{Coker} b'a, \operatorname{Ker} ba')|}.\end{aligned}$$

Furthermore, there is an action of the group $\operatorname{Aut} X \times \operatorname{Aut} Y$ on $Q(E,m)^*$ given by

$$(g_1,g_2).(a,b,a',b')^* = (a,b,a'g_2^{-1}, g_1 b')^*,$$

for $(g_1,g_2) \in \operatorname{Aut} X \times \operatorname{Aut} Y$ and $(a,b,a',b')^* \in Q(E,m)^*$. The stabilizer $G((a,b,a',b')^*)$ of $(a,b,a',b')^*$ is

$$\begin{aligned}&\{(g_1,\ g_2) \in \operatorname{Aut} X \times \operatorname{Aut} Y \mid\\ &\qquad ga' = a'g_2,\ b'g = g_1 b' \text{ for some } g \in 1 + a\operatorname{Hom}(M,N)b\}.\end{aligned}$$

The orbit space is denoted by $Q(E,m)^\wedge$, and the orbit of $(a,b,a',b')^*$ is denoted by $(a,b,a',b')^\wedge$. We have

$$\frac{1}{a_X a_Y}|Q(E,m)^*| = \sum_{(a,b,a',b')^\wedge\in Q(E,m)^\wedge} \frac{1}{|G((a,b,a',b')^*)|}.$$

3.3 Counting the Squares Under Group Actions

Let $\mathcal{D}(X, Y, M, N)^*$ be the set of $(B, D, e_1, e_2, e_3, e_4)$ such that the following diagram has exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & D \arrow[r, "e_1"] \arrow[d, "u'"] & Y \arrow[r, "e_2"] & B \arrow[r] \arrow[d, "x"] & 0 \\
 & N \arrow[d, "v'"] & & M \arrow[d, "y"] & \\
0 \arrow[r] & C \arrow[r, "e_3"] \arrow[d] & X \arrow[r, "e_4"] & A \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd} \tag{4}$$

where B, D are submodules of M, N, respectively and $A = M/B, C = N/D$. The maps u', v' and x, y are naturally induced. We have

$$|\mathcal{D}(X, Y, M, N)^*| = \sum_{\alpha,\beta,\gamma,\delta} g^{\xi}_{\gamma\alpha} g^{\xi'}_{\gamma\delta} g^{\eta}_{\delta\beta} g^{\eta'}_{\alpha\beta} a_\alpha a_\beta a_\delta a_\gamma.$$

There is an action of the group $\mathrm{Aut}_\Lambda X \times \mathrm{Aut}_\Lambda Y$ on $\mathcal{D}(X, Y, M, N)^*$ given by

$$(g_1, g_2).(B, D, e_1, e_2, e_3, e_4) = (B, D, g_2 e_1, e_2 g_2^{-1}, g_1 e_3, e_4 g_1^{-1})$$

for $(g_1, g_2) \in \mathrm{Aut}_\Lambda X \times \mathrm{Aut}_\Lambda Y$. The orbit space is denoted by $\mathcal{D}(X, Y, M, N)^\wedge$. We have

$$\begin{aligned}
&|\mathcal{D}(X, Y, M, N)^\wedge| \\
&\quad = \frac{1}{a_X a_Y} \sum_{\alpha,\beta,\gamma,\delta} |\operatorname{Hom}(A, C)||\operatorname{Hom}(B, D)| g^{\xi}_{\gamma\alpha} g^{\xi'}_{\gamma\delta} g^{\eta}_{\delta\beta} g^{\eta'}_{\alpha\beta} a_\alpha a_\beta a_\delta a_\gamma.
\end{aligned}$$

Fix a square as above, let $T = X \times_A M = \{(x \oplus m) \in X \oplus M \mid e_4(x) = y(m)\}$ and $S = Y \bigsqcup_D N = Y \oplus N/\{e_1(d) \oplus u'(d) \mid d \in D\}$. There is a unique map $f : S \to T$ (see [Rin2]) such that the natural long sequence

$$0 \to D \to S \xrightarrow{f} T \to A \to 0 \tag{5}$$

is exact.

Let (c, d) be a pair of maps such that c is surjective, d is injective and $cd = f$. The number of such pairs can be computed as follows. We have the following commutative diagram:

$$\begin{tikzcd} & 0 \arrow[r] & S \arrow[r, "d"] \arrow[d] & (E,m) \arrow[r, "d_1"] \arrow[d, "c"] & A \arrow[r] \arrow[d, equal] & 0 \\ \varepsilon_0 : & 0 \arrow[r] & \operatorname{Im} f \arrow[r] & T \arrow[r] & A \arrow[r] & 0 \end{tikzcd} \tag{6}$$

The exact sequence

$$0 \longrightarrow D \longrightarrow S \longrightarrow \operatorname{Im} f \longrightarrow 0$$

induces the following long exact sequence:

$$\begin{aligned} &0 \longrightarrow \operatorname{Hom}(A, D) \longrightarrow \operatorname{Hom}(A, S) \longrightarrow \operatorname{Hom}(A, \operatorname{Im} f) \longrightarrow \\ &\longrightarrow \operatorname{Ext}^1(A, D) \longrightarrow \operatorname{Ext}^1(A, S) \xrightarrow{\phi} \operatorname{Ext}^1(A, \operatorname{Im} f) \longrightarrow 0\,. \end{aligned} \tag{7}$$

We set $\varepsilon_0 \in \operatorname{Ext}^1(A, \operatorname{Im} f)$ corresponding to the canonical exact sequence

$$0 \longrightarrow \operatorname{Im} f \longrightarrow T \longrightarrow A \longrightarrow 0$$

and denote $\phi^{-1}(\varepsilon_0) \cap \operatorname{Ext}^1(A, S)_{(E,m)}$ by $\phi_m^{-1}(\varepsilon_0)$. Let $\mathcal{F}(f; m)$ be the set of (c, d) induced by diagram (6) with centre term (E, m). Let

$$\mathcal{F}(f) = \bigcup_{m:\Lambda \to \operatorname{End}_k E} \mathcal{F}(f; m).$$

Then

$$|\mathcal{F}(f; m)| = |\phi_m^{-1}(\varepsilon_0)| \frac{|\operatorname{Aut}_\Lambda(E, m)|}{|\operatorname{Hom}(A, S)|} |\operatorname{Hom}(A, \operatorname{Im} f)|,$$

$$|\mathcal{F}(f)| = |\operatorname{Aut}_k(E)| \frac{|\operatorname{Ext}^1(A, D)|}{|\operatorname{Hom}(A, D)|}.$$

Let $\mathcal{O}(E,m)$ be the set of $(B, D, e_1, e_2, e_3, e_4, c, d)$ such that the following diagram is commutative and has exact rows and columns:

$$
\begin{tikzcd}
 & 0 \arrow[d] & & & & 0 \arrow[d] & \\
0 \arrow[r] & D \arrow[rr, "e_1"] \arrow[dd, "u'"] & & Y \arrow[rr, "e_2"] \arrow[dd, dotted] \arrow[dl, dotted, "u_Y"'] & & B \arrow[r] \arrow[dd, "x"] & 0 \\
 & & S \arrow[dr, dotted, "d"] & & & & \\
 & N \arrow[rr, dotted] \arrow[ur, dotted, "u_N"] \arrow[dd, "v'"] & & (E,m) \arrow[rr, dotted] \arrow[dr, dotted, "c"] \arrow[dd, dotted] & & M \arrow[dd, "y"] & \\
 & & & & T \arrow[ur, dotted, "q_M"'] \arrow[dl, dotted, "q_X"] & & \\
0 \arrow[r] & C \arrow[rr, "e_3"] \arrow[d] & & X \arrow[rr, "e_4"] & & A \arrow[r] \arrow[d] & 0 \\
 & 0 & & & & 0 &
\end{tikzcd}
\tag{8}
$$

where the maps q_X, u_Y and q_M, u_N are naturally induced. In fact, the long exact sequence (5) has the following explicit form:

$$
0 \longrightarrow D \xrightarrow{u_Y e_1} S \xrightarrow{cd} T \xrightarrow{e_4 q_X} A \longrightarrow 0\,. \tag{9}
$$

We have

$$
\begin{aligned}
&|\mathcal{O}(E,m)| \\
&\quad = \sum_{\alpha,\beta,\gamma,\delta} |\phi_m^{-1}(\varepsilon_0)| \frac{|\operatorname{Aut}_\Lambda(E,m)|}{|\operatorname{Hom}(A,S)|} |\operatorname{Hom}(A, \operatorname{Im} f)| g_{\gamma\alpha}^{\xi} g_{\gamma\delta}^{\xi'} g_{\delta\beta}^{\eta} g_{\alpha\beta}^{\eta'} a_\alpha a_\beta a_\delta a_\gamma.
\end{aligned}
$$

Let $\mathcal{O}(X,Y,M,N) = \bigcup_{m:\Lambda\to\operatorname{End}_k E} \mathcal{O}(E,m)$

$$
|\mathcal{O}(X,Y,M,N)| = |\operatorname{Aut}_k(E)| \sum_{\alpha,\beta,\gamma,\delta} \frac{|\operatorname{Ext}^1(A,D)|}{|\operatorname{Hom}(A,D)|} g_{\gamma\alpha}^{\xi} g_{\gamma\delta}^{\xi'} g_{\delta\beta}^{\eta} g_{\alpha\beta}^{\eta'} a_\alpha a_\beta a_\delta a_\gamma
$$

The group $\operatorname{Aut}_\Lambda(E,m)$ naturally acts on $\mathcal{O}(E,m)$ and $\mathcal{O}(X,Y,M,N)$ as follows:

$$
g.(B, D, e_1, e_2, e_3, e_4, c, d) = (B, D, e_1, e_2, e_3, e_4, cg^{-1}, gd).
$$

We denote the orbit space by $\mathcal{O}(E,m)^*$ and $\mathcal{O}(X,Y,M,N)^*$. The orbit of $(B,D,e_1,e_2,e_3,e_4,c,d)$ is denoted by $(B,D,e_1,e_2,e_3,e_4,c,d)^*$. Then

$$|\mathcal{O}(X,Y,M,N)^*| = \sum_{\alpha,\beta,\gamma,\delta} |\operatorname{Ext}^1(A,D)| g^{\xi}_{\gamma\alpha} g^{\xi'}_{\gamma\delta} g^{\eta}_{\delta\beta} g^{\eta'}_{\alpha\beta} a_\alpha a_\beta a_\delta a_\gamma.$$

Similar to that on $\mathcal{D}(X,Y,M,N)^*$, there is an action of $\operatorname{Aut} X \times \operatorname{Aut} Y$ on $\mathcal{O}(X,Y,M,N)^*$ given by

$$(g_1,g_2).(B,D,e_1,e_2,e_3,e_4,c,d)^* = (B,D,g_2e_1,e_2g_2^{-1},g_1e_3,e_4g_1^{-1},c',d')^*.$$

Let us determine the relation between (c',d') and (c,d).

It is clear that there are isomorphisms:

$$a_1 : S \to S' \quad \text{and} \quad a_2 : T \to T'$$

induced by isomorphisms:

$$\begin{pmatrix} g_2 & 0 \\ 0 & \mathrm{id} \end{pmatrix} : Y \oplus N \to Y \oplus N \quad \text{and} \quad \begin{pmatrix} g_1 & 0 \\ 0 & \mathrm{id} \end{pmatrix} : X \oplus M \to X \oplus M.$$

Hence, $c' = a_2 c$, $d' = d a_1^{-1}$.

The stabilizer of $(B,D,e_1,e_2,e_3,e_4,c,d)^*$ is denoted by $G((B,D,e_1,e_2,e_3,e_4,c,d)^*)$, which is

$$\{(g_1,g_2) \in \operatorname{Aut} X \times \operatorname{Aut} Y \mid g_1 \in e_3 \operatorname{Hom}(A,C)e_4, g_2 \in e_1 \operatorname{Hom}(B,D)e_2,$$
$$cg^{-1} = c', gd = d' \text{ for some } g \in \operatorname{Aut}(E,m)\}.$$

The orbit space is denoted by $\mathcal{O}(X,Y,M,N)^\wedge$ and the orbit is denoted by $(B,D,e_1,e_2,e_3,e_4,c,d)^\wedge$. Then

$$\frac{1}{a_X a_Y} |\mathcal{O}(X,Y,M,N)^*| = \sum_{\substack{(B,D,e_1,e_2,e_3,e_4,c,d)^\wedge \\ \in \mathcal{O}(X,Y,M,N)^\wedge}} \frac{1}{|G((B,D,e_1,e_2,e_3,e_4,c,d)^*)|}.$$

3.4 Induced Bijections

There is a bijection $\Omega : Q(E,m) \to \mathcal{O}(E,m)$ which induces Green's formula. In the same way, we also have the following proposition.

Proposition 3.2. *There exist bijections* $\Omega^* : Q(E,m)^* \to \mathcal{O}(E,m)^*$ *and* $\Omega^\wedge : Q(E,m)^\wedge \to \mathcal{O}(E,m)^\wedge$.

Proof. For any $(a,b,a',b') \in Q(E,m)$,

$$\Omega(a,b,a',b') = (\operatorname{Ker} b'a, \operatorname{Im} ba', a'^{-1}a, ba', b'a, bb'^{-1}, c, d),$$

where c, d are defined by

$$du_N = a, du_Y = a', q_M c = b, q_X c = b'.$$

Hence,

$$\begin{aligned}\Omega(g.(a,b,a',b')) &= (ga, bg^{-1}, ga', b'g^{-1})\\ &= (\mathrm{Ker}\, b'a, \mathrm{Im}\, ba', a'^{-1}a, ba', b'a, bb'^{-1}, cg^{-1}, gd)\\ &= g.(\mathrm{Ker}\, b'a, \mathrm{Im}\, ba', a'^{-1}a, ba', b'a, bb'^{-1}, c, d),\end{aligned}$$

i.e.

$$\Omega^*((a,b,a',b')^*) = ((\mathrm{Ker}\, b'a, \mathrm{Im}\, ba', a'^{-1}a, ba', b'a, bb'^{-1}, c, d))^*$$

for $g \in \mathrm{Aut}(E, m)$. Similarly,

$$\begin{aligned}&\Omega^*((g_1,g_2).(a,b,a',b')^*) = (a, b, a'g_2^{-1}, g_1 b')^*\\ &= (\mathrm{Ker}\, g_1 b'a, \mathrm{Im}\, ba'g_2^{-1}, g_2 a'^{-1}a, ba'g_2^{-1}, g_1 b'a, bb'^{-1}g_1^{-1}, c', d')^*\\ &= (g_1, g_2).(\mathrm{Ker}\, b'a, \mathrm{Im}\, ba', a'^{-1}a, ba', b'a, bb'^{-1}, c, d)^*\end{aligned}$$

for $(g_1, g_2) \in \mathrm{Aut}\, X \times \mathrm{Aut}\, Y$. Hence,

$$\Omega^\wedge((a,b,a',b')^\wedge) = ((\mathrm{Ker}\, b'a, \mathrm{Im}\, ba', a'^{-1}a, ba', b'a, bb'^{-1}, c, d))^\wedge.$$

□

In particular, if $(a,b,a',b')^*$ corresponds to $(B, D, e_1, e_2, e_3, e_4, c, d)^*$, then

$$G((a,b,a',b')^*) = G((B, D, e_1, e_2, e_3, e_4, c, d)^*).$$

We also give the following variant of Green's formula, which is suggestive for the projective Green's formula over the complex numbers in the next section.

$$\begin{aligned}&\sum_{\lambda;\lambda\neq\xi'\oplus\eta'} \frac{1}{q-1} h_\lambda^{\xi'\eta'} g_{\xi\eta}^\lambda\\ &= \sum_{\substack{\alpha,\beta,\gamma,\delta;\\ \alpha\oplus\gamma\neq\xi \text{ or } \beta\oplus\delta\neq\eta}} \frac{|\mathrm{Ext}^1(A,D)||\mathrm{Hom}(M,N)|}{|\mathrm{Hom}(A,D)||\mathrm{Hom}(A,C)||\mathrm{Hom}(B,D)|} \frac{1}{q-1} h_\xi^{\gamma\alpha} h_\eta^{\delta\beta} g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'}\\ &+ \sum_{\substack{\alpha,\beta,\gamma,\delta;\\ \alpha\oplus\gamma=\xi,\beta\oplus\delta=\eta}} \frac{1}{q-1}\left(\frac{|\mathrm{Ext}^1(A,D)||\mathrm{Hom}(M,N)|}{|\mathrm{Hom}(A,D)||\mathrm{Hom}(A,C)||\mathrm{Hom}(B,D)|} - 1\right) g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'}\\ &+ \frac{1}{q-1}\left(\sum_{\substack{\alpha,\beta,\gamma,\delta;\\ \alpha\oplus\gamma=\xi,\beta\oplus\delta=\eta}} g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} - g_{\xi\eta}^{\xi'\oplus\eta'}\right).\end{aligned}$$

4 Green's Formula Over the Complex Numbers

4.1 Flags and Extensions

From now on, we consider $A = \mathbb{C}Q$, where $\mathbb{C}$ is the field of complex numbers. Let $\mathcal{O}_1, \mathcal{O}_2$ be G-invariant constructible subsets in $\mathbb{E}_{\underline{d}_1}(Q), \mathbb{E}_{\underline{d}_2}(Q)$, respectively, and let $\underline{d} = \underline{d}_1 + \underline{d}_2$. Define

$$\mathcal{V}(\mathcal{O}_1, \mathcal{O}_2; L) = \{0 = X_0 \subseteq X_1 \subseteq X_2 = L \mid \\ X_i \in \operatorname{mod} A, X_1 \in \mathcal{O}_2, \text{ and } L/X_1 \in \mathcal{O}_1\},$$

where $L \in \mathbb{E}_{\underline{d}}(Q)$. In particular, when $\mathcal{O}_1, \mathcal{O}_2$ are the orbits of A-modules X, Y respectively, we write $\mathcal{V}(X, Y; L)$ instead of $\mathcal{V}(\mathcal{O}_1, \mathcal{O}_2; L)$.

Let α be the image of X in $\mathbb{E}_{\underline{d}_\alpha}(Q)/G_{\underline{d}_\alpha}$. We write $X \in \alpha$, sometimes we also use the notation $\overline{X}$ to denote the image of X and the notation V_α to denote a representative of α. Instead of $\underline{d}_\alpha$, we use $\underline{\alpha}$ to denote the dimension vector of α. Put

$$g^\lambda_{\alpha\beta} = \chi(\mathcal{V}(X, Y; L))$$

for $X \in \alpha, Y \in \beta$ and $L \in \lambda$. Both are well defined and independent of the choice of objects in the orbits.

Definition 4.1 ([Rie]). For any $L \in \operatorname{mod} A$, let $L = \bigoplus_{i=1}^r L_i$ be the decomposition into indecomposables, then an action of $\mathbb{C}^*$ on L is defined by

$$t.(v_1, \ldots, v_r) = (tv_1, \ldots, t^r v_r)$$

for $t \in \mathbb{C}^*$ and $v_i \in L_i$ for $i = 1, \ldots, r$.

It induces an action of $\mathbb{C}^*$ on $\mathcal{V}(X, Y; L)$ for any A-modules X, Y and L. Let $(X_1 \subseteq L) \in \mathcal{V}(X, Y; L)$ and $t.X_1$ be the action of $\mathbb{C}^*$ on X_1 as above under the decomposition of L, then there is a natural isomorphism between A-modules $t_{X_1} : X_1 \simeq t.X_1$. Define $t.(X_1 \subseteq L) = (t.X_1 \subseteq L)$.

Let $D(X, Y)$ be the vector space over $\mathbb{C}$ of all tuples $d = (d(\alpha))_{\alpha \in Q_1}$ such that each linear map $d(\alpha)$ belongs to $\operatorname{Hom}_{\mathbb{C}}(X_{s(\alpha)}, Y_{t(\alpha)})$. Define $\pi : D(X, Y) \to \operatorname{Ext}^1(X, Y)$ by sending d to the short exact sequence

$$\varepsilon: \quad 0 \longrightarrow Y \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} L(d) \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} X \longrightarrow 0,$$

where $L(d)$ is the direct sum of X and Y as a vector space and for any $\alpha \in Q_1$,

$$L(d)_\alpha = \begin{pmatrix} Y_\alpha & d(\alpha) \\ 0 & X_\alpha \end{pmatrix}.$$

Fix a vector space decomposition $D(X,Y) = \operatorname{Ker}\pi \oplus E(X,Y)$, then we can identify $\operatorname{Ext}^1(X,Y)$ with $E(X,Y)$ (see [Rie], [DXX] or [GLS]). There is a natural $\mathbb{C}^*$-action on $E(X,Y)$ given by $t.d = (td(\alpha))$ for any $t \in \mathbb{C}^*$. This induces an action of $\mathbb{C}^*$ on $\operatorname{Ext}^1(X,Y)$. By the isomorphism of $\mathbb{C}Q$-modules between $L(d)$ and $L(t.d)$, $t.\varepsilon$ is the following short exact sequence:

$$0 \longrightarrow Y \xrightarrow{\begin{pmatrix} t \\ 0 \end{pmatrix}} L(d) \xrightarrow{(0\ 1)} X \longrightarrow 0$$

for any $t \in \mathbb{C}^*$. Let $\operatorname{Ext}^1(X,Y)_L$ be the subset of $\operatorname{Ext}^1(X,Y)$ of the equivalence classes of short exact sequences whose middle term is isomorphic to L. Then $\operatorname{Ext}^1(X,Y)_L$ can be viewed as a constructible subset of $\operatorname{Ext}^1(X,Y)$ under the identification between $\operatorname{Ext}^1(X,Y)$ and $E(X,Y)$. Put

$$h_\lambda^{\alpha\beta} = \chi(\operatorname{Ext}^1_A(X,Y)_L)$$

for $X \in \alpha, Y \in \beta$ and $L \in \lambda$. The following is known, for example, see [DXX].

Lemma 4.2. *For $A, B, X \in \operatorname{mod}\Lambda$, $\chi(\operatorname{Ext}^1_\Lambda(A,B)_X) = 0$ unless $X \simeq A \oplus B$.*

We remark that both $\mathcal{V}(X,Y;L)$ and $\operatorname{Ext}^1(X,Y)_L$ can be viewed as the orbit spaces of

$$W(X,Y;L) := \{(f,g) \mid 0 \longrightarrow Y \xrightarrow{f} L \xrightarrow{g} X \longrightarrow 0 \text{ is an exact sequence}\}$$

under the actions of $G_{\underline{\alpha}} \times G_{\underline{\beta}}$ and $G_{\underline{\lambda}}$, respectively, for $X \in \alpha, Y \in \beta$ and $L \in \lambda$.

4.2 Higher Order Associativity

For fixed ξ, η, ξ', η', consider the following canonical embedding:

$$\bigcup_{\alpha,\beta,\gamma,\delta;\alpha\oplus\gamma=\xi,\beta\oplus\delta=\eta} \mathcal{V}(V_\alpha, V_\beta; V_{\eta'}) \times \mathcal{V}(V_\gamma, V_\delta; V_{\xi'}) \xrightarrow{i} \mathcal{V}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'}) \quad (10)$$

sending $(V^1_{\eta'} \subseteq V_{\eta'}, V^1_{\xi'} \subseteq V_{\xi'})$ to $(V^1_{\xi'} \oplus V^1_{\eta'} \subseteq V_{\xi'} \oplus V_{\eta'})$ in a natural way. We set

$$\overline{\mathcal{V}}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'}) := V(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'}) \setminus \operatorname{Im} i,$$

i.e.

$$\mathcal{V}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'}) = \overline{\mathcal{V}}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'}) \cup \mathcal{V}_1, \quad (11)$$

where $\mathcal{V}_1 = \operatorname{Im} i$. Define

$$\mathcal{V}_1(\delta,\beta) := \operatorname{Im}(\mathcal{V}(V_\alpha, V_\beta; V_{\eta'}) \times \mathcal{V}(V_\gamma, V_\delta; V_{\xi'})).$$

Consider the $\mathbb{C}$-space $M_G(A) = \bigoplus_{\underline{d}\in\mathbb{N}^n} M_{G_{\underline{d}}}(Q)$ where $M_{G_{\underline{d}}}(Q)$ is the $\mathbb{C}$-space of $G_{\underline{d}}$-invariant constructible function on $\mathbb{E}_{\underline{d}}(Q)$. Define the convolution multiplication on $M_G(A)$ by

$$f \bullet g(L) = \sum_{c,d\in\mathbb{C}} \chi(\mathcal{V}(f^{-1}(c), g^{-1}(d); L))cd$$

for any $f \in M_{G_{\underline{d}}}(Q), g \in M_{G_{\underline{d}'}}(Q)$ and $L \in \mathbb{E}_{\underline{d}+\underline{d}'}$.

As usual for an algebraic variety V and a constructible function f on V, using the notation (1) in Sect. 2, we have

$$f \bullet g(L) = \int_{\mathcal{V}(\mathrm{supp}(f),\mathrm{supp}(g);L)} f(x')g(x'').$$

The following is well known (see [Lu], [Rie]), see a proof in [DXX].

Proposition 4.3. *The space $M_G(\Lambda)$ under the convolution multiplication $\bullet$ is an associative $\mathbb{C}$-algebra with unit element.*

The above proposition implies the following identity

Theorem 4.4. *For fixed A-modules X, Y, Z and M with dimension vectors $\underline{d}_X, \underline{d}_Y, \underline{d}_Z$ and $\underline{d}_M$ such that $\underline{d}_M = \underline{d}_X + \underline{d}_Y + \underline{d}_Z$, we have*

$$\int_{\overline{L}\in\mathbb{E}_{\underline{d}_X+\underline{d}_Y}(A)/G_{\underline{d}_X+\underline{d}_Y}} g^L_{XY} g^M_{LZ} = \int_{\overline{L'}\in\mathbb{E}_{\underline{d}_Y+\underline{d}_Z}(A)/G_{\underline{d}_Y+\underline{d}_Z}} g^M_{XL'} g^{L'}_{YZ}.$$

Define

$$W(X,Y;L_1,L_2) := \{(f,g,h) \mid$$
$$0 \longrightarrow Y \xrightarrow{f} L_1 \xrightarrow{g} L_2 \xrightarrow{h} X \longrightarrow 0 \quad \text{is an exact sequence}\}.$$

Under the action of $G_{\underline{\alpha}} \times G_{\underline{\beta}}$, where $\underline{\alpha} = \underline{\dim}\, X$ and $\underline{\beta} = \underline{\dim}\, Y$, the orbit space is denoted by $\mathcal{V}(X,Y;L_1,L_2)$. In fact,

$$\mathcal{V}(X,Y;L_1,L_2) = \{g : L_1 \to L_2 \mid \mathrm{Ker}\, g \cong Y \text{ and } \mathrm{Coker}\, g \cong X\}.$$

Put

$$h^{L_1L_2}_{XY} = \chi(\mathcal{V}(X,Y;L_1L_2)).$$

We have the following "higher order" associativity, which is similar to the associativity of multiplication in derived Hall algebras (see [To], [XX])

Theorem 4.5. *For fixed A-modules X, Y_i, L_i for $i = 1, 2$, we have*

$$\int_{\overline{Y}} g^Y_{Y_2Y_1} h^{L_1L_2}_{XY} = \int_{\overline{L'_1}} g^{L_1}_{L'_1Y_1} h^{L'_1L_2}_{XY_2}.$$

Dually, for fixed A-modules X_i, Y, L_i for $i = 1, 2$, we have

$$\int_{\overline{X}} g^{X}_{X_2X_1} h^{L_1L_2}_{XY} = \int_{\overline{L'_2}} g^{L_2}_{X_2L'_2} h^{L_1L'_2}_{X_1Y}.$$

Proof. Define

$$\begin{aligned} EF(X, Y_1, Y_2; L_1, L_2) = \{(g, Y^\bullet) \mid g : L_1 \to L_2,\ Y^\bullet = (\operatorname{Ker} g \supseteq Y' \supseteq 0) \\ \text{such that } \operatorname{Coker} g \simeq X,\ Y' \simeq Y_1,\ \operatorname{Ker} g/Y' \simeq Y_2\} \end{aligned}$$

and

$$\begin{aligned} EF'(X, Y_1, Y_2; L'_1, L_2) = \{(g', L^\bullet_1) \mid g' : L'_1 \to L_2,\ L^\bullet = (L_1 \supseteq Y' \supseteq 0) \\ \text{such that } \operatorname{Ker} g' \simeq Y_2,\ \operatorname{Coker} g' \simeq X,\ Y' \simeq Y_1,\ L_1/Y' \simeq L'_1\}. \end{aligned}$$

Consider the following diagram:

$$\begin{tikzcd} & Y_1 \arrow[r, equal] \arrow[d] & Y_1 \arrow[d] & & & \\ 0 \arrow[r] & \operatorname{Ker} g \arrow[r] \arrow[d] & L_1 \arrow[r, "g"] \arrow[d] & L_2 \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y_2 \arrow[r] & L'_1 \arrow[r, "g'"] & L_2 \arrow[r] & X \arrow[r] & 0 \end{tikzcd} \tag{12}$$

where $L'_1 = L_1/Y'$ is the pushout. This gives the following morphism of varieties:

$$EF(X, Y_1, Y_2; L_1, L_2) \to EF'(X, Y_1, Y_2; L'_1, L_2),$$

sending $(g, Y^\bullet)$ to $(g', L^\bullet_1)$ where $g' : L_1/Y' \to L_2$. Conversely, we also have the morphism

$$EF'(X, Y_1, Y_2; L'_1, L_2) \to EF(X, Y_1, Y_2; L_1, L_2)$$

sending $(g', L^\bullet_1)$ to $(g, Y^\bullet)$ where g is the composition: $L_1 \to L_1/Y' \simeq L'_1 \xrightarrow{g'} L_2$ (this implies $Y' \subseteq \operatorname{Ker} g$). A simple check shows that there exists a homeomorphism between $EF(X, Y_1, Y_2; L_1, L_2)$ and $EF'(X, Y_1, Y_2; L'_1, L_2)$. By Proposition 2.5, we have

$$\chi(EF(X, Y_1, Y_2; L_1, L_2)) = \int_{\overline{Y}} g^{Y}_{Y_2Y_1} h^{L_1L_2}_{XY}$$

and

$$\chi(EF'(X, Y_1, Y_2; L'_1, L_2)) = \int_{\overline{L'_1}} g^{L_1}_{L'_1Y_1} h^{L'_1L_2}_{XY_2}.$$

This completes the proof. ∎

We define

$$\mathrm{Hom}(L_1, L_2)_{Y[1]\oplus X} = \{g \in \mathrm{Hom}(L_1, L_2) \mid \mathrm{Ker}\, g \simeq Y, \mathrm{Coker}\, g \simeq X\}.$$

Then, it is easy to identify that

$$\mathcal{V}(X, Y; L_1, L_2) = \mathrm{Hom}(L_1, L_2)_{Y[1]\oplus X}.$$

We can consider a $\mathbb{C}^*$-action on $\mathrm{Hom}(L_1, L_2)_{Y[1]\oplus X}$ or $\mathcal{V}(X, Y; L_1, L_2)$ simply by $t.(f, g, h)^* = (f, tg, h)^*$ for $t \in \mathbb{C}^*$ and $(f, g, h)^* \in \mathcal{V}(X, Y; L_1, L_2)$. We also have a projective version of Theorem 4.5, where $\mathbb{P}$ indicates the corresponding orbit space under the $\mathbb{C}^*$-action.

Theorem 4.6. *For fixed A-modules X, Y_i, L_i for $i = 1, 2$, we have*

$$\int_{\overline{Y}} g^{Y}_{Y_2 Y_1} \chi(\mathbb{P}\mathrm{Hom}(L_1, L_2)_{Y[1]\oplus X}) = \int_{\overline{L'}_1} g^{L_1}_{L'_1 Y_1} \chi(\mathbb{P}\mathrm{Hom}(L'_1, L_2)_{Y_2[1]\oplus X}).$$

Dually, for fixed A-modules X_i, Y, L_i for $i = 1, 2$, we have

$$\int_{\overline{X}} g^{X}_{X_2 X_1} \chi(\mathbb{P}\mathrm{Hom}(L_1, L_2)_{Y[1]\oplus X}) = \int_{\overline{L'}_2} g^{L_2}_{X_2 L'_2} \chi(\mathbb{P}\mathrm{Hom}(L_1, L'_2)_{Y[1]\oplus X_1}).$$

4.3 Geometry Over Crossings

For fixed ξ, η and ξ', η' with $\underline{\xi} + \underline{\eta} = \underline{\xi'} + \underline{\eta'} = \underline{\lambda}$, let $V_\lambda \in \mathbb{E}_{\underline{\lambda}}$ and $Q(V_\lambda)$ be the set of (a, b, a', b') such that the row and the column of the following diagram are exact:

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ & & V_\eta \arrow[d, "a'"] & & \\ 0 \arrow[r] & V_{\eta'} \arrow[r, "a"] & V_\lambda \arrow[r, "b"] \arrow[d, "b'"] & V_{\xi'} \arrow[r] & 0 \\ & & V_\xi \arrow[d] & & \\ & & 0 & & \end{tikzcd} \tag{13}$$

Set

$$Q(\xi, \eta, \xi', \eta') := \bigcup_{V_\lambda \in \mathbb{E}_{\underline{\lambda}}} Q(V_\lambda).$$

We remark that $Q(\xi, \eta, \xi', \eta')$ can be viewed as a constructible subset of the module variety $\mathbb{E}_{(\underline{\xi}, \underline{\eta}, \underline{\xi'}, \underline{\eta'}, \underline{\lambda})}$ with $\underline{\xi} + \underline{\eta} = \underline{\xi'} + \underline{\eta'} = \underline{\lambda}$ of the following quiver:

$$\begin{tikzcd} & 2 \arrow[d] & \\ 4 \arrow[r] & 5 \arrow[r] \arrow[d] & 3 \\ & 1 & \end{tikzcd} \tag{14}$$

We have the following action of $G_{\underline{\lambda}}$ on $Q(\xi, \eta, \xi', \eta')$:

$$g.(a, b, a', b') = (ga, bg^{-1}, ga', b'g^{-1}).$$

The orbit space of $Q(\xi, \eta, \xi', \eta')$ is denoted by $Q(\xi, \eta, \xi', \eta')^*$, and the orbit of (a, b, a', b') in $Q(\xi, \eta, \xi', \eta')^*$ is denoted by $(a, b, a', b')^*$. We also have the following action of $G_{\underline{\lambda}}$ on $W(V_{\xi'}, V_{\eta'}; \mathbb{E}_{\underline{\lambda}})$: $g.(a, b) = (ga, bg^{-1})$. In the induced orbit space $\mathrm{Ext}^1(V_{\xi'}, V_{\eta'})$, the orbit of (a, b) is denoted by $(a, b)^*$. Hence, we have

$$\begin{tikzcd} W(V_{\xi'}, V_{\eta'}; \mathbb{E}_{\underline{\lambda}}) \times W(\xi, \eta; \mathbb{E}_{\underline{\lambda}}) = Q(\xi, \eta, \xi', \eta') \arrow[r, "\phi_2"] \arrow[d, "\phi_1"'] & \mathrm{Ext}^1(V_{\xi'}, V_{\eta'}) \\ Q(\xi, \eta, \xi', \eta')^* \arrow[ur, "\phi"'] & \end{tikzcd}, \tag{15}$$

where $\phi((a, b, a', b')^*) = (a, b)^*$ is well defined.

Let $(a, b, a', b') \in Q(V_\lambda)$. We claim that the stabilizer of (a, b, a', b') in ϕ_1 is

$$a'e_1 \operatorname{Hom}(\operatorname{Coker} b'a, \operatorname{Ker} ba')e_4 b',$$

which is isomorphic to $\operatorname{Hom}(\operatorname{Coker} b'a, \operatorname{Ker} ba')$, where the injection $e_1 : \operatorname{Ker} ba' \to V_\lambda$ is induced naturally by a', and the surjection $e_4 : V_\lambda \to \operatorname{Coker} b'a$ is induced naturally by b'. In fact, consider the action of $G_{\underline{\lambda}}$ on $W(V_\xi, V_\eta; \mathbb{E}_{\underline{\lambda}})$ given by $g.(a', b') = (ga, bg^{-1})$, the stabilizer of (a', b') is $1 + a' \operatorname{Hom}(V_\xi, V_\eta)b'$ ([Rin2]). It is clear that the stabilizer of (a, b, a', b') under the action given by ϕ_1 is the following subgroup:

$$\{1 + a'fb' \mid f \in \operatorname{Hom}(V_\xi, V_\eta), ba'fb' = 0, a'fb'a = 0\}.$$

Since b' is surjective and a' is injective, $ba'fb' = 0, a'fb'a = 0$ imply $ba'f = 0$, $fb'a = 0$. This means $\operatorname{Im} f \in \operatorname{Ker} ba'$ and $f(\operatorname{Ker} b'a) = 0$. We easily deduce the above claim by this conclusion. In the same way, the stabilizer of (a, b) under the action given by ϕ_2 is $1 + a \operatorname{Hom}(V_{\xi'}, V_{\eta'})b$, which is isomorphic to $\operatorname{Hom}(V_{\xi'}, V_{\eta'})$ just for a is injective and b is surjective. We now compute the fibre $\phi_1(\phi_2^{-1}((a, b)^*))$ of ϕ over $(a, b)^*$.

$$\phi_2^{-1}((a, b)^*) = (ga, bg^{-1}, a', b'),$$

where $(a', b') \in W(V_\xi, V_\eta; V_\lambda)$. Fix a, b, and let $U = \{(a, b, a', b')\}$. Then we have

$$U \subset \phi_2^{-1}((a,b)^*), \quad \phi_1(U) = \phi_1(\phi_2^{-1}((a,b)^*)).$$

The restriction $\phi_1|_U : U \to \phi_1(U)$ can be viewed as the action of the group $a \operatorname{Hom}(V_{\xi'}, V_{\eta'})b$ with stabilizer $a'e_1 \operatorname{Hom}(\operatorname{Coker} b'a, \operatorname{Ker} ba')e_4 b'$, i.e. the fibre of $\phi_1|_U$ is isomorphic to $a \operatorname{Hom}(V_{\xi'}, V_{\eta'})b / a'e_1 \operatorname{Hom}(\operatorname{Coker} b'a, \operatorname{Ker} ba')e_4 b'$. Hence, by Corollary 2.6,

$$\chi(W(V_\xi, V_\eta; V_\lambda)) = \chi(U) = \chi(\phi_1(U)) = \chi(\phi_1(\phi_2^{-1}((a,b)^*))).$$

Moreover, consider the action of $G_{\underline{\xi}} \times G_{\underline{\eta}}$ on $Q(\xi, \eta, \xi', \eta')^*$ and the induced orbit space, denoted by $Q(\xi, \eta, \xi', \eta')^\wedge$. The stabilizer $\operatorname{Stab}_G((a, b, a', b')^*)$ of $(a, b, a', b')^*$ is

$$\{(g_1, g_2) \in G_{\underline{\xi}} \times G_{\underline{\eta}} \mid ga' = a'g_2, b'g = g_1 b' \text{ for some } g \in 1 + a \operatorname{Hom}(V_{\xi'}, V_{\eta'})b\},$$

also denoted by $G((a, b, a', b')^*)$. This determines the group embedding

$$\operatorname{Stab}_G((a, b, a', b')^*) \longrightarrow (1 + a \operatorname{Hom}(V_{\xi'}, V_{\eta'})b) / (1 + ae_1 \operatorname{Hom}(\operatorname{Coker} b'a, \operatorname{Ker} ba')e_4 b).$$

The group $G((a, b, a', b')^*)$ is isomorphic to a vector space since $ba = 0$. We know that $1 + a \operatorname{Hom}(V_{\xi'}, V_{\eta'})b$ is the subgroup of $\operatorname{Aut} V_\lambda$, and it acts on $W(V_\xi, V_\eta; \mathbb{E}_{\underline{\lambda}})$ naturally. The orbit space of $W(V_\xi, V_\eta; \mathbb{E}_{\underline{\lambda}})$ under the action of $1 + a\operatorname{Hom}(V_{\xi'}, V_{\eta'})b$ is denoted by $\widetilde{W}(V_\xi, V_\eta; \mathbb{E}_{\underline{\lambda}})$, and similar considerations hold for $\mathcal{V}(V_\xi, V_\eta; \mathbb{E}_{\underline{\lambda}})$. Combined with the discussion above, we have the following commutative diagram of actions of groups:

$$\begin{CD} W(V_\xi, V_\eta; \mathbb{E}_{\underline{\lambda}}) @>{1+a \operatorname{Hom}(V_{\xi'}, V_{\eta'})b}>> \widetilde{W}(V_\xi, V_\eta; \mathbb{E}_{\underline{\lambda}}) \\ @V{G_{\underline{\xi}} \times G_{\underline{\eta}}}VV @VV{G_{\underline{\xi}} \times G_{\underline{\eta}}}V \\ \mathcal{V}(V_\xi, V_\eta; \mathbb{E}_{\underline{\lambda}}) @>{1+a \operatorname{Hom}(V_{\xi'}, V_{\eta'})b}>> \widetilde{\mathcal{V}}(V_\xi, V_\eta; \mathbb{E}_{\underline{\lambda}}) \end{CD} \tag{16}$$

The stabilizer of $(a', b')^\wedge$ in the bottom map is:

$$\{g \in 1 + a \operatorname{Hom}(V_{\xi'}, V_{\eta'})b \mid ga' = a'g_2, b'g = g_1 b' \text{ for some } (g_1, g_2) \in G_{\underline{\xi}} \times G_{\underline{\eta}}\},$$

which is isomorphic to a vector space too, and it is denoted by $V(a, b, a', b')$. We can construct the map from $V(a, b, a', b')$ to $\operatorname{Stab}_G((a, b, a', b')^*)$ sending g to (g_1, g_2). It is well defined since a' is injective and b' is surjective. We have

$$V(a, b, a', b') / \operatorname{Hom}(\operatorname{Coker} b'a, \operatorname{Ker} ba') \cong \operatorname{Stab}_G((a, b, a', b')^*).$$

We have the following proposition.

Proposition 4.7. *The fiber over $(a,b)^* \in \mathrm{Ext}^1(V_{\xi'}, V_{\eta'})_\lambda$ of the surjective map*

$$\phi^\wedge : Q(\xi,\eta,\xi',\eta')^\wedge \to \mathrm{Ext}^1(V_{\xi'}, V_{\eta'})$$

is isomorphic to $\widetilde{\mathcal{V}}(V_\xi, V_\eta; \mathbb{E}_\lambda)$, where $\widetilde{\mathcal{V}}(V_\xi, V_\eta; \mathbb{E}_\lambda)$ is such that there exists a surjective morphism from $\mathcal{V}(V_\xi, V_\eta; V_\lambda)$ to $\widetilde{\mathcal{V}}(V_\xi, V_\eta; \mathbb{E}_\lambda)$ such that any fibre is isomorphic to an affine space of dimension

$$\dim_{\mathbb{C}}\mathrm{Hom}(V_{\xi'}, V_{\eta'}) - \dim_{\mathbb{C}}\mathrm{Hom}(\mathrm{Coker}\, b'a, \mathrm{Ker}\, ba') - \dim_{\mathbb{C}}\mathrm{Stab}_G((a,b,a',b')^*).$$

Also, we also have a commutative diagram induced by (15). By Proposition 2.5, we have

Corollary 4.8. *The following equality holds.*

$$\sum_\lambda \chi(Q(\lambda)^\wedge) = \sum_\lambda g^\lambda_{\beta\alpha} h^{\xi'\eta'}_\lambda. \tag{17}$$

4.4 Geometry Over Squares

Let $\mathcal{O}(\xi,\eta,\xi',\eta')$ be the set of $(V_\delta, V_\beta, e_1, e_2, e_3, e_4, c, d)$ such that the following commutative diagram has exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & & 0 \arrow[d] & \\
0 \arrow[r] & V_\beta \arrow[rr, "e_1"] \arrow[dd, "u'"] & & V_\eta \arrow[rr, "e_2"] \arrow[dl, dotted, "u_{V_\eta}"'] \arrow[dd, dotted] & & V_\delta \arrow[r] \arrow[dd, "x"] & 0 \\
 & & S \arrow[dr, dotted, "d"] & & & & \\
 & V_{\eta'} \arrow[ur, dotted, "u_{V_{\eta'}}"'] \arrow[rr, dotted] \arrow[dd, "v'"] & & V_\lambda \arrow[rr, dotted] \arrow[dr, dotted, "c"] \arrow[dd, dotted] & & V_{\xi'} \arrow[dd, "y"] & \\
 & & & & T \arrow[ur, dotted, "q_{V_{\xi'}}"'] \arrow[dl, dotted, "q_{V_\xi}"'] & & \\
0 \arrow[r] & V_\alpha \arrow[rr, "e_3"] \arrow[d] & & V_\xi \arrow[rr, "e_4"] & & V_\gamma \arrow[r] \arrow[d] & 0 \\
 & 0 & & & & 0 &
\end{tikzcd} \tag{18}$$

where V_δ, V_β are submodules of $V_{\xi'}, V_{\eta'}$, respectively; $V_\gamma = V_{\xi'}/V_\delta$, $V_\alpha = V_{\eta'}/V_\beta$, u', x, v', y are the canonical morphisms, and V_λ is the centre induced by the above square, $T = V_\xi \times_{V_\gamma} V_{\xi'} = \{(x \oplus m) \in V_\xi \oplus V_{\xi'} \mid e_4(x) = y(m)\}$ and $S = V_\eta \bigsqcup_{V_\beta} V_{\eta'} = V_\eta \oplus V_{\eta'}/\{e_1(v_\beta) \oplus u'(v_\beta) \mid v_\beta \in V_\beta\}$. Then there is unique map $f : S \to T$ for the fixed square. Let (c, d) be a pair of maps such that c is surjective, d is injective and $cd = f$. In particular, for fixed submodules V_δ and V_β of $V_{\xi'}$ and $V_{\eta'}$, respectively, the subset of $\mathcal{O}(\xi, \eta, \xi', \eta')$

$$\{(V_1, V_2, e_1, e_2, e_3, e_4, c, d) \in \mathcal{O}(\xi, \eta, \xi', \eta') \mid V_1 = V_\delta, V_2 = V_\beta\}$$

is denoted by $\mathcal{O}_{(V_\gamma, V_\delta, V_\alpha, V_\beta)}$, where $V_\gamma = V_{\xi'}/V_\delta$ and $V_\alpha = V_{\eta'}/V_\beta$. There is a natural action of the group $G_{\underline{\lambda}}$ on $\mathcal{O}_{(V_\gamma, V_\delta, V_\alpha, V_\beta)}$ as follows:

$$g.(V_\delta, V_\beta, e_1, e_2, e_3, e_4, c, d) = (V_\delta, V_\beta, e_1, e_2, e_3, e_4, cg^{-1}, gd).$$

We denote by $\mathcal{O}^*_{(V_\gamma, V_\delta, V_\alpha, V_\beta)}$ and $\mathcal{O}(\xi, \eta, \xi', \eta')^*$ the orbit spaces under the actions of $G_{\underline{\lambda}}$.

4.5 Geometric Analogue of Green's Formula

There is a homeomorphism between $Q(\xi, \eta, \xi', \eta')^*$ and $\mathcal{O}(\xi, \eta, \xi', \eta')^*$ (see [DXX]):

$$\theta^* : Q(\xi, \eta, \xi', \eta')^* \to \mathcal{O}(\xi, \eta, \xi', \eta')^*$$

induced by the map between $Q(\xi, \eta, \xi', \eta')$ and $\mathcal{O}(\xi, \eta, \xi', \eta')$ defined as follows:

$$V_\beta = \mathrm{Ker}\, b'a \simeq \mathrm{Ker}\, ba', \qquad V_\delta = \mathrm{Im}\, ba',$$

$$e_1 = (a')^{-1}a, \quad e_2 = ba', \quad e_3 = b'a, \quad e_4 = b(b')^{-1}$$

and c, d are induced by the maps:

$$V_\eta \oplus V_{\eta'} \to V_\lambda \quad \text{and} \quad V_\lambda \to V_\xi \oplus V_{\xi'}.$$

There is an action of $G_{\underline{\xi}} \times G_{\underline{\eta}}$ on $\mathcal{O}(\xi, \eta, \xi', \eta')^*$, defined as follows: for $(g_1, g_2) \in G_\xi \times G_\eta$,

$$(g_1, g_2).(V_\delta, V_\beta, e_1, e_2, e_3, e_4, c, d)^* = (V_\delta, V_\beta, g_2e_1, e_2g_2^{-1}, g_1e_3, e_4g_1^{-1}, c', d')^*.$$

Let us determine the relation between (c', d') and (c, d).

Suppose that $(V_\delta, V_\beta, g_2e_1, e_2g_2^{-1}, g_1e_3, e_4g_1^{-1})$ induces S', T' and the unique map $f' : S' \to T'$, then it is clear that there are isomorphisms:

$$a_1 : S \to S' \quad \text{and} \quad a_2 : T \to T'$$

induced by isomorphisms:

$$\begin{pmatrix} g_2 & 0 \\ 0 & \mathrm{id} \end{pmatrix} : V_\eta \oplus V_{\eta'} \to V_\eta \oplus V_{\eta'} \quad \text{and} \quad \begin{pmatrix} g_1 & 0 \\ 0 & \mathrm{id} \end{pmatrix} : V_\xi \oplus V_{\xi'} \to V_\xi \oplus V_{\xi'}.$$

So $f' = a_2 f a_1^{-1}$, and we have the following commutative diagram:

$$\begin{tikzcd}
S' \arrow[r, "d'"] & V_\lambda \arrow[r, "d'_1"] & V_\gamma \\
S \arrow[u, "a_1"] \arrow[r, "d"] \arrow[d] & V_\lambda \arrow[u, "g"] \arrow[r, "d_1"] \arrow[d, "c"] & V_\gamma \arrow[u, equal] \arrow[d, equal] \\
\mathrm{Im}\, f \arrow[r] \arrow[d, "a_2"] & T \arrow[r] \arrow[d, "a_2"] & V_\gamma \arrow[d, equal] \\
\mathrm{Im}\, f' \arrow[r] & T' \arrow[r] & V_\gamma
\end{tikzcd} \tag{19}$$

Hence, $c' = a_2 c g^{-1}$ and $d' = g d a_1^{-1}$. In particular, $c = c'$ and $d = d'$ if and only if $g_1 = \mathrm{id}_{V_\xi}$ and $g_2 = \mathrm{id}_{V_\eta}$. This shows that the action of $G_{\underline{\xi}} \times G_{\underline{\eta}}$ is free.

Its orbit space is denoted by $\mathcal{O}(\xi, \eta, \xi', \eta')^\wedge$. The homeomorphism θ^* above induces the homeomorphism in the following Proposition:

Proposition 4.9. *There exists a homeomorphism under quotient topology*

$$\theta^\wedge : Q(\xi, \eta, \xi', \eta')^\wedge \to \mathcal{O}(\xi, \eta, \xi', \eta')^\wedge.$$

Let $\mathcal{D}(\xi, \eta, \xi', \eta')^*$ be the set of $(V_\delta, V_\beta, e_1, e_2, e_3, e_4)$ such that the diagram (18) is commutative and has exact rows and columns. In particular, for fixed V_δ and V_β, its subset

$$\{(V_1, V_2, e_1, e_2, e_3, e_4) \mid V_1 = V_\delta, V_2 = V_\beta\}$$

is denoted by $D^*_{(V_\gamma, V_\delta, V_\alpha, V_\beta)}$, where $V_\gamma = V_{\xi'}/V_\delta$ and $V_\alpha = V_{\eta'}/V_\beta$. Then we have a projection:

$$\varphi^* : \mathcal{O}(\xi, \eta, \xi', \eta')^* \to \mathcal{D}(\xi, \eta, \xi', \eta')^*.$$

We claim that the fibre of this morphism is isomorphic to a vector space which has the same dimension as $\mathrm{Ext}^1(V_\gamma, V_\beta)$ for any element in $\mathcal{D}^*_{(V_\gamma, V_\delta, V_\alpha, V_\beta)}$.

Fix an element $(V_\delta, V_\beta, e_1, e_2, e_3, e_4) \in \mathcal{D}^*_{(V_\gamma, V_\delta, V_\alpha, V_\beta)}$, and let V be the set consisting of the equivalence classes $(c, d)^*$of elements (c, d) under the action of $G_{\underline{\lambda}}$ such that the following diagram is commutative:

$$\begin{tikzcd} V_\beta \arrow[r, equal] \arrow[d, "u_1"] & V_\beta \arrow[r] \arrow[d, "s"] & 0 \arrow[d] \\ S \arrow[r, "d"] \arrow[d, "v_1"] & V_\lambda \arrow[r, "t"] \arrow[d, "c"] & V_\gamma \arrow[d, equal] \\ L \arrow[r, "u_2"] & T \arrow[r, "v_2"] & V_\gamma \end{tikzcd} \tag{20}$$

where u_1, u_2, v_1, v_2 are fixed and come from the long exact sequence:

$$\begin{tikzcd} & & L = \operatorname{Im} f \arrow[dr, "u_2"] & & & \\ 0 \arrow[r] & V_\beta \arrow[r, "u_1"] & S \arrow[u, "v_1"] \arrow[r, "f"] & T \arrow[r, "v_2"] & V_\gamma \arrow[r] & 0 \end{tikzcd} \tag{21}$$

and where s, t are naturally induced by c, d, respectively. Then V is the fibre of φ^*. We note that c is in diagram (20) if and only if $c \in u_2 \operatorname{Hom}(V_\gamma, L)t$. Of course, $u_2 \operatorname{Hom}(V_\gamma, L)t$ is isomorphic to $\operatorname{Hom}(V_\gamma, L)$.

Let $\varepsilon_0 \in \operatorname{Ext}^1(V_\gamma, L)$ be the class of the following exact sequence:

$$0 \longrightarrow L \xrightarrow{u_2} T \xrightarrow{v_2} V_\gamma \longrightarrow 0\ .$$

The above long exact sequence induces the following long exact sequence:

$$\begin{aligned} 0 &\longrightarrow \operatorname{Hom}(V_\gamma, V_\beta) \longrightarrow \operatorname{Hom}(V_\gamma, S) \longrightarrow \operatorname{Hom}(V_\gamma, L) \longrightarrow \\ &\longrightarrow \operatorname{Ext}^1(V_\gamma, V_\beta) \longrightarrow \operatorname{Ext}^1(V_\gamma, S) \xrightarrow{\phi} \operatorname{Ext}^1(V_\gamma, L) \longrightarrow 0\ . \end{aligned} \tag{22}$$

Consider the morphism $\omega : V \to \phi^{-1}(\varepsilon_0)$ sending $(c, d)^*$ to $(d, t)^*$.

$$\omega^{-1}((d,t)^*) = \{(cg^{-1}, gd)^* \mid g \in G_{\underline{\lambda}}\} = \{(cg^{-1}, d)^* \mid g \in 1+d \operatorname{Hom}(V_\gamma, S)t\}.$$

Hence, the fibre of ω can be viewed as the orbit space of $u_2 \operatorname{Hom}(V_\gamma, L)t$ under the action of $1 + d \operatorname{Hom}(V_\gamma, S)t$ given by $g.u_2 f t = u_2 f g_\gamma^{-1} t$, where $g \in 1 + d \operatorname{Hom}(V_\gamma, S)t$ and g_γ is the isomorphism on V_γ induced by g, with the stabilizer isomorphic to the vector space $\operatorname{Hom}(V_\gamma, V_\beta)$. Hence, up to a translation from ε_0 to 0, V is isomorphic to the affine space:

$$\phi^{-1}(\varepsilon_0) \times \operatorname{Hom}(V_\gamma, L) \times \operatorname{Hom}(V_\gamma, V_\beta) / \operatorname{Hom}(V_\gamma, S),$$

which is denoted by $W(V_\delta, V_\beta, e_1, e_2, e_3, e_4)$, and whose dimension is $\dim_{\mathbb{C}} \operatorname{Ext}^1(V_\gamma, V_\beta)$.

There is also an action of the group $G_{\underline{\xi}} \times G_{\underline{\eta}}$ on $\mathcal{D}(\xi,\eta,\xi',\eta')^*$ with stabilizer isomorphic to the vector space $\mathrm{Hom}(V_\gamma, V_\alpha) \times \mathrm{Hom}(V_\delta, V_\beta)$. The orbit space is denoted by $\mathcal{D}(\xi,\eta,\xi',\eta')^\wedge$. The projection φ^* naturally induces the projection:

$$\varphi^\wedge : \mathcal{O}(\xi,\eta,\xi',\eta')^\wedge \to \mathcal{D}(\xi,\eta,\xi',\eta')^\wedge.$$

Its fibre over $(V_\delta, V_\beta, e_1, e_2, e_3, e_4)^\wedge$ is isomorphic to the quotient space of

$$(\varphi^*)^{-1}(V_\delta, V_\beta, e_1, e_2, e_3, e_4)$$

under the action of $\mathrm{Hom}(V_\gamma, V_\alpha) \times \mathrm{Hom}(V_\delta, V_\beta)$. The corresponding stabilizer of $(V_\delta, V_\beta, e_1, e_2, e_3, e_4, c, d)^* \in (\varphi^*)^{-1}(V_\delta, V_\beta, e_1, e_2, e_3, e_4)$ is

$$\begin{aligned}\{(g_1, g_2) \in 1 + e_3\mathrm{Hom}(V_\gamma, V_\alpha)e_4 \times 1 + e_1\mathrm{Hom}(V_\delta, V_\beta)e_2 \mid \\ ga' = a'g_2, b'g = g_1b' \quad \text{for some } g \in 1 + a\mathrm{Hom}(V_{\xi'}, V_{\eta'})b\},\end{aligned}$$

where $(a, b, a', b')^*$ is induced by θ^* as showed in diagram (18). It is isomorphic to the vector space $\mathrm{Stab}_G((a, b, a', b')^*)$. Therefore, we have

Proposition 4.10. *There exists a projection*

$$\varphi^\wedge : \mathcal{O}(\xi,\eta,\xi',\eta')^\wedge \to \mathcal{D}(\xi,\eta,\xi',\eta')^\wedge$$

such that any fibre for $(V_\delta, V_\beta, e_1, e_2, e_3, e_4)^\wedge$ is isomorphic to an affine space of dimension

$$\begin{aligned}\dim_{\mathbb{C}}\mathrm{Ext}^1(V_\gamma, V_\beta) + \dim_{\mathbb{C}}\mathrm{Stab}_G((a, b, a', b')^*) \\ - \dim_{\mathbb{C}}\mathrm{Hom}(V_\gamma, V_\alpha) - \dim_{\mathbb{C}}\mathrm{Hom}(V_\delta, V_\beta),\end{aligned}$$

where $V_\gamma \simeq V_{\xi'}/V_\delta$ and $V_\alpha \simeq V_{\eta'}/V_\beta$.

Let us summarize the discussion above in the following diagram:

$$\mathrm{Ext}^1(V_{\xi'}, V_{\eta'}) \xleftarrow{\phi^\wedge} Q(\xi,\eta,\xi',\eta')^\wedge \xrightarrow{\theta^\wedge} \mathcal{O}(\xi,\eta,\xi',\eta')^\wedge \xrightarrow{\varphi^\wedge} \mathcal{D}(\xi,\eta,\xi',\eta')^\wedge. \tag{23}$$

The following theorem can be viewed as a degenerated version of Green's formula.

Theorem 4.11. *For fixed ξ, η, ξ', η', we have*

$$g_{\xi\eta}^{\xi'\oplus\eta'} = \int_{\alpha,\beta,\delta,\gamma;\alpha\oplus\gamma=\xi,\beta\oplus\delta=\eta} g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'}.$$

Proof. We note that

$$\chi(\mathcal{D}(\xi,\eta,\xi',\eta')^\wedge) = \int_{\alpha,\beta,\delta,\gamma} g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} h_\xi^{\gamma\alpha} h_\eta^{\delta\beta}.$$

Because the Euler characteristic of an affine space is 1, we have

$$\int_{\lambda} h_{\lambda}^{\xi'\eta'} g_{\xi\eta}^{\lambda} = \int_{\alpha,\beta,\delta,\gamma} g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} h_{\xi}^{\gamma\alpha} h_{\eta}^{\delta\beta}.$$

Using Proposition 2.5 and Lemma 4.2, we simplify the identity as

$$h_{\xi'\oplus\eta'}^{\xi'\eta'} g_{\xi\eta}^{\xi'\oplus\eta'} = \int_{\alpha,\beta,\delta,\gamma,\alpha\oplus\gamma=\xi,\beta\oplus\gamma=\eta} g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} h_{\xi}^{\gamma\alpha} h_{\eta}^{\delta\beta}$$

i.e.

$$g_{\xi\eta}^{\xi'\oplus\eta'} = \int_{\beta,\delta} g_{\xi'/\delta,\delta}^{\xi'} g_{\eta'/\beta,\beta}^{\eta'}.$$

□

4.6 Green's formula under the $\mathbb{C}^*$-action

We define $EF(\xi, \eta, \xi', \eta')$ to be the set

$$\{(\varepsilon, L'(d)) \mid \varepsilon \in \mathrm{Ext}^1(V_{\xi'}, V_{\eta'})_{L(d)}, L'(d) \subseteq L(d), L'(d) \simeq V_\eta, L(d)/L'(d) \simeq V_\xi\},$$

and let $FE(\xi, \eta, \xi', \eta')$ be the set

$$\Big\{(V'_{\xi'}, V'_{\eta'}, \varepsilon_1, \varepsilon_2) \mid V'_{\xi'} \subseteq V_\xi, V'_{\eta'} \subseteq V_{\eta'}, \varepsilon_1 \in \mathrm{Ext}^1(V'_{\xi'}, V'_{\eta'})_{V_\eta}, \varepsilon_2 \in \mathrm{Ext}^1(V_{\xi'}/V'_{\xi'}, V_{\eta'}/V'_{\eta'})_{V_\xi}\Big\}.$$

The projection

$$p_1 : EF(\xi, \eta, \xi', \eta') \to \mathrm{Ext}^1(V_{\xi'}, V_{\eta'})$$

satisfies that the fibre of any $\varepsilon \in \mathrm{Ext}^1(V_{\xi'}, V_{\eta'})_{L(d)}$ is isomorphic to $\mathcal{V}(V_\xi, V_\eta; L(d))$.

Comparing with Proposition 4.7, we have a morphism

$$EF(\xi, \eta, \xi', \eta') \to Q(\xi, \eta, \xi', \eta')^\wedge$$

satisfying the fibre of $(a, b, a', b')^\wedge$ is isomorphic to an affine space of dimension

$$\dim_{\mathbb{C}} \mathrm{Hom}(V_{\xi'}, V_{\eta'}) - \dim_{\mathbb{C}} \mathrm{Hom}(\mathrm{Coker}\, b'a, \mathrm{Ker}\, ba') - \dim_{\mathbb{C}} G((a, b, a', b')^*).$$

We also have a natural homeomorphism

$$FE(\xi, \eta, \xi', \eta') \to \mathcal{D}(\xi, \eta, \xi', \eta')^\wedge.$$

Hence, using Proposition 4.7, 4.9 and 4.10, we have

Proposition 4.12. *There is a natural morphism*

$$\rho : EF(\xi, \eta, \xi', \eta') \to FE(\xi, \eta, \xi', \eta')$$

satisfying the fibre for $(V'_{\xi'}, V'_{\eta'}, \varepsilon_1, \varepsilon_2)$ *is isomorphic to an affine space of dimension*

$$\dim_{\mathbb{C}} \mathrm{Hom}(V_{\xi'}, V_{\eta'}) - \dim_{\mathbb{C}} \mathrm{Hom}(V_\gamma, V_\beta) + \dim_{\mathbb{C}} \mathrm{Ext}(V_\gamma, V_\beta) \\ - \dim_{\mathbb{C}} \mathrm{Hom}(V_\gamma, V_\alpha) - \dim_{\mathbb{C}} \mathrm{Hom}(V_\delta, V_\beta),$$

where $V_\beta \simeq V'_{\eta'}$, $V_\delta \simeq V'_{\xi'}$ *and* $V_\alpha \simeq V_{\eta'}/V'_{\eta'}$, $V_\gamma \simeq V_{\xi'}/V'_{\xi'}$.

Now we consider the action of $\mathbb{C}^*$ on $EF(\xi, \eta, \xi', \eta')$ and $FE(\xi, \eta, \xi', \eta')$.

(1) For $t \in \mathbb{C}^*$ and $(\varepsilon, L'(d)) \in EF(\xi, \eta, \xi', \eta')$, we know $t.\varepsilon \in \mathrm{Ext}^1(V_{\xi'}, V_{\eta'})_{L(t.d)}$ and $L(d) = V_{\xi'} \oplus V_{\eta'}$ as a direct sum of vector spaces. Recall that $L(t.d)$ is defined in Sect. 4.1. Define

$$L'(t.d) := \{(v', tv'') \mid (v', v'') \in L'(d)\}.$$

Then $L'(t.d) \subseteq L(t.d)$. Hence, we define

$$t.(\varepsilon, L'(d)) = (t.\varepsilon, L'(t.d)).$$

The orbit space is denoted by $\widehat{EF}(\xi, \eta, \xi', \eta')$. A point $(\varepsilon, L'(d))$ is stable, i.e. $t.(\varepsilon, L'(d)) = (\varepsilon, L'(d))$ for any $t \in \mathbb{C}^*$ if and only if

$$L(d) = V_{\xi'} \oplus V_{\eta'} \text{ and } L'(d) = (L'(d) \cap V_{\xi'}) \oplus (L'(d) \cap V_{\eta'}).$$

Note that the above direct sums are the direct sums of modules. The set of stable points in $EF(\xi, \eta, \xi', \eta')$ is denoted by $EF_s(\xi, \eta, \xi', \eta')$. The action of $\mathbb{C}^*$ on the set of non-stable points in $EF(\xi, \eta, \xi', \eta')$ is free. We denote the orbit space by $\mathbb{P}EF(\xi, \eta, \xi', \eta')$. Of course, we have

$$\widehat{EF}(\xi, \eta, \xi', \eta') = EF_s(\xi, \eta, \xi', \eta') \cup \mathbb{P}EF(\xi, \eta, \xi', \eta').$$

The orbit of $(\varepsilon, L'(d))$ in $\mathbb{P}EF(\xi, \eta, \xi', \eta')$ is denoted by $\mathbb{P}(\varepsilon, L'(d))$.

(2) For $t \in \mathbb{C}^*$ and $(V'_{\xi'}, V'_{\eta'}, \varepsilon_1, \varepsilon_2) \in FE(\xi, \eta, \xi', \eta')$, we define

$$t.(V'_{\xi'}, V'_{\eta'}, \varepsilon_1, \varepsilon_2) = (V'_{\xi'}, V'_{\eta'}, t.\varepsilon_1, t.\varepsilon_2).$$

The orbit space is denoted by $\widehat{FE}(\xi, \eta, \xi', \eta')$. A point $(V'_{\xi'}, V'_{\eta'}, \varepsilon_1, \varepsilon_2)$ in $FE(\xi, \eta, \xi', \eta')$ is stable if and only if $\varepsilon_1 = \varepsilon_2 = 0$. The set of stable points in $FE(\xi, \eta, \xi', \eta')$ is denoted by $FE_s(\xi, \eta, \xi', \eta')$. The action of $\mathbb{C}^*$ on the set of non-stable points in $FE(\xi, \eta, \xi', \eta')$ is free. We denote the orbit space by $\mathbb{P}FE(\xi, \eta, \xi', \eta')$. Of course, we have

$$\widehat{FE}(\xi, \eta, \xi', \eta') = FE_s(\xi, \eta, \xi', \eta') \cup \mathbb{P}FE(\xi, \eta, \xi', \eta').$$

The orbit of $(V'_{\xi'}, V'_{\eta'}, \varepsilon_1, \varepsilon_2)$ in $\mathbb{P}FE(\xi, \eta, \xi', \eta')$ is denoted by $\mathbb{P}(V'_{\xi'}, V'_{\eta'}, \varepsilon_1, \varepsilon_2)$.

The morphism ρ induces the morphism

$$\hat{\rho}:\widehat{EF}(\xi,\eta,\xi',\eta')\to\widehat{FE}(\xi,\eta,\xi',\eta').$$

We consider its restriction to $\mathbb{P}EF(\xi,\eta,\xi',\eta')$,

$$\hat{\rho}\mid_{\mathbb{P}EF(\xi,\eta,\xi',\eta')}:\mathbb{P}EF(\xi,\eta,\xi',\eta')\to\widehat{FE}(\xi,\eta,\xi',\eta').$$

For any $(V'_{\xi'},V'_{\eta'},0,0)\in FE_s(\xi,\eta,\xi',\eta')\subseteq\widehat{FE}(\xi,\eta,\xi',\eta')$, we have

$$(\hat{\rho}\mid_{\mathbb{P}EF(\xi,\eta,\xi',\eta')})^{-1}((V'_{\xi'},V'_{\eta'},0,0))=\mathbb{P}(\rho^{-1}(V'_{\xi'},V'_{\eta'},0,0)\setminus(V'_{\xi'}\oplus V'_{\eta'},0)).$$

It is actually the projective space of the affine space in Proposition 4.12. For any $\mathbb{P}(V'_{\xi'},V'_{\eta'},\varepsilon_1,\varepsilon_2)\in\mathbb{P}FE(\xi,\eta,\xi',\eta')$, $(\hat{\rho}\mid_{\mathbb{P}EF(\xi,\eta,\xi',\eta')})^{-1}(\mathbb{P}(V'_{\xi'},V'_{\eta'},\varepsilon_1,\varepsilon_2))$ is isomorphic to the affine space in Proposition 4.12. Now we compute the Euler characteristics. By Proposition 2.1 and the above discussion of the fibres, we have

$$\begin{aligned}&\chi(\mathbb{P}EF(\xi,\eta,\xi',\eta'))\\&=\int_{\substack{(V'_{\xi'},V'_{\eta'},0,0)\\\in FE_s(\xi,\eta,\xi',\eta')}}[d(\xi',\eta')-d(\gamma,\alpha)-d(\delta,\beta)-\langle\gamma,\beta\rangle]g^{\xi'}_{\gamma\delta}g^{\eta'}_{\alpha\beta}+\chi(\mathbb{P}FE(\xi,\eta,\xi',\eta')),\end{aligned}$$

where $d(\gamma,\alpha)=\dim_{\mathbb{C}}\mathrm{Hom}_{\Lambda}(V_\gamma,V_\alpha)$ and the Euler form $\langle\gamma,\beta\rangle=\dim_{\mathbb{C}}\mathrm{Hom}(V_\gamma,V_\beta)-\dim_{\mathbb{C}}\mathrm{Ext}^1(V_\gamma,V_\beta)$. On the other hand, we know

$$\begin{aligned}&\chi(\mathbb{P}EF(\xi,\eta,\xi',\eta'))\\&=\int_{\substack{\alpha,\beta,\delta,\gamma,\\\alpha\oplus\gamma=\xi,\beta\oplus\delta=\eta}}\chi(\mathbb{P}\overline{\mathcal{V}}(V_\xi,V_\eta;V_{\xi'}\oplus V_{\eta'}))+\int_{\lambda\neq\xi'\oplus\eta'}\chi(\mathbb{P}\,\mathrm{Ext}^1(V_{\xi'},V_{\eta'})_\lambda)g^{\lambda}_{\xi\eta}.\end{aligned}$$

Therefore, we have the following theorem, which can be viewed as a geometric version of Green's formula under the $\mathbb{C}^*$-action.

Theorem 4.13. *For fixed ξ,η,ξ',η', we have*

$$\begin{aligned}&\int_{\lambda\neq\xi'\oplus\eta'}\chi(\mathbb{P}\mathrm{Ext}^1(V_{\xi'},V_{\eta'})_\lambda)g^{\lambda}_{\xi\eta}\\&=\int_{\alpha,\beta,\delta,\gamma,\alpha\oplus\gamma\neq\xi\text{ or }\beta\oplus\delta\neq\eta}\chi(\mathbb{P}(\mathrm{Ext}^1(V_\gamma,V_\alpha)_\xi\times\mathrm{Ext}^1(V_\delta,V_\beta)_\eta))g^{\xi'}_{\gamma\delta}g^{\eta'}_{\alpha\beta}\\&\quad+\int_{\alpha,\beta,\delta,\gamma,\alpha\oplus\gamma=\xi,\beta\oplus\delta=\eta}[d(\xi',\eta')-d(\gamma,\alpha)-d(\delta,\beta)-\langle\gamma,\beta\rangle]g^{\xi'}_{\gamma\delta}g^{\eta'}_{\alpha\beta}\\&\quad-\int_{\alpha,\beta,\delta,\gamma,\alpha\oplus\gamma=\xi,\beta\oplus\delta=\eta}\chi(\mathbb{P}\overline{\mathcal{V}}(V_\xi,V_\eta;V_{\xi'}\oplus V_{\eta'})).\end{aligned}$$

5 Application to Caldero–Keller Formula

5.1 Constructibility

Let Q be a quiver with vertex set $Q_0 = \{1, 2, \ldots, n\}$ containing no oriented cycles and $A = \mathbb{C}Q$ be the path algebra of Q. For $i \in Q_0$, we denote by P_i the corresponding indecomposable projective $\mathbb{C}Q$-module and by S_i the corresponding simple module. Let $\mathbb{Q}(x_1, \ldots, x_n)$ be a transcendental extension of $\mathbb{Q}$. Define the map [CC]

$$X_? : \mathrm{obj}(\mathrm{mod}\, A) \to \mathbb{Q}(x_1, \ldots, x_n)$$

by:

$$X_M = \sum_{\underline{e}} \chi(Gr_{\underline{e}}(M)) x^{\tau(\underline{e}) - \underline{\dim}\, M + \underline{e}},$$

where τ is the Auslander–Reiten translation on the Grothendieck group $K_0(\mathcal{D}^b(Q))$ and, for $v \in \mathbb{Z}^n$, we put

$$x^v = \prod_{i=1}^{n} x_i^{\langle \underline{\dim}\, S_i, v \rangle}$$

and $Gr_{\underline{e}}(M)$ is the $\underline{e}$-Grassmannian of M, i.e. the variety of submodules of M with dimension vector $\underline{e}$. This definition is equivalent to [Hu2]

$$X_M = \int_{\alpha,\beta} g^M_{\alpha\beta} x^{\underline{\beta} R + \underline{\alpha} R' - \underline{\dim}\, M},$$

where the matrices $R = (r_{ij})$ and $R' = (r'_{ij})$ satisfy $r_{ij} = \dim_{\mathbb{C}} \mathrm{Ext}^1(S_i, S_j)$ and $r'_{ij} = \dim_{\mathbb{C}} \mathrm{Ext}^1(S_j, S_i)$ for $i, j \in Q_0$. Here, we recall $g^M_{\alpha\beta} := \chi(\mathcal{V}((V_\alpha, V_\beta; M)))$ which is defined in Sect. 4.1. Note that (see [Hu2])

$$(\underline{\dim}\, P)R = \underline{\dim}\, \mathrm{rad}\, P, \qquad (\underline{\dim}\, I)R' = \underline{\dim}\, I - \underline{\dim}\, \mathrm{soc}\, I.$$

We consider the set

$$Gr_{\underline{e}}(\mathbb{E}_{\underline{d}}) := \{(M, M_1) \mid M \in \mathbb{E}_{\underline{d}}, M_1 \in Gr_{\underline{e}}(M)\}.$$

This is a closed subset of $\mathbb{E}_{\underline{d}} \times \prod_{i=1,\ldots,n} Gr_{e_i}(k^{d_i})$. Here, we simply use the notation $\mathbb{E}_{\underline{d}}$ instead of $\mathbb{E}_{\underline{d}}(Q)$ without confusion.

Proposition 5.1. *The function $X_?|_{\mathbb{E}_{\underline{d}}}$ is G-invariant constructible.*

Proof. Obviously it is G-invariant. Consider the canonical morphism $\pi : Gr_{\underline{e}}(\mathbb{E}_{\underline{d}}) \to \mathbb{E}_{\underline{d}}$ sending (M, M_1) to M. It is clear that $\pi^{-1}(M) = Gr_{\underline{e}}(M)$. Let $1_{Gr_{\underline{e}}(\mathbb{E}_{\underline{d}})}$ be the constant function on $Gr_{\underline{e}}(\mathbb{E}_{\underline{d}})$, by Theorem 2.2, $(\pi)_*(1_{Gr_{\underline{e}}(\mathbb{E}_{\underline{d}})})$ is constructible. We know that

$$(\pi)_*(1_{Gr_{\underline{e}}(\mathbb{E}_{\underline{d}})})(M) = \chi(Gr_{\underline{e}}(M)).$$

So, there are finitely many $\chi(Gr_{\underline{e}}(M))$ for $M \in \mathbb{E}_{\underline{d}}$. □

Proposition 5.2. *For fixed dimension vectors* $\underline{e}$ *and* $\underline{d}$, *the set*

$$\{g_{XY}^{E} \mid E \in \mathbb{E}_{\underline{d}}, Y \in \mathbb{E}_{\underline{e}}, X \in \mathbb{E}_{\underline{d}-\underline{e}}\}$$

is a finite set.

Proof. Let $M \in \mathbb{E}_{\underline{d}}$. For any submodule M_1 of dimension vector $\underline{e}$ of M, by the knowledge of linear algebra, there exist unique $(\mathbb{C}^{\underline{e}}, x) \in \mathbb{E}_{\underline{e}}$ isomorphic to M_1 and $(\mathbb{C}^{\underline{d}-\underline{e}}, x') \in \mathbb{E}_{\underline{d}-\underline{e}}$ isomorphic to M/M_1. We have $Gr_{\underline{e}}(\mathbb{E}_{\underline{d}}) = \bigcup_{M\in\mathbb{E}_{\underline{d}}} Gr_{\underline{e}}(M)$. Then this deduces the following morphisms:

$$Gr_{\underline{e}}(\mathbb{E}_{\underline{d}}) = \bigcup_{M\in\mathbb{E}_{\underline{d}}} Gr_{\underline{e}}(M) \xrightarrow{\pi_1} \mathbb{E}_{\underline{e}} \times \mathbb{E}_{\underline{d}-\underline{e}} \times \mathbb{E}_{\underline{d}} \xrightarrow{\pi_2} \bigcup_i \phi_i(U_i)\ ,$$

where $\mathbb{E}_{\underline{e}} \times \mathbb{E}_{\underline{d}-\underline{e}} \times \mathbb{E}_{\underline{d}} = \bigcup_i U_i$ is a finite stratification with respect to the action of the algebraic group $G_{\underline{e}} \times G_{\underline{d}-\underline{e}} \times G_{\underline{d}}$ and $\phi_i : U_i \to \phi_i(U_i)$ is the geometric quotient for any i, and $\pi_2 = \bigcup_i \phi_i$. For any $(Y, X, M) \in \mathbb{E}_{\underline{e}} \times \mathbb{E}_{\underline{d}-\underline{e}} \times \mathbb{E}_{\underline{d}}$,

$$\chi((\pi_2\pi_1)^{-1}(\pi_2((Y, X, M)))) = g_{XY}^{M}.$$

Consider the constant function $1_{Gr_{\underline{e}}(M)}$ on $Gr_{\underline{e}}(M)$, by Theorem 2.2, $(\pi_2\pi_1)_*(1_{Gr_{\underline{e}}(M)})$ is constructible. Hence, there are finitely many g_{XY}^{M} for $(X, Y, M) \in \mathbb{E}_{\underline{e}} \times \mathbb{E}_{\underline{d}-\underline{e}} \times \mathbb{E}_{\underline{d}}$. □

Proposition 5.3. *For fixed* $M \in \mathbb{E}_{\underline{\xi'}}, N \in \mathbb{E}_{\underline{\eta'}}$, *the set*

$$\{\chi(\mathrm{Ext}^1(M, N)_E) \mid E \in \mathbb{E}_{\underline{\xi'}+\underline{\eta'}}\}$$

is a finite set.

Proof. Consider the morphism:

$$\mathrm{Ext}^1(M, N) \xrightarrow{f} \mathbb{E}_{\underline{\xi'}+\underline{\eta'}} \xrightarrow{g} \bigcup_j \phi_j(V_j),$$

where $\mathbb{E}_{\underline{\xi'}+\underline{\eta'}} = \bigcup_j V_j$ is a finite stratification with respect to the action of the algebraic group $G_{\underline{\xi'}+\underline{\eta'}}$ and $\phi_j : V_j \to \phi_j(V_j)$ is a geometric quotient for any i, and f sends any extension to the middle term of the extension. Here, f is a morphism by identification between $\mathrm{Ext}^1(M, N)$ and $E(M, N)$ at the beginning of Sect. 4. The remaining discussion is almost the same as in Proposition 5.2. We omit it. □

5.2 The Multiplication Theorem

We now consider the cluster category, i.e. the orbit category $\mathcal{D}^b(Q)/F$ with $F = [1]\tau^{-1}$, where τ is the AR-translation of $\mathcal{D}^b(Q)$. Each object M in $\mathcal{D}^b(Q)/F$ can be uniquely decomposed into the form: $M = M_0 \oplus P_M[1] = M_0 \oplus \tau P_M$, where $M_0 \in \mathrm{mod}\, A$ and P_M is projective in $\mathrm{mod}\, A$. Now we can extend the map $X_?$ as in [CK], see also [Hu2]: by the rule: $X_{\tau P} = x^{\underline{\dim}\, P/\mathrm{rad}\, P}$ for projective A-module and $X_{M\oplus N} = X_M X_N$. Then we have a well-defined map

$$X_? : \mathrm{obj}(\mathcal{D}^b(Q)/F) \to \mathbb{Q}(x_1, \ldots, x_n).$$

Let $\overline{\mathbb{E}}_{\underline{d}}$ be the orbit space of $\mathbb{E}_{\underline{d}}$ under the action of $G_{\underline{d}}$. Note that all the integrals below are over $\overline{\mathbb{E}}_{\underline{d}}$ for some corresponding dimension vector $\underline{d}$. Note also that in $\mathrm{mod}\, A$ we have $d(\gamma, \alpha) = \dim \mathrm{Hom}(V_\gamma, V_\alpha)$ and $d^1(\gamma, \alpha) = \dim \mathrm{Ext}(V_\gamma, V_\alpha)$. We say that P_0 is the projective direct summand of $V_{\xi'}$ if $V_{\xi'} \simeq V'_{\xi'} \oplus P_0$ and no direct summand of $V'_{\xi'}$ is projective.

The cluster algebra corresponding to the cluster category $\mathcal{D}^b(Q)/F$ is the subalgebra of $\mathbb{Q}(x_1, \ldots, x_n)$ generated by $\{X_M, X_{\tau P} \mid M \in \mathrm{mod}\, A, P \in \mathrm{mod}\, A \text{ is projective}\}$. The following theorem gives a generalization of the cluster multiplication formula in [CK]. The idea of the proof follows the work [Hu2] of Hubery.

Theorem 5.4. (1) *For any A-modules $V_{\xi'}$, $V_{\eta'}$ we have*

$$d^1(\xi', \eta') X_{V_{\xi'}} X_{V_{\eta'}} = \int_{\lambda \neq \xi' \oplus \eta'} \chi(\mathbb{P}\mathrm{Ext}^1(V_{\xi'}, V_{\eta'})_{V_\lambda}) X_{V_\lambda}$$
$$+ \int_{\gamma, \beta, \iota} \chi(\mathbb{P}\mathrm{Hom}(V_{\eta'}, \tau V_{\xi'})_{V_\beta[1] \oplus \tau V'_\gamma \oplus I_0}) X_{V_\gamma} X_{V_\beta} x^{\underline{\dim}\, \mathrm{soc}\, I_0},$$

where $I_0 \in \iota$ is injective and $V_\gamma = V'_\gamma \oplus P_0$, P_0 is the projective direct summand of $V_{\xi'}$.

(2) *For any A-module $V_{\xi'}$ and $P \in \rho$ is projective, let $I = D\mathrm{Hom}(P, A)$. Here, $D\mathrm{Hom}(-, A)$ is the Nakajima functor. Then*

$$d(\rho, \xi') X_{V_{\xi'}} x^{\underline{\dim}\, P/\mathrm{rad}\, P} = \int_{\delta, \iota'} \chi(\mathbb{P}\mathrm{Hom}(V_{\xi'}, I)_{V_\delta[1] \oplus I'}) X_{V_\delta} x^{\underline{\dim}\, \mathrm{soc}\, I'}$$
$$+ \int_{\gamma, \rho'} \chi(\mathbb{P}\mathrm{Hom}(P, V_{\xi'})_{P'[1] \oplus V_\gamma}) X_{V_\gamma} x^{\underline{\dim}\, P'/\mathrm{rad}\, P'},$$

where $I' \in \iota'$ is injective and $P' \in \rho'$ is projective.

Proof. We set

$$S_1 := \int_{\lambda \in \overline{\mathbb{E}}_{\underline{\xi'}+\underline{\eta'}}, \lambda \neq \xi' \oplus \eta'} \chi(\mathbb{P}\,\mathrm{Ext}^1(V_{\xi'}, V_{\eta'})_{V_\lambda}) X_{V_\lambda}.$$

By Proposition 5.2,

$$S_1 = \int_{\xi,\eta,\lambda\neq\xi'\oplus\eta'} \chi(\mathbb{P}\,\mathrm{Ext}^1(V_{\xi'},V_{\eta'})_{V_\lambda}) g_{\xi\eta}^{\lambda} x^{\underline{\eta}R+\underline{\xi}R'-(\underline{\xi'}+\underline{\eta'})}.$$

Using Theorem 4.13, we have

$$\begin{aligned}
&\int_{\xi,\eta,\lambda\neq\xi'\oplus\eta'} \chi(\mathbb{P}\,\mathrm{Ext}^1(V_{\xi'},V_{\eta'})_{V_\lambda}) g_{\xi\eta}^{\lambda} x^{\underline{\eta}R+\underline{\xi}R'-(\underline{\xi'}+\underline{\eta'})}\\
&\quad=\int_{\substack{\alpha,\beta,\delta,\gamma,\xi,\eta,\\ \alpha\oplus\gamma\neq\xi \text{ or } \beta\oplus\delta\neq\eta}} \chi(\mathbb{P}(\mathrm{Ext}^1(V_\gamma,V_\alpha)_{V_\xi}\times \mathrm{Ext}^1(V_\delta,V_\beta)_{V_\eta})) g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} x^{\underline{\eta}R+\underline{\xi}R'-(\underline{\xi'}+\underline{\eta'})}\\
&\qquad+\int_{\substack{\alpha,\beta,\delta,\gamma,\xi,\eta,\\ \alpha\oplus\gamma=\xi,\beta\oplus\delta=\eta}} [d(\xi',\eta')-d(\gamma,\alpha)-d(\delta,\beta)-\langle\gamma,\beta\rangle] g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} x^{\underline{\eta}R+\underline{\xi}R'-(\underline{\xi'}+\underline{\eta'})}\\
&\qquad-\int_{\substack{\alpha,\beta,\delta,\gamma,\xi,\eta,\\ \alpha\oplus\gamma=\xi,\beta\oplus\delta=\eta}} \chi(\mathbb{P}\overline{V}(V_\xi,V_\eta;V_{\xi'}\oplus V_{\eta'})) x^{\underline{\eta}R+\underline{\xi}R'-(\underline{\xi'}+\underline{\eta'})}.
\end{aligned}$$

We come to simplify every term following [Hu2]. For fixed $\alpha,\beta,\delta,\gamma$,

$$\int_{\substack{\xi,\eta,\alpha\oplus\gamma\neq\xi\\ \text{or } \beta\oplus\delta\neq\eta}} \chi(\mathbb{P}(\mathrm{Ext}^1(V_\gamma,V_\alpha)_{V_\xi}\times \mathrm{Ext}^1(V_\delta,V_\beta)_{V_\eta})) = d^1(\gamma,\alpha)+d^1(\delta,\beta),$$

$$\begin{aligned}
&\int_{\substack{\alpha,\beta,\delta,\gamma,\xi,\eta,\\ \alpha\oplus\gamma\neq\xi \text{ or } \beta\oplus\delta\neq\eta}} \chi(\mathbb{P}(\mathrm{Ext}^1(V_\gamma,V_\alpha)_{V_\xi}\times \mathrm{Ext}^1(V_\delta,V_\beta)_{V_\eta})) g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} x^{\underline{\eta}R+\underline{\xi}R'-(\underline{\xi'}+\underline{\eta'})}\\
&\quad=\int_{\alpha,\beta,\delta,\gamma} [d^1(\gamma,\alpha)+d^1(\delta,\beta)] g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} x^{\underline{\eta}R+\underline{\xi}R'-(\underline{\xi'}+\underline{\eta'})}.
\end{aligned}$$

Moreover,

$$d^1(\gamma,\alpha)+d^1(\delta,\beta)+d(\xi',\eta')-d(\gamma,\alpha)-d(\delta,\beta)-\langle\gamma,\beta\rangle = d^1(\xi',\eta')+\langle\delta,\alpha\rangle.$$

Hence,

$$\begin{aligned}
&\int_{\xi,\eta,\lambda\neq\xi'\oplus\eta'} \chi(\mathbb{P}\,\mathrm{Ext}^1(V_{\xi'},V_{\eta'})_{V_\lambda}) g_{\xi\eta}^{\lambda} x^{\underline{\eta}R+\underline{\xi}R'-(\underline{\xi'}+\underline{\eta'})}\\
&\quad=\int_{\alpha,\beta,\delta,\gamma} [d^1(\xi',\eta')+\langle\delta,\alpha\rangle] g_{\gamma\delta}^{\xi'} g_{\alpha\beta}^{\eta'} x^{\underline{\eta}R+\underline{\xi}R'-(\underline{\xi'}+\underline{\eta'})}\\
&\qquad-\int_{\alpha,\beta,\delta,\gamma,\xi,\eta,\alpha\oplus\gamma=\xi,\beta\oplus\delta=\eta} \chi(\mathbb{P}\overline{V}(V_\xi,V_\eta;V_{\xi'}\oplus V_{\eta'})) x^{\underline{\eta}R+\underline{\xi}R'-(\underline{\xi'}+\underline{\eta'})}.
\end{aligned}$$

As for the last term, consider the following diagram, it may be compared with diagram (10).

$$\bigcup_{\alpha,\beta,\gamma,\delta} \mathcal{V}(V_\alpha, V_\beta; V_{\eta'}) \times \mathcal{V}(V_\gamma, V_\delta; V_{\xi'}) \xrightarrow{j_1} \bigcup_{\xi,\eta} \mathcal{V}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'}) \tag{24}$$

sending $(V^1_{\eta'} \subseteq V_{\eta'}, V^1_{\xi'} \subseteq V_{\xi'})$ to $(V^1_{\xi'} \oplus V^1_{\eta'} \subseteq V_{\xi'} \oplus V_{\eta'})$. And

$$\bigcup_{\xi,\eta} \mathcal{V}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'}) \xrightarrow{j_2} \bigcup_{\alpha,\beta,\gamma,\delta} \mathcal{V}(V_\alpha, V_\beta; V_{\eta'}) \times \mathcal{V}(V_\gamma, V_\delta; V_{\xi'}) \tag{25}$$

sending $(V^1 \subseteq V_{\xi'} \oplus V_{\eta'})$ to $(V^1 \cap V_{\eta'} \subseteq V_{\eta'},\ V^1/V^1 \cap V_{\eta'} \subseteq V_{\xi'})$. The map j_1 is an embedding and

$$\bigcup_{\xi,\eta} \mathcal{V}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'}) \setminus \operatorname{Im} j_1 = \bigcup_{\xi,\eta} \overline{\mathcal{V}}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'}).$$

The fibre of j_2 is isomorphic to a vector space $V(\delta,\alpha)$ of dimension $d(\delta,\alpha)$ (see [Hu2, Corollary 8]). If we restrict j_2 to $\bigcup_{\xi,\eta} \overline{\mathcal{V}}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'})$, then the fibre is isomorphic to $V(\delta,\alpha) \setminus \{0\}$. Under the action of $\mathbb{C}^*$, by Proposition 2.1, we have

$$\begin{aligned}&\int_{\alpha,\beta,\delta,\gamma,\xi,\eta,\alpha\oplus\gamma=\xi,\beta\oplus\delta=\eta} \chi(\mathbb{P}\overline{\mathcal{V}}(V_\xi, V_\eta; V_{\xi'} \oplus V_{\eta'}))x^{\underline{\eta}R+\underline{\xi}R'-(\underline{\xi'}+\underline{\eta'})}\\ &\quad= \int_{\alpha,\beta,\delta,\gamma} d(\delta,\alpha) g^{\xi'}_{\gamma\delta} g^{\eta'}_{\alpha\beta} x^{\underline{\eta}R+\underline{\xi}R'-(\underline{\xi'}+\underline{\eta'})}.\end{aligned}$$

Therefore,

$$S_1 = \int_{\alpha,\beta,\delta,\gamma} [d^1(\xi',\eta') - d^1(\delta,\alpha)] g^{\xi'}_{\gamma\delta} g^{\eta'}_{\alpha\beta} x^{\underline{\eta}R+\underline{\xi}R'-(\underline{\xi'}+\underline{\eta'})}.$$

There is a natural $\mathbb{C}^*$-action on $\operatorname{Hom}(V_{\eta'}, \tau V_{\xi'})_{V_\beta[1]\oplus\tau V'_\gamma\oplus I_0} \setminus \{0\}$ by left multiplication, the orbit space is $\mathbb{P}\operatorname{Hom}(V_{\eta'}, \tau V_{\xi'})_{V_\beta[1]\oplus\tau V'_\gamma\oplus I_0}$. Define

$$S_2 := \int_{\substack{\gamma,\beta,\iota,\\ \kappa,l,\mu,\theta}} \chi(\mathbb{P}\operatorname{Hom}(V_{\eta'}, \tau V_{\xi'})_{V_\beta[1]\oplus\tau V'_\gamma\oplus I_0}) g^{\gamma}_{\kappa l} g^{\beta}_{\theta\mu} x^{(\underline{l}+\underline{\mu})R+(\underline{\kappa}+\underline{\theta})R'-(\underline{\beta}+\underline{\gamma})+\underline{\dim}\operatorname{soc} I_0},$$

where $I_0 \in \iota$. The above definition is well defined by Propositions 5.2 and 5.3. We note that

$$\underline{\dim}\operatorname{soc} I_0 - (\underline{\beta}+\underline{\gamma}) = (\underline{\xi'}-\underline{\gamma})R + (\underline{\eta'}-\underline{\beta})R' - (\underline{\xi'}+\underline{\eta'}).$$

Since $\mathrm{Hom}(V_{\eta'}, \tau V'_{\xi'})_{V_\beta[1]\oplus\tau V'_\gamma\oplus I_0} = \mathcal{V}(\tau V'_\gamma \oplus I_0, V_\beta; V_{\eta'}, \tau V_{\xi'})$, we can apply Theorem 4.6 to the following diagram twice:

$$\begin{tikzcd}
 & V_\mu \arrow[r, equal] \arrow[d] & V_\mu \arrow[d] & \tau V_\kappa \arrow[r, equal] & \tau V_\kappa & \\
{[4pt]0} \arrow[r] & V_\beta \arrow[r] \arrow[d] & V_{\eta'} \arrow[r] \arrow[d] & \tau V'_{\xi'} \arrow[r] \arrow[u] \arrow[d, equal] & \tau V'_\gamma \oplus I_0 \arrow[r] \arrow[u] \arrow[d, equal] & 0 \\
{[4pt]0} \arrow[r] & X \arrow[r] \arrow[d, equal] & V_{\widetilde{\beta}} \arrow[r] \arrow[d, equal] & \tau V'_{\xi'} \arrow[r] & \tau V'_\gamma \oplus I_0 \arrow[r] & 0 \\
{[4pt]0} \arrow[r] & X \arrow[r] & V_{\widetilde{\beta}} \arrow[r] & \tau V'_{\widetilde{\gamma}} \arrow[r] \arrow[u] & \tau L \oplus I_0 \arrow[r] \arrow[u] & 0
\end{tikzcd} \tag{26}$$

where $V_{\widetilde{\beta}}$ and $\tau V'_{\widetilde{\gamma}}$ are the corresponding pullback and pushout. Moreover,

$$(\underline{\xi}'-\underline{\gamma}+\underline{l}+\underline{\mu})R+(\underline{\eta}'-\underline{\beta}+\underline{\kappa}+\underline{\theta})R'-(\underline{\xi}'+\underline{\eta}') = (\underline{\widetilde{\gamma}}+\underline{\mu})R+(\underline{\kappa}+\underline{\widetilde{\beta}})R'-(\underline{\xi}'+\underline{\eta}').$$

Hence,

$$\begin{aligned}
S_2 &= \int_{\widetilde{\gamma},\widetilde{\beta},\kappa,\mu,\iota,l,\theta} \chi(\mathbb{P}\,\mathrm{Hom}(V_{\widetilde{\beta}}, \tau V'_{\widetilde{\gamma}})_{X[1]\oplus\tau L\oplus I_0}) g^{\xi'}_{\kappa\widetilde{\gamma}} g^{\eta'}_{\widetilde{\beta}\mu} x^{(\underline{\widetilde{\gamma}}+\underline{\mu})R+(\underline{\kappa}+\underline{\widetilde{\beta}})R'-(\underline{\xi}'+\underline{\eta}')} \\
&= \int_{\widetilde{\gamma},\widetilde{\beta},\kappa,\mu} \chi(\mathbb{P}\,\mathrm{Hom}(V_{\widetilde{\beta}}, \tau V'_{\widetilde{\gamma}})) g^{\xi'}_{\kappa\widetilde{\gamma}} g^{\eta'}_{\widetilde{\beta}\mu} x^{(\underline{\widetilde{\gamma}}+\underline{\mu})R+(\underline{\kappa}+\underline{\widetilde{\beta}})R'-(\underline{\xi}'+\underline{\eta}')} \\
&= \int_{\alpha,\beta,\delta,\gamma} d^1(\delta,\alpha) g^{\xi'}_{\gamma\delta} g^{\eta'}_{\alpha\beta} x^{\underline{\eta}R+\underline{\xi}R'-(\underline{\xi}'+\underline{\eta}')}.
\end{aligned}$$

Hence,

$$S_1 + S_2 = d^1(\xi', \eta') \int_{\alpha,\beta,\delta,\gamma} g^{\xi'}_{\gamma\delta} g^{\eta'}_{\alpha\beta} x^{\underline{\eta}R+\underline{\xi}R'-(\underline{\xi}'+\underline{\eta}')}.$$

The first assertion is proved. In order to prove the second part, by Theorem 4.6, we have

$$\begin{aligned}
&\int_{\delta,\delta_1,\delta_2,\iota'} g^{\delta}_{\delta_1\delta_2} \chi(\mathbb{P}\,\mathrm{Hom}(V_{\xi'}, I)_{V_\delta[1]\oplus I'}) x^{\underline{\delta_2}R+\underline{\delta_1}R'-\underline{\delta}+\underline{\dim}\,\mathrm{soc}\,I'} \\
&= \int_{\widetilde{\xi'},\delta_1,\delta_2,\iota'} g^{\xi'}_{\widetilde{\xi'}\delta_2} \chi(\mathbb{P}\,\mathrm{Hom}(V_{\widetilde{\xi'}}, I)_{V_{\delta_1}[1]\oplus I'}) x^{\underline{\delta_2}R+\underline{\widetilde{\xi'}}R'-\underline{\xi}'+\underline{\dim}\,\mathrm{soc}\,I} \\
&= \int_{\widetilde{\xi'},\delta_2} g^{\xi'}_{\widetilde{\xi'}\delta_2} \chi(\mathbb{P}\,\mathrm{Hom}(V_{\widetilde{\xi'}}, I)) x^{\underline{\delta_2}R+\underline{\widetilde{\xi'}}R'-\underline{\xi}'+\underline{\dim}\,\mathrm{soc}\,I}
\end{aligned}$$

and

$$\int_{\gamma,\gamma_1,\gamma_2,\rho'} g^{\gamma}_{\gamma_1\gamma_2}\chi(\mathbb{P}\operatorname{Hom}(P, V_{\xi'})_{P'[1]\oplus V_\gamma})x^{\underline{\gamma_2}R+\underline{\gamma_1}R'-\underline{\gamma}+\underline{\dim}\,P'/\operatorname{rad}P'}$$
$$=\int_{\widetilde{\xi'},\gamma_1,\gamma_2,\rho'} g^{\xi'}_{\gamma_1\widetilde{\xi'}}\chi(\mathbb{P}\operatorname{Hom}(P, V_{\widetilde{\xi'}})_{P'[1]\oplus V_{\gamma_2}})x^{\underline{\gamma_1}R+\underline{\widetilde{\xi'}}R'-\underline{\xi'}+\underline{\dim}\,P/\operatorname{rad}P}$$
$$=\int_{\widetilde{\xi'},\gamma_1} g^{\xi'}_{\widetilde{\xi'}\gamma_1}\chi(\mathbb{P}\operatorname{Hom}(P, V_{\widetilde{\xi'}}))x^{\underline{\gamma_1}R+\underline{\widetilde{\xi'}}R'-\underline{\xi'}+\underline{\dim}\,P/\operatorname{rad}P}.$$

We note that

$$\underline{\dim}\operatorname{soc} I = \underline{\dim}\,P/\operatorname{rad}P,$$

and

$$\chi(\mathbb{P}\operatorname{Hom}(P, M)) = \chi(\mathbb{P}\operatorname{Hom}(P, V_{\widetilde{\xi'}})) + \chi(\mathbb{P}\operatorname{Hom}(V_{\widetilde{\xi'}}, I)).$$

The second assertion is proved. □

5.3 An Example

We illustrate Theorem 5.4 by the following example.

Let Q be the Kronecker quiver $1 \rightrightarrows 2$. Let S_1 and S_2 be the simple modules associated with vertices 1 and 2, respectively. Hence,

$$R = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad R' = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

and

$$X_{S_1} = x^{\underline{\dim}\,S_1 R'-\underline{\dim}\,S_1} + x^{\underline{\dim}\,S_1 R-\underline{\dim}\,S_1} = x_1^{-1}(1+x_2^2),$$
$$X_{S_2} = x^{\underline{\dim}\,S_2 R'-\underline{\dim}\,S_2} + x^{\underline{\dim}\,S_2 R-\underline{\dim}\,S_2} = x_2^{-1}(1+x_1^2).$$

For $\lambda \in \mathbb{P}^1(\mathbb{C})$, let u_λ be the regular representation $\mathbb{C} \overset{1}{\underset{\lambda}{\rightrightarrows}} \mathbb{C}$. Then

$$X_{u_\lambda} = x^{(1,1)R'-(1,1)} + x^{(1,1)R-(1,1)} + x^{(0,1)R+(1,0)R'-(1,1)}$$
$$= x_1x_2^{-1} + x_1^{-1}x_2 + x_1^{-1}x_2^{-1}.$$

Let I_1 and I_2 be the indecomposable injective modules corresponding vertices 1 and 2, respectively; then

$$X_{(I_1\oplus I_2)[-1]} := x^{\underline{\dim}\operatorname{soc}(I_1\oplus I_2)} = x_1x_2.$$

The left side of the identity of Theorem 5.4 is

$$\dim_{\mathbb{C}}\operatorname{Ext}^1(S_1, S_2)X_{S_1}X_{S_2} = 2(x_1^{-1}x_2^{-1} + x_1x_2^{-1} + x_1^{-1}x_2 + x_1x_2).$$

The first term of the right side is

$$\int_{\lambda\in\mathbb{P}^1(\mathbb{C})} \chi(\mathbb{P}\,\mathrm{Ext}^1(S_1,S_2)_{u_\lambda})X_{u_\lambda} = 2(x_1^{-1}x_2^{-1} + x_1x_2^{-1} + x_1^{-1}x_2).$$

To compute the second term of the right side, we note that for any $f \neq 0 \in \mathrm{Hom}(S_2, \tau S_1)$, we have the following exact sequence:

$$0 \to S_2 \xrightarrow{f} \tau S_1 \to I_1 \oplus I_2 \to 0.$$

This implies $\mathrm{Hom}(S_2,\tau S_1)\setminus\{0\} = \mathrm{Hom}(S_2,\tau S_1)_{I_1\oplus I_2}$ Hence, the second term is equal to $2x_1x_2$.

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